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Seiberg-Witten Invariants of Lens Spaces

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Abstract. We show that the Seiberg-Witten invariants of a lens space determine and are determined by its Casson-Walker invariant and its Reidemeister-Turaev torsion.

Introduction

The Seiberg-Witten theory of rational homology spheres is particularly difficult since the usual count of monopoles leads to a metric *dependent* integer. Chen [2], Lim [10] and Marcolli-Wang [12] have shown that this count, suitably altered by a certain combination of eta invariants, leads to a topological invariant. For integral homology spheres, there is an unique spin^c structure and this altered count was shown to coincide with the Casson invariant; see [3, 11], and [20] in the special case of Brieskorn spheres. For a rational homology sphere N there are $\#H_1(N,\mathbb{Z})$ such invariants which are rational numbers. They define a function

$$\mathrm{sw} = \mathrm{sw}_N\colon\ \mathrm{Spin}^c(N) \to \mathbb{Q}, \quad \sigma \mapsto \mathrm{sw}(\sigma).$$

We will call sw_N the Seiberg-Witten invariant of N. This invariant can be further formalized as follows.

Recall that $H_1(N,\mathbb{Z}) \cong H^2(N,\mathbb{Z})$ acts freely and transitively on the space $\mathrm{Spin}^c(N)$ of spin^c structures on N

$$\mathrm{Spin}^c(N) \times H_1(N,\mathbb{Z}) \ni (\sigma, h) \mapsto \sigma \cdot h \in \mathrm{Spin}^c(N)$$

Thus each $\sigma_0 \in \mathrm{Spin}^c(N)$ defines an element $\mathrm{SW}_{\sigma_0} \in \mathbb{Q}[H]$ (= the rational group algebra of the multiplicative group $H = H_1(N,\mathbb{Z})$) defined by

$$\mathrm{SW}_{\sigma_0} = \sum_{h\in H} \mathrm{sw}_N(\sigma_0 \cdot h)h.$$

Clearly

$$\mathrm{SW}_{\sigma_0\cdot g} = \mathrm{SW}_{\sigma_0} \cdot g^{-1}, \quad \forall g \in H.$$

Thus, the collection $\mathrm{SW} := \{\mathrm{SW}_\sigma; \sigma \in \mathrm{Spin}^c(N)\} \subset \mathbb{Q}[H]$ coincides with an orbit of the right action of H on $\mathbb{Q}[H]$ so that the Seiberg-Witten invariant can be viewed as an element in $\mathbb{Q}[H]/H$.

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This Seiberg-Witten invariant is unchanged by natural involution

$$\operatorname{Spin}^c(N) \to \operatorname{Spin}^c(N), \quad \sigma \mapsto \bar{\sigma}.$$

The conditions $\operatorname{sw}(\sigma) = \operatorname{sw}(\bar{\sigma})$ and $\sigma \bar{\cdot} h = \bar{\sigma} \cdot h^{-1}$ imply

$$\mathrm{SW}_{\bar{\sigma}} = \overline{\mathrm{SW}}_{\sigma}$$

where $\bar{\ }\colon \mathbb{Q}[H] \to \mathbb{Q}[H]$ is the involution determined by $H \ni h \mapsto h^{-1} \in H$.

In [20] we have explicitly computed the invariant SW for Brieskorn homology spheres with at most 4 singular fibers and we have identified it with the Casson invariant.

In the present paper we use the results and techniques of [19] to produce a simple algorithm computing the SW. As in [19], these formulæ involve the Dedekind-Rademacher sums so, each concrete computation, although completely elementary, can be quite involved.

Denote by $\mathrm{SW}_{p,q}$ the Seiberg-Witten invariant of $L(p,q)$. It is an element of $\mathbb{Q}[\mathbb{Z}_p]/\mathbb{Z}_p$ and we will regard it as a polynomial in one variable t satisfying $t^p = 1$. The ring $\mathbb{Q}[\mathbb{Z}_p]$ is equipped with an augmentation map

$$\operatorname{aug}\colon \mathbb{Q}[\mathbb{Z}_p] \to \mathbb{Q}, \quad \sum_{k=0}^{p-1} a_k t^k \mapsto \sum_{k=0}^{p-1} a_k.$$

We prove in Section 3.2, Theorem 3.1 that

$$\operatorname{aug}(\mathrm{SW}_{p,q}) = \mathrm{CW}\big(L(p,q)\big). \tag{0.1}$$

where CW denotes the Casson-Walker invariant (see [31]) of a rational homology sphere normalized as in [9].

As explained in [1], the results of Meng-Taubes [14] imply that an analogous result is true for 3-manifolds with positive Betti numbers provided that the augmentation map is defined in a regularized sense. In this case we have an equality of the form

$$\sum_{\sigma} \operatorname{sw}_N(\sigma) = \mathrm{CWL}(N)$$

where CWL stands for the Casson-Walker-Lescop invariant of N and the sum on the left-hand-side should be understood in the ζ-regularized sense when $b_1(N) = 1$.

Following [15] we introduce the polynomial $\Sigma = \sum_{k=0}^{p-1} t^k$. It can be used to define a projection

$$\operatorname{Proj}\colon \mathbb{Q}[\mathbb{Z}_p] \to \Lambda_p := \ker \operatorname{aug}, \quad R \mapsto R - \frac{\operatorname{aug}(R)}{p}\Sigma.$$

Set

$$T_{p,q} = \operatorname{Proj}(\mathrm{SW}_{p,q}) = \mathrm{SW}_{p,q} - \frac{\mathrm{CW}\big(L(p,q)\big)}{p}\Sigma.$$

We can regard $T_{p,q}$ as an element of $\Lambda_p/\mathbb{Z}_p$. If A, B are two "polynomials" in Λ_p then $A \sim B$ will signify $A = t^n B$ for some $n \in \mathbb{Z}$.

The Reidemeister torsion of $L(p,q)$, which we denote by $\tau_{p,q}$, is also an element of Λ_p (see [15, 22]). More precisely, using the convention of [29] we have (see [15, 22, 29])

$$\tau_{p,q} \sim (1-t)^{-1}(1-t^q)^{-1}$$

i.e.

$$\tau_{p,q}(1-t)(1-t^q) \sim \hat{\mathbf{1}} := 1 - \frac{1}{p}\Sigma.$$

As explained in [15, 22] the "polynomial" $\hat{\mathbf{1}}$ represents $\mathbf{1}$ in Λ_p. We prove the following.

For any positive integers p, q such that $\gcd(p,q) = \gcd(p,q-1) = 1$ *we have*

$$T_{p,q}(1-t)(1-t^q) \sim \hat{\mathbf{1}} \tag{0.2}$$

The method we present works in the general case, when $\gcd(p,q-1) \geq 1$, but the additional arithmetical difficulties are not particularly enlightening so we have not included them.

The paper consists of three parts. The first part is a review of basic, known facts about Seifert manifolds. The second part explains how to use the results in[19] to compute the various eta invariants needed to compute the Seiberg-Witten invariants. The third part is devoted to the proof of (0.1) and (0.2).

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1 Seifert manifolds

1.1 Classification results

The literature on Seifert manifolds can be quite inconsistent as far as the meaning of Seifert invariant is concerned. This subsection is an informal comparative survey of the most frequently used descriptions of a Seifert manifold. In particular, we carefully keep track of the various sign conventions and carefully describe various natural metrics (Thurston geometries). This is particularly crucial in the case of lens spaces which admit *infinitely many* Seifert structures and thus *infinitely many* Thurston geometries. However *only two* (!!!) of them are Sasakian, which is exactly the geometric context needed to invoke the results in [17, 19].

A. The Equivariant Description In this paper, a *Seifert manifold (or fibration)* is a compact, *oriented*, smooth 3-manifold N without boundary, equipped with an infinitesimally free S^1 action. A fiber $S^1 \cdot x$ is called *regular* if the stabilizer St_x of x is trivial. Otherwise, the fiber is called *singular*. In this case St_x is a cyclic group $\mathbb{Z}_\alpha$ and the order of this stabilizer is called the *multiplicity* of the fiber. It is customary to identify St_x with the cyclic subgroup

$$C_\alpha = \Big\{ \exp\Big(\frac{2k\pi\mathbf{i}}{\alpha}\Big) ; k = 0, 1, \dots, \alpha - 1 \Big\} \subset S^1.$$

For brevity set $\rho_\alpha := \exp(\frac{2\pi\mathbf{i}}{\alpha})$. The *base* of the Seifert fibration is the space of orbits $\Sigma := N/S^1$. Topologically, it is a compact oriented surface but smoothly, it is a 2-dimensional orbifold. The orbifold singularities are all cone-like and correspond bijectively to the singular fibers. Equip N with an S^1-invariant Riemann metric h. Suppose $F \subset N$ is a singular fiber of multiplicity α containing the point x. The bundle $TN|_F$ splits orthogonally as

$$TN|_F = TF \oplus (TF)^\perp.$$

Both TF and $(TF)^\perp$ are S^1-equivariant bundles over F. The stabilizer C_α of x acts *effectively* on $(T_xF)^\perp$. Denote this action by

$$\tau \colon C_\alpha \to \mathrm{Aut}\big((T_xF)^\perp\big).$$

If we identify $(T_xF)^\perp$ as an oriented vector space with $\mathbb{C}$ then τ is completely described by an integer $0 < q < \alpha$, $\gcd(q, \alpha) = 1$ by the formula

$$\tau(\rho_\alpha)z = \rho_\alpha^q z.$$

We will denote this action by $\tau_{\alpha,q}$ or, when no confusion is possible, by τ_q. Following [23], we call the pair (α, q) the *orbit invariant* of the singular fiber F. Now denote by β the integer uniquely determined by the requirements

$$0 < \beta < 1, \quad \beta q \equiv 1 \pmod{\alpha}.$$

The pair (α, β) is called the *(oriented, normalized,) Seifert invariant* of the singular fiber F.

