A note on an identity of Jacobi's

By NANCY WALLS.

That the determinant

$$\begin{vmatrix} h_r & h_s & \dots & h_t \\ h_{r-1} & h_{s-1} & \dots & h_{t-1} \\ h_{r-2} & h_{s-2} & \dots & h_{t-2} \\ \dots & \dots & \dots & \dots \end{vmatrix},$$

where h_r is the r^{th} complete homogeneous symmetric function in a set of n arguments, is equal to the quotient of a particular pair of alternants was shown essentially by Jacobi in 1841 and by Trudi in 1864. The present note exhibits this well-known relation, (3), as the immediate consequence of a simple matrix equality.

The symmetric functions h_r are connected with the elementary symmetric functions a_r in the same n arguments $\alpha, \beta, \dots, \kappa$ by the Wronski relations

$$a_0h_1 - a_1h_0 = 0,$$
$$a_0h_2 - a_1h_1 + a_2h_0 = 0,$$
$$a_0h_3 - a_1h_2 + a_2h_1 - a_3h_0 = 0,$$
$$\dots\dots\dots\dots$$

obtained by equating the coefficients of powers of x in the identity

$$\mathcal{A}(\alpha, \beta, \dots, \kappa;\ x)\, \mathcal{H}(\alpha, \beta, \dots, \kappa;\ x) = 1,$$

where

$$\mathcal{A} \equiv (1 - \alpha x)(1 - \beta x) \dots (1 - \kappa x)$$

and

$$\mathcal{H} \equiv (1 + \alpha x + \alpha^2x^2 + \dots)(1 + \beta x + \beta^2x^2 + \dots)\dots(1 + \kappa x + \kappa^2x^2 + \dots)$$

are the generating functions of the a_r and the h_r respectively. Let us similarly equate coefficients in the identity

$$\mathcal{A}(\beta, \gamma, \dots, \kappa;\ x)\, \mathcal{H}(\alpha, \beta, \gamma, \dots, \kappa;\ x) = 1 + \alpha x + \alpha^2x^2 + \dots \qquad (1)$$

We obtain relations of the type

$$(\alpha)a_0h_r - (\alpha)a_1h_{r-1} + (\alpha)a_2h_{r-2} - \dots = \alpha^r,$$

where $(\alpha)a_r$ denotes the r^{th} elementary symmetric function in the arguments with α omitted, a set of $n - 1$ arguments, so that all $a_r = 0$ for $r \geqq n$.

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In virtue of such relations for the different arguments we have

$$\begin{bmatrix} 1 & -(\alpha)a_1 & (\alpha)a_2 & \dots & \pm(\alpha)a_{n-1} \\ 1 & -(\beta)a_1 & (\beta)a_2 & \dots & \pm(\beta)a_{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & -(\kappa)a_1 & (\kappa)a_2 & \dots & \pm(\kappa)a_{n-1} \end{bmatrix} \begin{bmatrix} h^r & h_s & \dots & h_t \\ h_{r-1} & h_{s-1} & \dots & h_{t-1} \\ h_{r-2} & h_{s-2} & \dots & h_{t-2} \\ \dots & \dots & \dots & \dots \end{bmatrix}$$

$$= \begin{bmatrix} \alpha^r & \alpha^s & \dots & \alpha^t \\ \beta^r & \beta^s & \dots & \beta^t \\ \dots & \dots & \dots & \dots \\ \kappa^r & \kappa^s & \dots & \kappa^t \end{bmatrix}. \qquad (2)$$

If we take as second factor on the left hand side the matrix

$$H \equiv \begin{bmatrix} h_0 & h_1 & h_2 & \dots & h_{n-1} \\ \cdot & h_0 & h_1 & \dots & h_{n-2} \\ \dots & \dots & \dots & \dots & \dots \\ \cdot & \cdot & \cdot & \dots & h_0 \end{bmatrix},$$

so that $| H | = 1$, the corresponding determinantal relation gives us

$$| 1 \quad -(\beta)a_1 \quad (\gamma)a_2 \quad \dots \quad \pm(\kappa)a_{n-1} | = | \alpha^0 \quad \beta^1 \quad \gamma^2 \quad \dots \quad \kappa^{n-1} |,$$

as is otherwise obvious from consideration of the linear factors of each determinant, and hence the determinantal form of (2) is equivalent to

$$| \alpha^0\beta^1\gamma^2 \dots \kappa^{n-1} | \; | h_0h_{r-1}h_{s-2} \dots h_{t-n+1} | = | \alpha^0\beta^r\gamma^s \dots \kappa^t |, \quad (3)$$

Jacobi's identity.

