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Fixed point index calculations on cones

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One of the most important tools in nonlinear functional analysis is the Leray–Schauder degree for compact vector fields defined on the closure of open subsets of some Banach space. The Leray–Schauder theory can be used to prove the existence of solutions of differential equations. In some cases in differential equations, we need to find the *positive* solutions. For example, if our equation is a partial differential equation for the concentration of a chemical, the only relevant solutions are the non-negative ones. In some cases, after some transformations, the problem is to find the fixed points in cones of positive mappings (mapping the cone itself). The fixed point index in cones has been used a great deal to study the fixed points of positive mappings in cones. Amann obtained the index relative to a cone at zero under an assumption of non-degeneracy. Dancer obtained a formula for the index of an isolated fixed point of positive mappings under a non-degeneracy assumption, which was extended by Du to strict set contractions. Also, Dancer and Du obtained an index formula for the boundary points of product cones. In this thesis, we extend these results for cones or products of cones to cover more general situations. We introduce new methods for index calculations not covered by the earlier theory. My results are not complete, but it seems very hard to obtain a complete calculation.

We arrange the thesis in the following way. In Chapter One, we consider two cones C_1 and C_2 in two Banach spaces E_1 and E_2, respectively. We assume that $C_2 - C_2$ is dense in E_2, that $E = E_1 \oplus E_2$ and that $C = C_1 \oplus C_2$. Let $A : \overline{\Omega} \to E$ and $A(u,v) = \big(A_1(u,v),\ A_2(u,v)\big)$, where $A_i(i = 1,2) : \overline{\Omega} \to E_i(i = 1,2)$ are completely continuous functions mapping the closure of an open set Ω in C containing zero to C_i. Furthermore assume that $A(u_0, 0) = (u_0, 0)$ and that $A'(u_0, 0) = \begin{pmatrix} B & C \\ D & F \end{pmatrix}$, where

$$\begin{aligned} B &= D_1 A_1(u_0, 0), \\ C &= D_2 A_1(u_0, 0), \\ D &= D_1 A_2(u_0, 0), \quad \text{and} \\ F &= D_2 A_2(u_0, 0). \end{aligned}$$

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Our main purpose is to find conditions such that the reduction property of the degree holds, that is, $\mathrm{ind}_C(A,(u_0,0)) = \mathrm{ind}_{C_1}(A_1|_{C_1},u_0)$. Note that this structure arises frequently in systems of algebraic or differential equations. We first discuss the case when $r(F) < 1$ (here $r(F)$ dentoes the spectral radius of F). We seek conditions such that $\mathrm{ind}_C(A,(u_0,0)) = \mathrm{ind}_{C_1}(A_1|_{C_1},u_0)$. In the case of $r(F) > 1$, we discuss conditions such that $\mathrm{ind}_C(A,(u_0,0)) = 0$. We then extend this result to the so called isolated set of fixed points relative to C. We also discuss several differential cases as follows:

(1) $D = 0$ and $r(F) < 1$,

(2) $D = 0$ and 1 is a simple eigenvalue of F with $r(F) \geqslant 1$.

The first case is discussed in Dancer. We give a detailed proof.

Next we discuss the case $D \neq 0$, but with the condition that the kernel and co-kernel are one dimensional with $N(I - A'(u_0,0)) = \mathrm{span}\left\{\begin{pmatrix} k \\ n \end{pmatrix}\right\}$, where $N(I - A'(u_0,0))$ is the kernel of $I - A'(u_0,0)$. The purpose of this section is to generalise Dancer's Theorem to the case $(0,h)^T \notin R(I - A'(u_0,0))$, where $R(I - A'(u-0,0))$ is the range if $I - A'(u_0,0)$. To use Dancer's formula, we need to calculate the spectral radius $r(\widetilde{A}'(u_0,0))$, where $\widetilde{A}'(u_0,0)$ is the resulting quotient operator of $A'(u_0,0)$. We discuss the spectrum of $A'(u_0,0)$ and we give a generalisation of Amann's result mentioned earlier.

In Chapter Two we first discuss the case of Dancer's isolated fixed point index formula on cones in the degenerate case when the kernel and co-kernel are 1-dimensional. By discussing the case when $r(\widetilde{A}'(u_0,0))$, and the case when $r(\widetilde{A}'(x-0)) = 1$, we obtain similar results to those in Chapter One. That is, we reduct the index calculation to the calculation on a low dimensional subspace. In the last part of this chapter, we generalise Dancer's formula to the situation when the fixed points of a positive completely continuous map are not isolated but connected. We assume that the set of fixed points is compact and isolated in the cone.

The main purpose of Chapter Three is to calculate the index of the isolated fixed point zero of A. For that, we do not assume that $\dim N(I - A'(0)) = 1$. We assume that $A(x)$ has a Taylor expansion of the form

$$A(x) = A'(0)x + \frac{A^{(s)}(0)}{s!}x^s + D(x)$$

with $\|D(x)\| = 0(\|x\|^s)$ for some $s \in \mathbb{N}$ and $A^{(s)}(0) \neq 0$. If $E = N(I - A'(0)) \oplus R(I - A'(0))$, we obtain results similar to those in Dancer. Also, we show that $\mathrm{ind}_C(A,0)$ is the sum of the indices of the points close to $(0,1)$ on each branch in C. Finally we consider

$$E = N(I - A'(0))^p \oplus R(I - A'(0))^p$$

with $p > 1$.

In Chapter Four, we discuss bifurcations for $A(x) = \lambda x$ with $A(x_0) = x_0$ and $\lambda_0 = 1$. Here $x_0 \neq 0$ and $A \in C^n(U,E)$, where U is a neighbourhood of x_0 and $n \geqslant 2$. We suppose that $A(x)$ has a Taylor expansion of the form

$$A(x) = x_0 + B(x - x_0) + C(x - x_0)^s + D(x)(x - x_0)^s$$

where

$$B = A'(x_0) \neq 0, \quad C = \frac{A^{(s)}(x_0)}{s!},$$

for some $2 \leqslant s \leqslant n$ and $D(x)$ is a bounded symmetric s-linear function with $D(x) \to 0$ as $x \to x_0$. We discuss two cases:

(1) $E = N(I - B) \oplus R(I - B)$, $x_0 \in R(I - B) = R$, and

(2) $x_0 \notin R(I - B)$, $\dim N(I - B) > 1$.

Under some additional conditions, we show that the solutions of the above equation near $(x_0, 1)$ form a finite number of curves. In the last theoretical part of this chapter, we discuss the bifurcation theory of the general equation $G(\lambda, x) = 0$ with $G(1, x_0) = 0$ near $(1, x_0)$ by applying the same techniques. We obtain results similar to those in the case $A(x) = \lambda x$. In ths last section of this chapter, we give some applications of the theory by analysing several examples of equations in $\mathbb{R}^n$.

In Chapter Five, we discuss some applications of Chapter Four to the fixed point index calculations in spaces or cones. We give some examples of fixed point index calculations for positive mappings in cones. The index in the full space is the sum of the indices of each point near $(x_0, 1)$ on each curve either to the left side or to the right side of 1. The index in the cone is the sum of all the indices of each point near $(x_0, 1)$ on each curve which lies in the cone either to the left side or to the right side of 1.

In Chapter Six, we discuss the index of a continuous arc of solutions for the fixed point equations $A(\lambda, x) = x$. We prove that either the whole connected arc is in the cone or the index of each isolated section relative to the cone is zero.

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