Using the principal C_α-bundle $P_\alpha = (S^1 \to S^1)$, $z \mapsto z^\alpha$, and the representation τ_q we can form the associated S^1-equivariant line bundle

$$E_{\alpha,q} := P_\alpha \times_{\tau_q} \mathbb{C} \to S^1.$$

The S^1-action on $E_{\alpha,q}$ is induced from the obvious action on $S^1 \times \mathbb{C}$

$$e^{\mathbf{i}\theta} \cdot (z_1, z_2) = (e^{\mathbf{i}\theta} z_1, z_2), \quad |z_1| = 1,\ z_2 \in \mathbb{C}$$

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which commutes with the action of C_α

$$\rho_\alpha(z_1, z) = (\rho_\alpha z_1, \rho_\alpha^{-q} z_2).$$

To describe this S^1-action on $E_{\alpha,q}$ more explicitly note first that $E_{\alpha,q}$ is diffeomorphic to $S^1 \times \mathbb{C}$. Such a diffeomorphism can be obtained using the C_α-invariant map

$$T\colon S^1 \times \mathbb{C} \to S^1 \times \mathbb{C}, (z_1, z_2) \overset{T}{\mapsto} (\zeta_1, \zeta_2) = (z_1^\alpha, z_1^q z_2).$$

Then we can regard (ζ_1, ζ_2) as global coordinates on $E_{\alpha,q}$ and we can describe the S^1-action by

$$e^{\mathbf{i}\theta}(\zeta_1, \zeta_2) = Te^{\mathbf{i}\theta} \cdot (z_1, z_2) = (e^{\mathbf{i}\alpha\theta}\zeta_1, e^{\mathbf{i}q\theta} z_2).$$

We have the following result (see [23]).

The Slice Theorem *There exists an S^1-invariant open neighborhood U of F in N, an S^1-invariant open neighborhood V of the zero section of $E_{\alpha,q}$ and an S^1-equivariant diffeomorphism $\phi\colon V \to U$ which maps the zero section to F and $\mathbf{1} \in S^1$ to a given fixed point $x \in F$.*

Denote D_r denotes the disk of radius r in the fiber of $E_{\alpha,q}$ over $\mathbf{1} \in S^1$, *i.e.*,

$$D_r = \{(1, \zeta_2) \in E_{\alpha,q}; |\zeta_2| \le r\}.$$

The surface $\phi(D_r)$ will be called a *slice* of the S^1-action. For simplicity, we will continue to denote it by D_r. Its boundary, equipped with the induced orientation, will be denoted by $\vec{\sigma}$. It can be explicitly described by the parametrization

$$(\zeta_1, \zeta_2) = (1, re^{\mathbf{i}t}), \quad t \in [0, 2\pi].$$

Denote by $\Delta(r) = \Delta_{\alpha,\beta}$ the bundle of disks of radius r determined by $E_{\alpha,q}$ and set $S(r) = S_{\alpha,\beta} := \partial\Delta_{\alpha,\beta}$. $\Delta(r)$ is usually known as the *fibered torus* corresponding to the Seifert invariants (α, β). Endow $S(r)$ with the induced orientation. $S(r)$ is equipped with a *free* S^1-action. Denote by $\vec{f}$ an orbit in $S(r)$ equipped with the induced orientation. It can be parametrized explicitly by

$$(\zeta_1, \zeta_2) = (e^{\mathbf{i}\alpha t}, e^{\mathbf{i}qt}), \quad t \in [0, 2\pi].$$

$\vec{f}$ meets $\vec{\sigma}$ geometrically α-times. In fact, when we use the outer-normal-first orientation convention for manifolds with boundary, we also have $\vec{\sigma} \cdot \vec{f} = \alpha$, algebraically as well.

A *section* of the S^1-action on $S(r)$ is a closed, oriented curve $\vec{s}$ such that $\vec{s} \cdot \vec{f} = 1$ both algebraically and geometrically. There exists a *canonical section* satisfying the homological condition

$$\vec{\sigma} = \alpha\vec{s} + \beta\vec{f}. \tag{1.1}$$

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Clearly the above condition uniquely determines the homology class of $\vec{s}$ in S_r.

We can now use these notions to describe the structure of Seifert fibrations. Suppose the Seifert fibration has $m \geq 1$ singular fibers $F_{x_1}, \ldots, F_{x_m}$ with normalized Seifert invariants

$$(\alpha_1, \beta_1), \ldots, (\alpha_m, \beta_m).$$

Delete small, pairwise disjoint, S^1-invariant neighborhoods $U_1, \ldots, U_m$ of the singular fibers, determined by the Slice Theorem. We get a 3-manifold with boundary

$$N' = N \setminus \Big(\bigcup_{i=1}^{m} U_i \Big) \tag{1.2}$$

equipped with a free S^1-action. This is a principal S^1-bundle $S^1 \hookrightarrow N' \to \Sigma' := N'/S^1$. The restriction of this bundle to $\partial\Sigma'$ has canonical sections, determined by (1.1). In other words, it is trivialized along the boundary. Such a bundle is completely determined topologically by an integer b, the relative degree (or Euler number). Here we have to warn the reader that our b differs by a sign from the conventions in [8, 16]. Denote by ℓ the rational number

$$\ell = b - \sum \frac{\beta_i}{\alpha_i}.$$

It is called the *rational Euler number* or *degree* of the Seifert fibration. The *normalized Seifert invariant* of N is defined as the collection

$$\big(g, b, \{(\alpha_1, \beta_1), \ldots, (\alpha_m, \beta_m)\} \big) \tag{1.3}$$

where g denotes the genus of Σ'. Two Seifert fibrations are equivalent iff they have identical normalized Seifert invariants.

B. Surgery Description To obtain a Seifert fibration with Seifert invariant (1.3) start with a S^1-bundle over an *oriented* Riemann surface Σ' of genus g with m small disjoint disks removed. We assume this is trivialized over $\partial\Sigma'$ and has relative degree b. N' has several boundary components, $\partial_i N'$, $i = 1, \ldots, m$, all diffeomorphic to a torus and oriented using the orientation conventions in the introduction. The Seifert manifold with the above Seifert invariants can be obtain by attaching, a solid torus $D^2 \times S^1$ to each $\partial_i N'$ using the orientation reversing homeomorphism

$$\Gamma_{\alpha_i, \beta_i} \colon \partial(D^2 \times S^1) \to \partial_i N'$$

homologically described by the matrix with integral entries and det $= -1$

$$\Gamma_{\alpha_i, \beta_i} := \begin{bmatrix} -\alpha_i & q_i \\ \beta_i & x_i \end{bmatrix}. \tag{1.4}$$

In the above description we assumed that $H_1(S^1 \times D^2; \mathbb{Z})$ is equipped with the natural basis $\{* \times \partial D^2, S^1 \times *\}$ while $H_1(\partial_i N', \mathbb{Z})$ is equipped with the basis $\{\vec{s}, \vec{f}\}$ given by the trivialization of N' along the boundary of Σ' and respectively a fiber. This gluing map implements the homological equation (1.1).

Often it is useful to work with un-normalized Seifert invariants. These are collections

$$\mathbf{S} = \big(g, b, m; (\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m)\big)$$

such that $\gcd(\beta_i, \alpha_i) = 1$, $\alpha_i \neq 0$. Two collections $\mathbf{S}$ and $\mathbf{S}'$ are called *equivalent* if $g = g'$, the collection of α_i-s not equal to 1 coincides (including multiplicities) with the collection of α'_j-s not equal to 1 and

$$b - \sum_i \frac{\beta_i}{\alpha_i} = b' - \sum_i \frac{\beta'_j}{\alpha'_j}.$$

We can carry out the above surgeries using the prescriptions given by these new invariants. We refer the reader to [8, 23] for a proof of the fact that equivalent un-normalized Seifert invariants lead to S^1-diffeomorphic Seifert manifolds.

C. Orbifold Description, [**5**] Start with a V-surface Σ with m singular points $x_1, \dots, x_m$ with isotropies $C_{\alpha_1}, \dots, C_{\alpha_m}$. Pick a complex line V-bundle $L \to \Sigma$ such that the isotropies in the fibers over the singular points are given by the representations

$$\tau_{\alpha_i,\omega_i} \colon C_{\alpha_i} \to U(1), \quad \tau_{\alpha_i,\omega_i}(\rho_{\alpha_i}) = \rho_{\alpha_i}^{\omega_i}.$$

Above, ω_i are integers satisfying the conditions

$$0 < \omega_i < \alpha_i, \quad \gcd(\alpha_i, \omega_i) = 1. \tag{1.5}$$

Then the unit circle bundle $N = S(L)$ determined by L is a Seifert manifold. In [19] we defined the Seifert invariants as the collection

$$\big(g, \ell, m; (\alpha_1, \omega_1), \dots, (\alpha_m, \omega_m)\big)$$

where ℓ is the rational degree of L. We will refer to these as the *Seifert V-invariants*. The normalized Seifert invariants (as defined in this paper) of N are

$$\beta_i := \alpha_i - \omega_i \tag{1.6}$$

and

$$b = \ell + \sum_i \frac{\beta_i}{\alpha_i}. \tag{1.7}$$

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We want to clarify one point. Denote by $|L|$ the desingularization of L (described in [19]). Then

$$\deg|L| = \deg L - \sum_i \frac{\omega_i}{\alpha_i} = \ell + \sum_i \frac{\beta_i}{\alpha_i} - m = b - m. \tag{1.8}$$

The description of Seifert fibrations via line V-bundles has its computational advantages. It allows a very convenient description of the cohomology group $H^2(N,\mathbb{Z})$. We include it here for later use.