It may be remarked in passing that, since the reciprocal of H is

$$A \equiv \begin{bmatrix} a_0 & -a_1 & a_2 & \dots & \pm a_{n-1} \\ \cdot & a_0 & -a_1 & \dots & \mp a_{n-2} \\ \dots & \dots & \dots & \dots & \dots \\ \cdot & \cdot & \cdot & \dots & a_0 \end{bmatrix},$$

because of the Wronski relations, the particular form of (2) with $r, s, \dots, t = 0, 1, \dots, n-1$ may also be written

$$J \equiv \begin{bmatrix} 1 & -(\alpha)a_1 & (\alpha)a_2 & \dots & \pm(\alpha)a_{n-1} \\ 1 & -(\beta)a_1 & (\beta)a_2 & \dots & \pm(\beta)a_{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & -(\kappa)a_1 & (\kappa)a_2 & \dots & \pm(\kappa)a_{n-1} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \alpha & \alpha^2 & \dots & \alpha^{n-1} \\ 1 & \beta & \beta^2 & \dots & \beta^{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \kappa & \kappa^2 & \dots & \kappa^{n-1} \end{bmatrix} \begin{bmatrix} a_0 & -a_1 & a_2 & \dots & \pm a_{n-1} \\ \cdot & a_0 & -a_1 & \dots & \mp a_{n-2} \\ \dots & \dots & \dots & \dots & \dots \\ \cdot & \cdot & \cdot & \dots & a_0 \end{bmatrix}, \qquad (4)$$

the determinantal form of which has been noted by Muir (*Theory of*

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Determinants, IV, p. 148) in referring to a paper in which $(-)^{\frac{1}{2}n(n-1)} | J |$ arises as the Jacobian of the functions $a_1, a_2, \ldots, a_n$.

The identity (1) may be extended to

$$\mathcal{A}(\ldots.;\ x)\ \mathcal{H}(\ldots., \alpha, \beta, \ldots.;\ x) = \mathcal{H}(\alpha, \beta, \ldots.;\ x).$$

and similarly we have

$$\mathcal{A}(\ldots., \alpha, \beta, \ldots.;\ x)\ \mathcal{H}(\ldots.;\ x) = \mathcal{A}(\alpha, \beta, \ldots.:\ x).$$

[In particular, (4) is an immediate consequence of

$$\mathcal{A}(\alpha, \beta, \ldots., \kappa;\ x)\ \mathcal{H}(\alpha;\ x) = \mathcal{A}(\beta, \ldots., \kappa;\ x).]$$

Further, $\mathcal{A}(\ldots., \alpha, \beta, \ldots.;\ x)\ \mathcal{H}(\ldots., \lambda, \mu, \ldots.;\ x)$
$$= \mathcal{A}(\alpha, \beta, \ldots.;\ x)\ \mathcal{H}(\lambda, \mu, \ldots.;\ x).$$

Generalizations of (2) may hence be obtained.

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A note on the "probleme des rencontres."

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1. This celebrated problem is treated in nearly all the textbooks on probability; for example in Bertrand's *Calcul des Probabilités*, 1889, pp. 15-17, in Poincaré's of the same title, 1896, pp. 36-38, and in most of the recent textbooks. The problem may be stated in abstract terms as follows: Among the $n!$ permutations $(\alpha_1\alpha_2\alpha_3\ldots\alpha_n)$ of the natural order $(123\ldots n)$, how many have no α_j equal to j? The problem has been clothed in many picturesque (and highly unlikely) "representations"; for example, by imagining n letters placed at random in n addressed envelopes, and inquiring what is the chance that no letter is in its correct envelope; or by imagining n gentlemen returning at random to their n houses; and so on, *ad risum*. Various derivations have also been given of the probability in question, namely

$$p(0;\ n) = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-)^n \frac{1}{n!} \tag{1}$$