Consider a Seifert fibration N over a 2-orbifold Σ defined as the unit circle bundle determined by a line V-bundle $L_0 \to \Sigma$. Suppose the singularities of Σ have isotropies $\alpha_1,\dots,\alpha_m$ while the isotropies of L_0 over the singular points are described by $\rho_{\alpha_i}^{\omega_i}$ as explained above. Denote by $\mathrm{Pic}^t(\Sigma)$ the space (Abelian group more precisely) of isomorphism classes of line V-bundles over Σ. Define a group morphism

$$\tau\colon\ \mathrm{Pic}^t(\Sigma) \to \mathbb{Q}\oplus\mathbb{Z}_{\alpha_1}\oplus\cdots\oplus\mathbb{Z}_{\alpha_m}$$

by

$$\tau(L) = (\deg L, \gamma_1 \bmod \alpha_1,\dots,\gamma_m \bmod \alpha_m)$$

where $\deg L$ is the rational degree of L and γ_i describe the isotropies of L over the singular points of Σ. Next, define

$$\delta\colon \mathbb{Q}\oplus\mathbb{Z}_{\alpha_1}\oplus\cdots\oplus\mathbb{Z}_{\alpha_m} \to \mathbb{Q}/\mathbb{Z}$$

by

$$\delta(c,\gamma_1,\dots,\gamma_m) = \Big(c - \sum_i \frac{\gamma_i}{\alpha_i}\Big) \bmod \mathbb{Z}.$$

In [5] it is shown that the sequence below is exact

$$0 \to \mathrm{Pic}^t(\Sigma) \xrightarrow{\tau} \mathbb{Q}\oplus\mathbb{Z}_{\alpha_1}\oplus\cdots\oplus\mathbb{Z}_{\alpha_m} \xrightarrow{\delta} \mathbb{Q}/\mathbb{Z} \to 0. \tag{1.9}$$

Moreover, there exists an isomorphism of groups

$$H^2\big(S(L_0),\mathbb{Z}\big) \cong \mathbb{Z}^{2g}\oplus \mathrm{Pic}^t(\Sigma)/\mathbb{Z}[L_0], \tag{1.10}$$

where g is the genus of Σ and $\mathbb{Z}[L_0]$ denotes the cyclic subgroup of $\mathrm{Pic}^t(\Sigma)$ generated by L_0. The subgroup $\mathrm{Pic}^t(\Sigma)/\mathbb{Z}[L_0]$ of $H^2\big(S(L_0),\mathbb{Z}\big)$ consists of the Chern classes of the line bundles on $S(L_0)$ obtained by pullback from line V-bundles on Σ.

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1.2 Geometric Seifert Structures on Lens Spaces

We now want to apply the general considerations in the previous subsection to lens spaces.

If p, q are two coprime integers, $p > 1$ we define the lens space $L(p,q)$ as the quotient of

$$S^3 := \{(z_1, z_2) \in \mathbb{C}^2; |z_1|^2 + |z_2|^2 = 1\}$$

via the action of C_p given by

$$\rho_p(z_1, z_2) = (\rho_p z_1, \rho_p^q z_2). \tag{1.11}$$

Alternatively, we can describe $L(p,q)$ as a result of gluing two solid tori $D \times S^1$ along their boundaries using the gluing map $\Gamma_{q,p}$ (see [8]). This shows that we can regard a lens space as a Seifert manifold with (un-normalized) Seifert invariant $\big(g = 0, b = 0, (q,p)\big)$. In fact, as explained in [8, 28], any lens space admits infinitely many Seifert structures. They all have something in common. Their bases have zero genus and they have at most two singular fibers. Moreover, any Seifert fibration over S^2 with at most two singular fibers must be a Seifert fibration of a lens space. The Seifert invariants of all these Seifert fibrations are described in Section 4 of [8].

We will be interested only in those Seifert structure on a lens space such that the base is a good orbifold in the sense described in [28]. This can happen if and only if they have an (un-normalized) Seifert invariant

$$\big(g = 0, b = 0, (\alpha_1, \beta_1), (\alpha_2, \beta_2)\big)$$

satisfying $\alpha_1 = \alpha_2$. These Seifert structures were determined in [24] for any lens space $L(p,q)$. There are only two of them

$$\mathbf{S}_\pm(p,q) = \big(0, 0, (\alpha_\pm, \beta_1^\pm), (\alpha_\pm, \beta_2^\pm)\big) \tag{1.12}$$

which can be explicitly computed as follows:

- $\alpha_\pm = p/\gcd(p, q \pm 1)$
- $\beta_1^\pm + \beta_2^\pm = \mp \gcd(p, q \pm 1)$
- $\beta_2^\pm \cdot \frac{q \pm 1}{\gcd(p, q\pm 1)} \equiv -1 \bmod \alpha_\pm$.

We will refer to the above Seifert structures on $L(p,q)$ as the *geometric Seifert structures*. There is a more conceptual description of these structures. To present it, recall first the Hopf actions of S^1 on S^3 given by

$$h_\pm \colon (z_1, z_2) \stackrel{e^{i\theta}}{\mapsto} (e^{\pm i\theta} z_1, e^{i\theta} z_2).$$

The action (1.11) of C_p commutes with these action of S^1 and thus the Hopf actions descend to two infinitesimally free S^1-actions on the lens space $L(p,q)$. These define precisely the two geometric Seifert structures.

1.3 Sasakian Structures on Lens Spaces

All Seifert fibrations admit natural geometries, *i.e.* locally homogeneous Riemann metrics and their universal covers belong to a list of 6 homogeneous spaces (see [28]). In the case of lens spaces this geometry is induced from a round metric on their universal cover, S^3. We want to describe those Seifert structures which interact in a certain way with this metric. In the terminology of [17], we need a $(K, \lambda) \iff$ *Sasakian* structure. In this case this is equivalent to asking that the Seifert structures are the quotient of the Hopf actions on S^1 modulo the action (1.11) of C_p. In other words, we must restrict to geometric Seifert structures.

Consider a lens space $N = L(p, q)$ equipped with a geometric Seifert structure with (un-normalized) invariant

$$\big(g = 0, b = 0, (\alpha, \beta_1), (\alpha, \beta_2)\big).$$

The base $\Sigma = N/S^1$ is a 2-orbifold with at most two conical points of identical isotropies C_α. Denote by $g(R)$ the metric on N induced by the round metric on the 3-sphere of radius R. The radius R will be described below. The group S^1 acts by isometries of $g(R)$ so that ζ, the infinitesimal generator of this action, is a Killing vector field. ζ is nowhere vanishing and produces an orthogonal decomposition

$$TN = \operatorname{span}(\zeta) \oplus \operatorname{span}(\zeta)^{\perp}.$$

The action of S^1 is compatible with this splitting and thus, the metric on $\operatorname{span}(\zeta)^{\perp}$ induces an orbifold metric h on Σ. Now fix $R = R_0$ such that

$$\operatorname{vol}_h(\Sigma) = \pi. \tag{1.13}$$

The radius R_0 can be explicitly determined as follows. Observe first that the volume of N is equal to

$$\text{length regular fiber } \times \operatorname{vol}_h(\Sigma) = 2\pi^2 R_0/p$$

Since the regular fibers have length $(1/p) \times \big(\text{length of a great circle on } S^3(R_0)\big) = 2\pi R_0/p$. Hence

$$\operatorname{vol}(N) = 2\pi^2 R_0^2/p.$$

On the other hand

$$\operatorname{vol}(N) = \operatorname{vol}\big(S^3(R_0)\big)/p = 2\pi^2 R_0^3/p$$

from which we deduce $R_0 = 1$.

The regular fibers of N are geodesics and have the same length $2\pi/p$ so that ζ has length $1/p$. Denote by $\varphi \in \Omega^1(N)$ the $g(R_0)$-dual of ζ. The metric $g(R_0)$ can be described as

$$g(R_0) = \varphi^2 \oplus h.$$

For $0 < r < 1$ define the anisotropic rescaling

$$g_r = (pr)^2\varphi^2 \oplus h.$$

With respect to this metric the regular fibers have length $2\pi r$. Denote by ∇^r the Levi-Civita connection of the metric g_1. Following [18] we define for each $t \in (0, 1]$ an isometry

$$L_t\colon (TN, g_{rt}) \to (TN, g_r), \quad \zeta \mapsto t\zeta,\ X \mapsto X \quad \text{if } X \perp \zeta.$$

Now set

$$\tilde{\nabla}^{r,t} := L_t \nabla^{rt} L_t^{-1}.$$

The connection $\tilde{\nabla}^{r,t}$ is compatible with g_r but it is not symmetric. In [18] we have shown that the limit $\lim_{t\to 0} \tilde{\nabla}^{r,t}$ exists and defines a connection compatible with the metric g_r. We will call this limit the *adiabatic Levi-Civita* connection of the metric g_r and we will denote it by $\tilde{\nabla}^r$.

Observe that a lens space admits two geometric Seifert structures. Arguing as above we obtain two families of Riemann metrics g_r and h_ρ. Both have positive scalar curvature (for $r, \rho \ll 1$), and there exist values r_0, $\rho_0 > 0$ (which need not be equal) such that the metrics g_{r_0} is homothetic to the metric h_{ρ_0}.

2 Seiberg-Witten Invariants of Rational Homology Spheres

2.1 Definition

Suppose N is rational homology sphere equipped with a Riemann metric g. Pick a divergence free 1-form ν, thought of as a perturbation parameter for the 3-dimensional Seiberg-Witten equations $\mathrm{SW}(g, \nu, \sigma)$ on (N, g, σ), where σ is a spin^c structure on N. Denote by $\mathbb{S}_\sigma$ the bundle of complex spinors associated to σ and set $\det\sigma = \det\mathbb{S}_\sigma$. The pair (g, ν) is said to be *good* iff the following hold:

- The irreducible solutions of $\mathrm{SW}(g, \nu, \sigma)$ are nondegenerate for all σ.
- If $\theta = (\psi = 0, A_\sigma)$ is the reducible solution of $\mathrm{SW}(g, \nu, \sigma)$ then $\ker \mathfrak{D}_{A_\sigma} = 0$ where $\mathfrak{D}_{A_\sigma}$ denotes the Dirac operator on $\mathbb{S}_\sigma$ coupled with the connection A_σ on $\det\sigma$.

For any irreducible solution α of $\mathrm{SW}(g, \nu, \sigma)$ denote by $i(\alpha, \theta)$ the virtual dimension of the space of tunnelings (= connecting gradient flow lines) from α to θ.

Fix a spin^c structure σ on N and a *good* pair (g, ν). The set of gauge equivalence classes of monopoles is finite and it consists of a unique nondegenerate reducible monopole $\theta = (0, A_\sigma)$ and finitely many, nondegenerate irreducible ones $\{\mathrm{C}_k; i = 1, \ldots, n\}$. Set

$$n_k = n_k(g) = i(\mathrm{C}_k, \theta),$$

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and

$$F(\sigma) = F_g(\sigma) = 4\eta(\mathfrak{D}_{A_\sigma}) + \eta_{\text{sign}}(g),$$

where $\eta_{\text{sign}}(g)$ denotes the eta invariant of the odd-signature operator on N determined by the metric g.

The Seiberg-Witten invariant of (N, σ) is the rational number

$$\text{sw}(\sigma) = \text{sw}_N(\sigma) = \frac{1}{8}F_g(\sigma) - \sum_k (-1)^{n_k(g)}. \tag{2.1}$$

In [2, 10] it was proved that $\text{sw}(\sigma)$ is independent of the choice of the good pair (g, η) and

$$\text{sw}(\sigma) \in \frac{1}{8h_1}\mathbb{Z},$$

where $h_1 = \#H_1(N, \mathbb{Z})$. Observe that $\text{sw}(\sigma) = \text{sw}(\bar{\sigma})$ where $\sigma \mapsto \bar{\sigma}$ is the natural involution on $\text{Spin}^c(N)$. Set

$$\text{sw}(N) := \sum_\sigma \text{sw}(\sigma). \tag{2.2}$$

2.2 Computations of Eta Invariants

Consider a lens space $N = L(p, q)$ and fix a geometric Seifert fibration structure on it. The discussion in Section 1.4 shows that the Seifert invariants of this structure has the form

$$\big(g = 0, b = 0, (\alpha, \beta_1), (\alpha, \beta_2)\big), \quad \alpha > 0.$$

More explicitly, this is one of the Seifert structures $\mathbf{S}_\pm(p, q)$ described in (1.12).

If we regard N as the unit circle bundle determined by a line V-bundle over $\Sigma = S^2(\alpha, \alpha) = N/S^1$ then we deduce that

$$\ell := \deg L_0 = -\frac{\beta_1 + \beta_2}{\alpha} \tag{2.3}$$

and the isotropies of L_0 over the singular points are given by

$$\omega_i = (-\beta_i) \bmod \alpha_i, \quad i = 1, 2. \tag{2.4}$$

Above and in the sequel, for any $x, n \in \mathbb{Z}$ we denote by $x \bmod n$ the smallest non-negative integer $\equiv x \bmod n$. We want to warn the reader that when $\alpha = 1$ the above Seifert structure has no singular fibers and N is a genuine smooth S^1-bundle over S^2 of degree ℓ.

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The canonical line bundle K_Σ of Σ has rational degree

$$\kappa := -\frac{2}{\alpha} \tag{2.5}$$

so that the rational Euler characteristic is

$$\chi = -\kappa = \frac{2}{\alpha}. \tag{2.6}$$

Denote by $\eta_{\text{sign}}(r)$ the eta invariant of the odd signature operator of N equipped with the deformed metric g_r (described in Section 1.3). $\eta_{\text{sign}}(r)$ was computed in [24]. To describe it explicitly we need to introduce the *Dedekind-Rademacher sums* defined for the first time by Hans Rademacher in [25]. More precisely, for every pair of coprime integers $\alpha, \beta, \alpha > 1$ and any $x, y \in \mathbb{R}$ set

$$s(\beta, \alpha; x, y) := \sum_{r=1}^{\alpha} \Big(\Big(x + \beta \frac{r+y}{\alpha} \Big) \Big) \Big(\Big(\frac{r+y}{\alpha} \Big) \Big)$$

where for any $r \in \mathbb{R}$ we set

$$((r)) = \begin{cases} 0 & r \in \mathbb{Z} \\ \{q\} - \frac{1}{2} & r \in \mathbb{R} \setminus \mathbb{Z} \end{cases} \quad (\{r\} := r - [r]).$$

The sums $s(\beta, \alpha) := s(\beta, \alpha; 0, 0)$ are the Dedekind sums studied in great detail in [7, 26].

$$\eta_{\text{sign}}(r) = -\operatorname{sign}(\ell) + \frac{2\ell}{3}(\chi r^2 - \ell^2 r^4) + \frac{\ell}{3} - 4s(\omega_1, \alpha) - 4s(\omega_2, \alpha). \tag{2.7}$$

The canonical spin^c structure on the orbifold Σ (with determinant line bundle K_Σ^{-1}) determines by pullback a spin^c structure on N which we denote by σ_0. This allows us to bijectively identify the collection of spin^c structures on L with the space of isomorphism classes of complex line bundles. Since $H^2(N, \mathbb{Z}) = \mathbb{Z}_p$ is pure torsion, the discussion at the end of Section 1.1 shows that all the line bundles on N are pullbacks of line V-bundles. Thus

$$\operatorname{Spin}^c(N) \cong \operatorname{Pic}^t(\Sigma)/\mathbb{Z}[L_0] \tag{2.8}$$

where $\operatorname{Spin}^c(N)$ denotes the space of spin^c structures on N. If L is a line bundle on N then the spin^c structure $\sigma_0 \otimes L$ which corresponds to L has determinant line bundle

$$\det(\sigma_0 \otimes L) = L^{\otimes 2} \otimes \det \sigma_0 = L^{\otimes 2} \otimes \pi^* K_\Sigma^{-1}$$

where $\pi\colon N \to \Sigma$ is the natural projection. The associated bundle of complex spinors is

$$\mathbb{S}_L = L \oplus L \otimes \pi^* K_\Sigma^{-1}.$$

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In [19] it was shown that, up to gauge equivalence, there is a unique flat connection on $\det \sigma_L$ which we denote by A_L. The Levi-Civita connection of g_r and A_L canonically determine a connection on $\mathbb{S}_L$ compatible with the Clifford multiplication. Denote by $\mathfrak{D}_L$ the associated Dirac operator, by $\eta_{\mathrm{dir}}(L, r)$ its eta invariant, and

$$F_r(L) := 4\eta_{\mathrm{dir}}(L, r) + \eta_{\mathrm{sign}}(r).$$

The results of [19] show that for r sufficiently small, the unperturbed Seiberg-Witten equations corresponding to the spin^c structure L have only one solution which is reducible. It is also nondegenerate since the scalar curvature of g_r is positive. Thus, g_r is a good metric for $r \ll 1$ and since there is no Floer homology we deduce that

$$\mathrm{sw}(\sigma_0 \otimes L) = F_r(L).$$

We now show how one can use the results of [19, 18] to provide explicit descriptions of $F_r(L)$. We have to distinguish two cases.

A $\alpha = 1$ so that N is a degree ℓ line bundle over S^2 or, as a lens space, $N = L(\ell, -1) = L\big(|\ell|, |\ell| - \mathrm{sign}(\ell)\big)$. The signature eta invariant is

$$\eta_{\mathrm{sign}}(r) = -\mathrm{sign}(\ell) + \frac{2\ell}{3}(\chi r^2 - \ell^2 r^4) + \frac{\ell}{3} \tag{2.9}$$

In this case there is a unique spin structure on $\Sigma = S^2$ which corresponds to the unique holomorphic square root $K^{1/2}$ of K_Σ. This determines by pullback a spin structure on N and denote by σ_{spin} the spin^c structure associated to it. Then

$$\sigma_{\mathrm{spin}} = \sigma_0 \otimes \pi^* K_\Sigma^{1/2}.$$

For each integer $0 \le k < |\ell|$ we denote by L_k the line bundle of degree k over Σ and by σ_k the spin^c-structure

$$\sigma_{\mathrm{spin}} \otimes \pi^* L_k = \sigma_0 \otimes \pi^*(K^{1/2} \otimes L_k).$$

Also let $\mathfrak{D}_k$ denote the Dirac operator on $\mathbb{S}_{\sigma_k}$ determined by the unique flat connection on $\det \sigma_k$ and denote by $\eta_{\mathrm{dir}}(k, r)$ its eta invariant. Then

$$\mathrm{Spin}^c(N) = \{\sigma_k; 0 \le k < |\ell|\}.$$

In [19] we computed the eta invariants, not for the operator $\mathfrak{D}_k$, but for the adiabatic Dirac operators D_k. These are constructed using the connection on $\mathbb{S}_{\sigma_k}$ induced by the adiabatic Levi-Civita connection TN and the flat connection $\det \sigma_k$. The eta invariant of $\mathfrak{D}_k$ can be determined using variational formulæ corresponding to the affine deformation $(1-t)\mathfrak{D}_k + tD_k$. The difference $\eta_{\mathrm{dir}}(k, r) - \eta(D_k)$ can be expressed as the sum of a continuous (transgression) term and a discontinuity contribution (spectral flow). The transgression term is expressed in the second transgression formula of [18] while the analysis in Section 4 of [17] shows that the spectral flow contribution is zero if $r \ll 1$. We obtain the following results:

- $k = 0$ (use Theorem 2.4 of [18])

$$\eta_{\mathrm{dir}}(k, r) = \frac{\ell}{6} - \frac{\ell}{6}(\chi r^2 - \ell^2 r^4).$$

- $0 < k < |\ell|$ (use the equality (2.22) and the second transgression formula of [18])

$$\eta_{\mathrm{dir}}(k, r) = \frac{\ell}{6} + \frac{k^2}{\ell} - \operatorname{sign}(\ell)k - \frac{\ell}{6}(\chi r^2 - \ell^2 r^4).$$

Using (2.9) we deduce

$$F_r(k) = \frac{4}{\ell}k^2 - 4\operatorname{sign}(\ell)k + \ell - \operatorname{sign}(\ell).$$

We see that $F_r(k)$ is *independent of* r!!!

B $\alpha > 1$ The computations are similar in spirit to the ones in Case A but obviously they are more complex due to the presence of singular fibers.

Let $L \to N$ be a line bundle over $N = S(L_0)$ and set $\sigma = \sigma_0 \otimes L \in \mathrm{Spin}^c(N)$. To compute $\eta_{\mathrm{dir}}(\sigma, r) := \eta_{\mathrm{dir}}(L, r)$ we need to determine the *canonical representative* of L. This is the unique line V-bundle $\hat{L} = \hat{L}_\sigma \to \Sigma$ satisfying the conditions

$$\pi^*\hat{L} \cong L \tag{2.10}$$

$$\frac{\kappa - 2\deg \hat{L}}{2\ell} \in [0, 1). \tag{2.11}$$

Denote by $\rho = \rho(\sigma) \in [0, 1)$ the rational number sitting in the left-hand-side of (2.11) and by $0 \le \gamma_i = \gamma_i(\sigma) < \alpha$, $i = 1, 2$ the isotropy of the fibers of $\hat{L}_\sigma$ over the singular points. Finally set

$$d(\sigma) = \frac{\kappa}{2} - \ell\rho(\sigma) = \deg \hat{L}_\sigma.$$

In Proposition 1.10 of [19] we computed the eta invariant for the adiabatic Dirac operator $D_L = D_\sigma$ defined by using the adiabatic connection on $\mathbb{S}_\sigma$ and the flat connection on $\det \sigma$. To recover the eta invariant of $\mathfrak{D}_\sigma := \mathfrak{D}_L$ we use a deformation argument as in Case A and we deduce the following results:

- If $\rho(\sigma) = 0$ then

$$\begin{aligned}\eta_{\mathrm{dir}}(\sigma, r) = \frac{\ell}{6} - 2\sum_{i=1}^{2} s\big(\omega_i, \alpha; \gamma_i(\sigma)/\alpha, 0\big) \\ - \sum_{i=1}^{2}\left(\left(\frac{q_i\gamma_i(\sigma)}{\alpha}\right)\right) - \frac{\ell}{6}(\chi r^2 - \ell^2 r^4),\end{aligned} \tag{2.12}$$

where $0 \le q_i < \alpha$ denotes the inverse of $\omega_i \bmod \alpha$.

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- If $\rho(\sigma) > 0$ then

$$\begin{aligned}\eta_{\mathrm{dir}}(\sigma, r) = {} & \Big(1 - \frac{1}{\alpha}\Big)(1 - 2\rho) - \ell\rho(1-\rho) + 2\rho + \frac{\ell}{6} \\ & - 2\sum_{i=1}^{2} s\Big(\omega_i, \alpha; \frac{\gamma_i(\sigma) + \omega_i\rho}{\alpha}, -\rho\Big) \\ & - \sum_{i=1}^{2}\Big\{\frac{q_i\gamma_i(\sigma) + \rho}{\alpha}\Big\} - \frac{\ell}{6}(\chi r^2 - \ell^2 r^4),\end{aligned} \tag{2.13}$$

where $\{x\}$ denotes the fractional part of the real number x.

The above formulæ may seem hopelessly useless. Fortunately, the Dedekind-Rademacher sums satisfy a reciprocity law (see [25]) which makes them computationally very friendly. The reciprocity law, coupled with the identities

$$s(\beta, \alpha; x, y) = s(\beta - m\alpha, \alpha; x + my, y), \quad \forall m \in \mathbb{Z} \tag{2.14}$$

reduces the computation of any Dedekind-Rademacher sum to the special case $s(\beta, 1; x, y)$ which is

$$s(\beta, 1; x, y) = \big((\beta y + x)\big) \cdot ((y)) \tag{2.15}$$

The complexity of the computation is comparable with the complexity of Euclid's algorithm which is very fast.

Using (2.7), (2.12) and (2.13) we conclude that when $\rho(\sigma) = 0$ we have

$$\begin{aligned}F_r(\sigma) = {} & \ell - \mathrm{sign}(\ell) - 8\sum_{i=1}^{2} s\big(\omega_i, \alpha; \gamma_i(\sigma)/\alpha, 0\big) \\ & - 4\sum_{i=1}^{2}\Big(\Big(\frac{q_i\gamma_i(\sigma)}{\alpha}\Big)\Big) - 4\sum_{i=1}^{2} s(\omega_i, \alpha)\end{aligned} \tag{2.16}$$

and when $\rho(\sigma) > 0$ we have

$$\begin{aligned}F_r(\sigma) = {} & \ell - \mathrm{sign}(\ell) + 4\Big(1 - \frac{1}{\alpha}\Big)(1 - 2\rho) - 4\ell\rho(1-\rho) + 8\rho \\ & - 8\sum_{i=1}^{2} s\Big(\omega_i, \alpha; \frac{\gamma_i(\sigma) + \omega_i\rho}{\alpha}, -\rho\Big) \\ & - 4\sum_{i=1}^{2}\Big\{\frac{q_i\gamma_i(\sigma) + \rho}{\alpha}\Big\} - 4\sum_{i=1}^{2} s(\omega_i, \alpha).\end{aligned} \tag{2.17}$$

Note again the r has disappeared!!!

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To put the formulæ to work we need to have a complete list of the canonical representatives of the line bundles on N. Given the isomorphism (1.9) this reduces to an elementary number theoretic problem.

According to (1.9) any line V-bundle on Σ can be uniquely represented as a collection

$$\Big(\frac{i}{\alpha}, j \bmod \alpha, (i-j) \bmod \alpha\Big), \quad i, j \in \mathbb{Z}.$$

Set $n = (\beta_1 + \beta_2)$ so that $\ell = -n/\alpha$. A collection as above is the canonical representative of a line bundle as above if

$$\frac{\kappa - 2i/\alpha}{-2n/\alpha} = \frac{i+1}{n} \in [0,1).$$

Thus, when $\operatorname{sign}(n) = -1$ we deduce that the complete list of canonical representatives is

$$\mathcal{R}_n = \Big\{\Big(\frac{i}{\alpha}, j \bmod \alpha, (i-j) \bmod \alpha\Big); i = -1, -2, \dots, -|n|, 0 \le j < \alpha\Big\} \tag{2.18}$$

while when $\operatorname{sign}(n) = 1$ the complete set of canonical representatives is

$$\mathcal{R}_n = \Big\{\Big(\frac{i}{\alpha}, j \bmod \alpha, (i-j) \bmod \alpha\Big); i = -1, 0, \dots, |n| - 2, 0 \le j < \alpha\Big\}. \tag{2.19}$$

The invariant ρ of a canonical representative $\nu = (i/\alpha, j, i-j) \in \mathcal{R}$ is

$$\rho(\nu) = \frac{i+1}{n}. \tag{2.20}$$

Notice that we have a bijection

$$I_{n,\alpha} := \{-1, 0, \dots, |n| - 2\} \times \mathbb{Z}_\alpha \sim \mathcal{R}_n$$

given by the correspondence

$$(k, j \bmod \alpha) \sim \nu \mapsto \Big(\frac{\operatorname{sign}(n)k - c}{\alpha}, j, -\operatorname{sign}(n)k - c - j\Big),$$

where $c := 1 - \operatorname{sign}(n)$. The functions $\rho, \gamma_1, \gamma_2 \colon \mathcal{R} \to \mathbb{Q}$ can now be regarded as functions on $I_{n,\alpha}$. More precisely

$$\rho(k, j \bmod \alpha) = \frac{k+1}{|n|} \tag{2.21}$$

and

$$\gamma_1(k, j \bmod \alpha) = j, \quad \gamma_2(k, j \bmod \alpha) = \operatorname{sign}(n)k - c - j. \tag{2.22}$$

Finally we can now regard F_r as a function

$$F_r = F_r(k, j) \colon I_{n,\alpha} \to \mathbb{Q}$$

given by (2.16), (2.17), (2.21) and (2.22).

3 Seiberg-Witten ⇒ Casson-Walker + Reidemeister Torsion

In this section we describe a relationship between the Seiberg-Witten invariants of a lens space and other "classic" invariants.

If N is the lens space $L(p,q)$ then, as explained in Section 2.2, a geometric Seifert structure on N determines a spin^c-structure σ_0 on N. We will work with the geometric Seifert structure determined by $\alpha = p/\gcd(p,q-1)$ and we set

$$\mathrm{SW}_{p,q} = \sum_{j=0}^{p-1} \mathrm{sw}(\sigma_0 \cdot t)t^j,$$

where t is a generator of the cyclic group $\mathbb{Z}_p$. Observe that

$$\mathrm{sw}\big(L(p,q)\big) = \mathrm{aug}(\mathrm{SW}_{p,q}).$$

The *Casson-Walker* invariant of N is defined in [9, 31]. It is a rational number $\mathrm{CW}(N)$ uniquely determined by certain Dehn surgery properties. We will work with C. Lescop's normalization used in [9]. It is related to K. Walker's normalization used in [31] by the equality ([9, Property T5.0, p. 76]

$$\mathrm{CW}(N)_{\mathrm{Lescop}} = \frac{h_1}{2}\mathrm{CW}(N)_{\mathrm{Walker}}.$$

The Casson-Walker invariant of the lens space can be expressed in terms of the Dedekind sums. More precisely we have the equality (see [31])

$$\mathrm{CW}\big(L(p,q)\big) = -\frac{p}{2}s(q,p). \tag{3.1}$$

We can now state the first result of this section.

Theorem 3.1

$$\mathrm{sw}\big(L(p,q)\big) = \mathrm{CW}\big(L(p,q)\big).$$

3.1 Seiberg-Witten ⇒ Casson-Walker

Our proof of Theorem 3.1 is arithmetic in nature and relies on the computations in Section 2.2.

We will work with the same metric as in Section 2.2. Since it has positive scalar curvature we deduce there are no irreducible monopoles, the unique reducible is also nondegenerate and thus

$$\mathrm{sw}\big(L(p,q),\sigma\big) = \frac{1}{8}F_{p,q}(\sigma), \quad \forall \sigma \in \mathrm{Spin}^c\big(L(p,q)\big).$$

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To proceed further we need to organize the computational facts established in Section 2.2 in a form suitable to our current purposes.

Set $n = \gcd(p, q-1)$, $\alpha = p/n$

$$\beta_2 \cdot \frac{q-1}{n} \equiv -1 \bmod \alpha, \quad \beta_1 = n - \beta_2,$$
$$\omega_i = -\beta_i,\ q_i\omega_i \equiv 1 \bmod \alpha \quad \forall i = 1, 2.$$

The rational Euler number of $L(p, q)$ equipped with the above geometric Seifert structure is

$$\ell = -\frac{n}{\alpha} = -\frac{n^2}{p}.$$

For each positive integer m set

$$I_m := \{0, 1, \dots, m-1\} \quad \text{and} \quad I_m^* = \{1, \dots, m-1\}.$$

The set $\mathrm{Spin}^c\big(L(p, q)\big)$ can be identified with $I_n \times I_\alpha$ and we have several functions of interest

$$\rho\colon I_n \times I_\alpha \to \mathbb{Q}, \quad \rho(k, j) = \frac{k}{n},$$
$$\gamma_1, \gamma_2\colon I_n \times I_\alpha \to \mathbb{Z}, \quad \gamma_1(k, j) = j, \quad \gamma_2(k, j) = k - 1 - j.$$

The function $F_{p,q}(\sigma)$ can be regarded as a function $F\colon I_n \times I_\alpha \to \mathbb{Q}$. It is explicitly described by

$$\begin{aligned} F(k, j) = {} & \ell + 1 - 4\ell\rho(1-\rho) + 8\rho \\ & - 4\sum_{i=1}^{2} s(\omega_i, \alpha) - 8\sum_{i=1}^{2} s\Big(\omega_i, \alpha, \frac{\gamma_i + \omega_i\rho}{\alpha}, -\rho\Big) \\ & + 4\begin{cases} -\sum_{i=1}^{2}\big(\big(\frac{q_i\gamma_i}{\alpha}\big)\big) & \text{if } \rho = 0 \\ \big(1 - \frac{1}{\alpha}\big)(1 - 2\rho) - \sum_{i=1}^{2}\big\{\frac{q_i\gamma_i + \rho}{\alpha}\big\} & \text{if } \rho \neq 0 \end{cases}. \end{aligned} \tag{3.2}$$

We have to prove

$$\sum_{k\in I_n}\sum_{j\in I_\alpha} F(k, j) = -4ps(q, p). \tag{3.3}$$

The proof of (3.3) relies on two identities. The first one was proved by M. Ouyang, [24, p. 652]. More precisely, we have

$$\sum_{i=1}^{2} s(\omega_i, \alpha) = s(q, p) - \frac{1}{6p} - \frac{n^2}{12p} + \frac{1}{4}. \tag{3.4}$$

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The second one is central in the theory of Dedekind sums and has the form

$$\sum_{\mu\in I_m}\Big(\Big(\frac{\mu+w}{m}\Big)\Big)=((w)),\quad \forall m\in\mathbb{Z}_+,\ w\in\mathbb{R}. \tag{3.5}$$

For a proof we refer to [7].

Summing (3.4) over $(k, j)\in I_n\times I_\alpha$ and using the equality $p=n\alpha$ we deduce

$$4\sum_{k\in I_n}\sum_{j\in I_\alpha}s(\omega_i,\alpha)=4ps(q,p)-\frac{2}{3}-\frac{n^2}{3}+p. \tag{3.6}$$

We now proceed to sum over $(k, j)\in I_n\times I_\alpha$ all the terms entering into the definition of $F(k, j)$.

$$\sum_{k\in I_n}\sum_{j\in I_\alpha}(\ell+1)=-n^2+p. \tag{3.7}$$

$$8\sum_{k\in I_n}\sum_{j\in I_\alpha}\rho=8\sum_{j\in I_\alpha}\sum_{k\in I_n}\frac{k}{n}=\frac{8\alpha}{n}\frac{n(n-1}{2}=4(p-\alpha). \tag{3.8}$$

$$4\ell\sum_{k\in I_n}\sum_{j\in I_\alpha}\rho(1-\rho)=-\frac{4n}{\alpha}\sum_{j\in I_\alpha}\sum_{k\in I_n}\frac{k(n-k)}{n^2}=-\frac{4}{n}\sum_{k\in I_n}k(n-k)$$

$(\sum_{k\in I_n}k^2=\frac{n^3}{3}-\frac{n^2}{2}+\frac{n}{6})$

$$=-\frac{4}{n}\Big(\frac{n^3}{2}-\frac{n^2}{2}-\frac{n^3}{3}+\frac{n^2}{2}-\frac{n}{6}\Big)=-\frac{2}{3}n^2+\frac{2}{3}. \tag{3.9}$$

Next,

$$\begin{aligned}&\sum_{k\in I_n}\sum_{j\in I_\alpha}s\Big(\omega_i,\alpha,\frac{\gamma_i+\omega_i\rho}{\alpha},-\rho\Big)\\
&=\sum_{k\in I_n}\sum_{j\in I_\alpha}\sum_{\mu\in I_\alpha}\Big(\Big(\frac{\mu-\rho}{\alpha}\Big)\Big)\Big(\Big(\frac{(\omega_i(\mu-\rho)+\gamma_i+\omega_i\rho}{\alpha}\Big)\Big)\\
&=\sum_{\mu\in I_\alpha}\Big(\Big(\frac{\mu-\rho}{\alpha}\Big)\Big)\sum_{k\in I_n}\sum_{j\in I_\alpha}\Big(\Big(\frac{\gamma_i(k,j)+\omega_i\mu}{\alpha}\Big)\Big)\\
&=\sum_{\mu\in I_\alpha}\Big(\Big(\frac{\mu-\rho}{\alpha}\Big)\Big)\sum_{k\in I_n}\sum_{r\in I_\alpha}\Big(\Big(\frac{r+\omega_i\mu}{\alpha}\Big)\Big).\end{aligned}$$

According to (3.5), the last sum (over r) is equal to $\big((\omega_i\mu)\big)=0$. Hence

$$\sum_{k\in I_n}\sum_{j\in I_\alpha}s(\omega_i,\alpha,\frac{\gamma_i+\omega_i\rho}{\alpha},-\rho)=0. \tag{3.10}$$

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Using (3.5) again we deduce

$$\sum_{k\in I_n^*}\sum_{j\in I_\alpha}\left(\left(\frac{q_i\gamma_i(k,j)}{\alpha}\right)\right)=\sum_{k\in I_n^*}\sum_{r\in I_\alpha}\left(\left(\frac{r}{\alpha}\right)\right)\overset{(3.5)}{=\!=}0. \tag{3.11}$$

Observe that since $1-2\rho(k)=-\big(1-2\rho(n-k)\big)$ we have

$$\Big(1-\frac{1}{\alpha}\Big)\sum_{k\in I_n^*}\sum_{j\in I_\alpha}(1-2\rho)=0. \tag{3.12}$$

Finally, we have

$$\begin{aligned}\sum_{k\in I_n^*}\sum_{j\in I_\alpha}&\left\{\frac{q_i\gamma_i(k,j)+\rho(k)}{\alpha}\right\}\\ &=\sum_{k\in I_n}\sum_{j\in I_\alpha}\left\{\frac{q_i\gamma_i(k,j)+\rho(k)}{\alpha}\right\}-\sum_{j\in I_\alpha}\left\{\frac{q_i\gamma_i(0,j)}{\alpha}\right\}\\ &=\sum_{k\in I_n}\sum_{j\in I_\alpha}\left\{\frac{nq_i\gamma_i(k,j)+k}{p}\right\}-\sum_{j\in I_\alpha}\left\{\frac{q_i\gamma_i(0,j)}{\alpha}\right\}.\end{aligned}$$

Now observe that as k covers I_n and j covers I_α the quantity $\big(nq_i\gamma_i(k,j)+k \bmod p\big)$ covers I_p while $q_i\gamma_i(0,j)$ covers I_α. Hence

$$\sum_{k\in I_n}\sum_{j\in I_\alpha}\left\{\frac{nq_i\gamma_i(k,j)+k}{p}\right\}=\sum_{r\in I_p}\left\{\frac{r}{p}\right\}=\frac{p-1}{2}$$

and

$$\sum_{j\in I_\alpha}\left\{\frac{q_i\gamma_i(0,j)}{\alpha}\right\}=\sum_{r\in I_\alpha}\left\{\frac{r}{\alpha}\right\}=\frac{\alpha-1}{2}.$$

We conclude that

$$4\sum_{k\in I_n^*}\sum_{j\in I_\alpha}\sum_{i=1}^{2}\left\{\frac{q_i\gamma_i(k,j)+\rho(k)}{\alpha}\right\}=4(p-\alpha). \tag{3.13}$$

The identity (3.3) now follows from (3.6)–(3.13). Theorem 3.1 is proved. ■

Remark 3.2 As explained in [1], for any 3-manifold N we can define an invariant

$$\mathrm{sw}(N)=\sum_{\sigma}\mathrm{sw}_N(\sigma)$$

where the summation is carried over all spin^c structures of N. If $b_1(N) > 1$ then the above sum is finite. If $b_1(N) = 1$ then the above sum is infinite but admits a finite ζ-function regularization. When $b_1(N) > 0$ the results of [14] imply that $\mathrm{sw}(N)$ is equal to the Casson-Walker-Lescop invariant of N.

Recently, Marcolli and Wang [13] have proved that $\mathrm{sw}(N) = \mathrm{CW}(N)$ for any rational homology sphere. Theorem 3.1 is used as an initial step in their inductive proof.

3.2 Seiberg-Witten ⇒ Reidemeister Torsion

Consider the Reidemeister torsion $\tau_{p,q}$ of the lens space $L(p,q)$ described in the introduction. The goal of this section is to prove the following result.

Proposition 3.3 *If* $\gcd(p, q-1) = 1$ *then*

$$T_{p,q}(1-t)(1-t^q) \sim \hat{\mathbf{1}} \tag{3.14}$$

i.e., $T_{p,q} \sim \tau_{p,q}$.

Proof For a while we will not rely on the assumption $\gcd(p, q-1) = 1$. We will continue to use the notations in the previous subsection so that $n = \gcd(p, q-1)$.

As explained in Section 2.2, each $(k, j) \in I_n \times I_\alpha \cong I_{n,\alpha}$ defines a line bundle on $L_{k,j}$ on $L(p,q)$ and thus, via the first Chern class an element

$$e(k,j) = c_1(L_{k,j}) \in H^2\big(L(p,q), \mathbb{Z}\big) \cong \mathbb{Z}_p.$$

Moreover, the correspondence

$$e\colon I_n \times I_\alpha \to \mathbb{Z}_p, \quad (k,j) \mapsto e(k,j)$$

is a bijection.

Lemma 3.4 *There exists an isomorphism of abelian groups* $H^2\big(L(p,q), \mathbb{Z}\big) \to \mathbb{Z}_p$ *such that*

$$e(k,j) = q(k-1) - (q-1)j \bmod p.$$

Proof of the lemma $H^2\big(L(p,q), \mathbb{Z}\big)$ is torsion so according to the results in Section 1.1 it can be described in terms of the chosen geometric Seifert structure as follows.

Consider map $\mathbb{Q} \oplus \mathbb{Z}_\alpha \oplus \mathbb{Z}_\alpha \to \mathbb{Q}/\mathbb{Z}$

$$(d, \gamma_1, \gamma_2) \mapsto d - \frac{\gamma_1 + \gamma_2}{\alpha}$$

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and the element

$$L_0 = (-n, \omega_1, \omega_2) \in \ker \delta.$$

Recall that L_0 describes a line V-bundle over a genus 0 orbifold whose associated circle bundle coincides with the lens space equipped with the chosen Seifert structure. Then

$$H^2\big(L(p,q),\mathbb{Z}\big) \cong \ker \delta/\mathbb{Z}[L_0].$$

Now observe that $\ker \delta/\mathbb{Z}[L_0]$ has the presentation

$$0 \to \mathbb{Z}^2 \stackrel{A}{\to} \mathbb{Z}^2 \to \ker \delta/\mathbb{Z}[L_0] \to 0$$

where

$$A = \begin{bmatrix} -n & 0 \\ \omega_1 & \alpha \end{bmatrix}.$$

We let the reader verify that

$$\begin{bmatrix} 1 & 0 \\ 0 & p \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ q & 1-q \end{bmatrix} \cdot A \cdot \begin{bmatrix} y & -\alpha \\ -x & -\omega_2 \end{bmatrix} \tag{3.15}$$

where

$$y = -(q-1)/n \quad \text{and} \quad x = -\frac{\omega_2 y + 1}{\alpha}.$$

This shows that indeed

$$\ker \delta/\mathbb{Z}[L_0] \cong \mathbb{Z}_p.$$

To each pair $(k, j) \in I_n \times I_\alpha$ it corresponds the line bundle $L_{k,j}$ with Seifert data $(k-1, j, k-1-j) \in \ker \delta$. Its first Chern class is the image of the vector $\vec{v} = (k-1, j) \in \mathbb{Z}^2$ in the quotient $\mathbb{Z}^2/A\mathbb{Z}^2$. Using the equality (3.15) we deduce that this image is $(y_2 \bmod p)$ where

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ q & 1-q \end{bmatrix} \cdot \begin{bmatrix} k-1 \\ j \end{bmatrix}.$$

This establishes the assertion in the lemma. ∎

Denote by $c\colon \mathbb{Z}_p \to I_n \times I_\alpha$ the inverse of the map e described in the above lemma.

Lemma 3.5 *We have the following equalities.*

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(i) If $n = 1$ *then* $\alpha = p$ *and*

$$c(m) = (0, -\omega_2 m + \omega_1 \bmod p).$$

(ii) If $n \geq 1$ *then*

$$c(-1) = c(p-1) = (0, \alpha - 1)$$

and

$$c(-m) = c(p-m) = \big(r, (-m - s\omega_1) \bmod \alpha\big), \quad \forall m \in I_p$$

where $r \in I_n$ *and* $s \in \mathbb{Z}$ *are such that* $ns = (m-1) + r$ *so that*

$$r = -(m-1) \bmod n \quad \textit{and} \quad s = \left\lceil \frac{m-1}{n} \right\rceil$$

where $\lceil x \rceil$ *is the smallest integer* $\geq x$.

Proof We prove only part (i). The second part is left to the reader.

Observe that when $n = 1$ we have $I_n \times I_\alpha = \{0\} \times I_\alpha$. Thus we can write $c(m) = (0, j)$, where

$$m = -q - (q-1) j \bmod p.$$

Since $\omega_2 = (q-1)^{-1} \bmod p$ we have the following mod p equalities

$$\omega_2 m = -q\omega_2 - j = -(q - 1 + 1)\omega_2 - j = -\omega_2 - 1 - j.$$

The equality in (i) now follows form $\omega_1 + \omega_2 = -n = -1$. ■

In the remaining part of this section we assume

$$n = \gcd(p, q-1) = 1.$$

We can now write

$$\mathrm{SW}_{p,q} = \frac{1}{8} \sum_{m \in I_p} F\big(c(m)\big)\, t^m.$$

Since $\Sigma \cdot (1-t) = 0$ in $\mathbb{Q}[\mathbb{Z}_p]$ the equality (3.14) is equivalent to

$$\mathrm{SW}_{p,q}(1-t)(1-t^q) \sim \hat{\mathbf{1}}.$$

We will prove a slightly stronger statement, namely

$$\mathrm{SW}_{p,q}(1-t)(1-t^q) = \hat{\mathbf{1}}. \tag{3.16}$$

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Let us introduce the polynomial

$$f(t) = \sum_{j \in I_p} \Big(\Big(\frac{j}{p} \Big) \Big) t^j \in \mathbb{Q}[\mathbb{Z}_p].$$

A simple computation shows that $f(t^{-1}) = -f(t)$, and for all m coprime with p we have (see [22] for an interpretation using harmonic analysis)

$$\Big(\frac{1}{2} - f(t^m) \Big) (1 - t^m) = \hat{\mathbf{1}} \quad \text{in } \mathbb{Q}[\mathbb{Z}_p] \tag{3.17}$$

We want to express $\mathrm{SW}_{p,q}$ as a linear combination of polynomials of the form $t^a f(t^a)$, $t^a f(t^a) f(t^b)$ and Σ. Observe first that since $n = 1$, in the equality (3.2) of Section 3.1 we always have $\rho = 0$. Thus for all $(k, j) \in I_n \times I_\alpha$ we have

$$\begin{aligned} F(k, j) = \ell + 1 - 4 \sum_{i=1}^{2} s(\omega_i, \alpha) \\ - 8 \sum_{i=1}^{2} s\big(\omega_i, \alpha, \gamma_i(k, j)/\alpha, 0\big) - 4 \sum_{i=1}^{2} \Big(\Big(\frac{q_i \gamma_i(k, j)}{\alpha} \Big) \Big). \end{aligned}$$

Observe two things:

- Since $n = 1$ we always have $k = 0 \in I_1 = \{0\}$ so that we can write $\gamma_1(j)$ instead of $\gamma_i(k, j)$.
- The first term in the definition of $F(k, j)$ is independent of (k, j). Thus its contribution to $\mathrm{SW}_{p,q}$ will be of the form const.Σ which is cancelled upon multiplication by $(1 - t)$. Thus when computing $\mathrm{SW}_{p,q}(1 - t)(1 - t^q)$ we can neglect this first term.

For $i = 1, 2$ define

$$A_i = -8 \sum_{m \in I_p} s\Big(\omega_i, \alpha, \frac{\gamma_i\big(c(m)\big)}{\alpha}, 0 \Big) t^m, \quad B_i = \sum_{m \in I_p} \Big(\Big(\frac{q_i \gamma_i\big(c(m)\big)}{\alpha} \Big) \Big) t^m$$

where according to Section 3.2 we have

$$\gamma_1(j) = j, \quad \gamma_2(j) = -1 - j$$

so that according to Lemma 3.5 we have

$$\gamma_1\big(c(m)\big) = -\omega_2 m + \omega_1, \quad \gamma_2\big(c(m)\big) = \omega_2 m - \omega_1 - 1 = \omega_2(m + 1).$$

Observe that since $q_2 \omega_2 = 1 \bmod p$ and $\omega_2(q - 1) = 1 \bmod p$ we have

$$q_2 = (q - 1) \bmod p.$$

Lemma 3.6

$$B_1 = -t^{-q} f(t^{-q}), \tag{3.18}$$

$$B_2 = -t^{-1} f(t^{-1}), \tag{3.19}$$

$$A_1 = -t^{-q} f(t^{-q}) f(t^{q-1}), \tag{3.20}$$

$$A_2 = t^{-1} f(t^{-1}) f(t^{q-1}). \tag{3.21}$$

Proof For any $(m, p) = 1$ we will denote by $1/m$ the inverse of m mod p.

$$B_1 = -\sum_{m \in I_m} \left(\left(\frac{q_1(\omega_2 m - \omega_1)}{\mathcal{P}} \right) \right) t^m$$

$(\mu := q_1\omega_2 - q_1\omega_1 = q_1\omega_2 m - 1,\ m = \frac{\omega_1}{\omega_2}(\mu + 1))$

$$= -t^{\omega_1/\omega_2} \sum_{\mu \in I_p} \left(\left(\frac{\mu}{p} \right) \right) t^{\omega_1 \mu/\omega_2} = -t^{\omega_1/\omega_2} f(t^{\omega_1/\omega_2}).$$

Now observe that $1/\omega_2 = q_2 = q - 1$ and $\omega_1 = -1 - \omega_2$ so that $\omega_1/\omega_2 = -q$. This proves (3.18).

$$B_2 = \sum_{m \in I_m} \left(\left(\frac{q_2\omega_2(m+1)}{p} \right) \right) t^m = \sum_{\mu \in I_p} \left(\left(\frac{\mu}{p} \right) \right) t^{\mu - 1}$$
$$= t^{-1} f(t) = -t^{-1} f(t^{-1}).$$

This proves (3.19).

$$A_1 = \sum_{m \in I_p} \sum_{\mu \in I_p} \left(\left(\frac{\mu}{p} \right) \right) \left(\left(\frac{\omega_1\mu - \omega_2 m + \omega_1}{p} \right) \right) t^m$$
$$= \sum_{\mu \in I_p} \left(\left(\frac{\mu}{p} \right) \right) \sum_{m \in I_p} \left(\left(\frac{\omega_1\mu - \omega_2 m + \omega_1}{p} \right) \right) t^m$$

$(r = \omega_1\mu - \omega_2 m + \omega_1,\ m = -r/\omega_2 + \omega_1(\mu + 1)/\omega_2)$

$$= t^{\omega_1/\omega_2} \sum_{\mu \in I_p} \left(\left(\frac{\mu}{p} \right) \right) t^{\omega_1\mu/\omega_2} \sum_{r \in I_p} \left(\left(\frac{r}{p} \right) \right) t^{-r/\omega_2}$$
$$= t^{\omega_1/\omega_2} f(t^{\omega^1/\omega_2}) f(t^{-1/\omega_2})$$
$$= t^{-q} f(t^{-q}) f(t^{-(q-1)}) = -t^{-q} f(t^{-q}) f(t^{q-1}).$$

This proves (3.20). Finally, we have

$$\begin{aligned} A_2 &= \sum_{m\in I_p}\sum_{\mu\in I_p}\Big(\Big(\frac{\mu}{p}\Big)\Big)\Big(\Big(\frac{\omega_2\mu+\omega_2 m+\omega_2}{p}\Big)\Big)t^m \\ &= \sum_{\mu\in I_p}\Big(\Big(\frac{\mu}{p}\Big)\Big)\sum_{m\in I_p}\Big(\Big(\frac{\omega_2\mu+\omega_2 m+\omega_2}{p}\Big)\Big)t^m a \end{aligned}$$

$(r=\omega_2(m+\mu+1),\, m=r/\omega_2-\mu-1)$

$$= t^{-1}\sum_{\mu\in I_p}\Big(\Big(\frac{\mu}{p}\Big)\Big)t^{-\mu}\sum_{r\in I_p}\Big(\Big(\frac{r}{p}\Big)\Big)t^{r/\omega_2} = t^{-1}f(t^{-1})f(t^{q-1})$$

This proves (3.21). ■

We can now finish the proof of Proposition 3.3. Using Lemma 3.6 we deduce

$$\begin{aligned} &8\,\mathrm{SW}_{p,q}(1-t)(1-t^q) \\ &= (-8A_1-8A_2-4B_1-B_2+\mathrm{const.}\Sigma)(1-t)(1-t^q) \\ &= -4(2A_1+2A_2+B_1+B_2)(1-t)(1-t^q) \\ &= -4\big\{-t^{-q}f(t^{-q})\big(1+2f(t^{q-1})\big) - t^{-1}f(t^{-1})\big(1-2f(t^{q-1})\big)\big\} \\ &\quad\times(1-t)(1-t^q) \\ &= -8\Big\{-t^{-q}f(t^{-q})\Big(\frac{1}{2}-f(t^{-(q-1)})\Big) - t^{-1}f(t^{-1})\Big(\frac{1}{2}-f(t^{q-1})\Big)\Big\} \\ &\quad\times(1-t)(1-t^q) \\ &\overset{(3.17)}{=} 8\Big\{t^{-q}f(t^{-q})\cdot\frac{\hat{\mathbf{1}}}{1-t^{1-q}} + t^{-1}f(t^{-1})\cdot\frac{\hat{\mathbf{1}}}{1-t^{q-1}}\Big\}(1-t)(1-t^q) \\ &= 8\Big\{t^{-1}f(t^{-q})\cdot\frac{\hat{\mathbf{1}}}{t^{q-1}-1} + t^{-1}f(t^{-1})\cdot\frac{\hat{\mathbf{1}}}{1-t^{q-1}}\Big\}(1-t)(1-t^q) \\ &= 8t^{-1}\frac{\hat{\mathbf{1}}}{1-t^{q-1}}\big(f(t^{-1})-f(t^{-q})\big)(1-t)(1-t^q) \\ &= 8t^{-1}\frac{\hat{\mathbf{1}}}{1-t^{q-1}}\big(f(t^{q})-f(t)\big)(1-t)(1-t^q) \\ &\overset{(3.17)}{=} 8t^{-1}\frac{\hat{\mathbf{1}}}{1-t^{q-1}}\Big(\frac{\hat{\mathbf{1}}}{1-t}-\frac{\hat{\mathbf{1}}}{1-t^q}\Big)(1-t)(1-t^q) \\ &= 8t^{-1}\frac{\hat{\mathbf{1}}}{1-t^{q-1}}\{(1-t^q)-(1-t)\} \\ &= 8t^{-1}\frac{\hat{\mathbf{1}}}{1-t^{q-1}}(t-t^q) = 8\cdot\hat{\mathbf{1}}. \end{aligned}$$

The proof of Proposition 3.3 is now complete. ■

Remark 3.7

(a) The restriction $\gcd(p, q-1) = 1$ in Proposition 3.3 can be dropped but we will not present the details as they are not particularly revealing.
(b) The results of this paper were posted on the Internet as math.DG/9901071 in early 1999. Since then we have succeeded to extend the results in this paper to arbitrary rational homology spheres; see [21]. The paper [21] does not render the results in the present paper obsolete. On the contrary, Theorem 3.1 and Proposition 3.3 are needed as stepping stones in our inductive proof.

References

[1] M. Blau and G. Thompson, *On the relationship between the Rozanski-Witten and the* 3*-dimensional Seiberg-Witten invariants*. hep-th/0006244
[2] W. Chen, *Casson invariant and Seiberg-Witten gauge theory*. Turkish J. Math. **21**(1997), 61–81.
[3] ———, *Dehn surgery formula for Seiberg-Witten invariants of homology* 3*-spheres*. dg-ga/9708006
[4] R. Fintushel and R. Stern, *Instanton homology of Seifert fibered homology three spheres.* Proc. London Math. Soc. **61**(1990), 109–137.
[5] M. Furuta and B. Steer, *Seifert fibred homology* 3*-spheres and the Yang-Mills equations on Riemann surfaces with marked points.* Adv. in Math. **96**(1992), 38–102.
[6] F. Hirzebruch, W. D. Neumann and S. S. Koh, *Differentiable Manifolds and Quadratic Forms.* Lect. Notes in Pure and Appl. Math. **4**, Marcel Dekker, 1971.
[7] F. Hirzebruch and D. Zagier, *The Atiyah-Singer Index Theorem and Elementary Number Theory*. Math. Lect. Series **3**, Publish or Perish Inc., Boston, 1974.
[8] M. Jankins and W. D. Neumann, *Lectures on Seifert Manifolds.* Brandeis Lecture Notes, 1983.
[9] C. Lescop, *Global Surgery Formula for the Casson-Walker Invariant*. Annals of Math. Studies **140**, Princeton University Press, 1996.
[10] Y. Lim, *Seiberg-Witten invariants for* 3*-manifolds in the case* $b_1 = 0$ *or* 1. Pacific J. Math. **195**(2000), 179–204.
[11] ———, *The equivalence of Seiberg-Witten and Casson invariants for homology* 3*-spheres*. Math. Res. Letters **6**(1999), 631–644.
[12] M. Marcolli and B. L. Wang, *Equivariant Seiberg-Witten-Floer homology*. dg-ga/9606003
[13] ———, *Exact triangles in monopole homology and the Casson-Walker invariant*. math.DG/0101127
[14] G. Meng and C. H. Taubes, $\underline{SW}$ *= Milnor torsion*. Math. Res. Letters **3**(1996), 661–674.
[15] J. Milnor, *Whitehead torsion.* Bull. Amer. Math. Soc. **72**(1966), 358–426.
[16] W. D. Neumann and F. Raymond, *Seifert manifolds, plumbing,* μ*-invariant and orientation reversing maps*. In: Lecture Notes in Math. **644**, 161–195.
[17] L. I. Nicolaescu, *Adiabatic limits of the Seiberg-Witten equations on Seifert manifolds.* Comm. Anal. Geom. **6**(1998), 301–362.
[18] ———, *Eta invariants of Dirac operators on circle bundles over Riemann surfaces and virtual dimensions of finite energy Seiberg-Witten moduli spaces*. Israel. J. Math. **114**(1999), 61–123.
[19] ———, *Finite energy Seiberg-Witten moduli spaces on* 4*-manifolds bounding Seifert fibrations.* Comm. Anal. Geom. **8**(2000), 1027–1096.
[20] ———, *Lattice points inside rational simplices and the Casson invariant of Brieskorn spheres.* Geom. Dedicata, to appear.
[21] ———, *Seiberg-Witten invariants of rational homology spheres*. math.GJ/0103020
[22] ———, *Reidemeister Torsion*. notes available at: http://www.nd.edu/˜lnicolae/
[23] P. Orlik, *Seifert Manifolds.* Lect. Motes in Math. **291**, Springer-Verlag, 1972.
[24] M. Ouyang, *Geometric invariants for Seifert fibered* 3*-manifolds.* Trans. Amer. Math. Soc. **346**(1994), 641–659.
[25] H. Rademacher, *Some remarks on certain generalized Dedekind sums.* Acta Arith. **9**(1964), 97–105.
[26] H. Rademacher and E. Grosswald, *Dedekind Sums.* The Carus Math. Monographs, MAA, 1972.
[27] R. von Randow, *Zür Topologie von dreidimensionalen Baummanigfatigkeiten.* Bonner Math. Schriften **14**(1962).
[28] P. Scott, *The geometries of* 3*-manifolds*. Bull. London. Math. Soc. **15**(1983), 401–487.

https://doi.org/10.4153/CJM-2001-032-7 Published online by Cambridge University Press

[29] V. G. Turaev, *Euler structures, nonsingular vector fields and torsions of Reidemeister type*. Izv. Akad. Nauk. USSR **53**(1989); English Transl. Math. USSR-Izv. **34**(1990), 627–662.
[30] ______, *Torsion invariants of* spin^c *structures on* 3*-manifolds*. Math. Res. Letters **4**(1997), 679–695.
[31] K. Walker, *An Extension of Casson's Invariant*. Annals of Math. Studies **126**, Princeton University Press, 1992.

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