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ON A UNIQUENESS THEOREM

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The classical uniqueness theorems of Riesz and Koebe show an important characteristic of boundary behavior of analytic functions in the unit disc. The purpose of this note is to discuss these uniqueness theorems on hyperbolic Riemann surfaces. It will be necessary to give additional hypotheses because Riemann surfaces can be very nasty. So, in this note the Wiener compactification will be used as ideal boundary of Riemann surfaces. The Theorem, Corollaries 1, 2 and 3 are of Riesz type, Riesz-Nevanlinna type, Koebe type and Koebe-Nevanlinna type respectively. Corollaries 4 and 5 are general forms of Corollaries 2 and 3 respectively.

Let f be a mapping from an open Riemann surface R into a Riemann surface R'. For a positive superharmonic function s on a hyperbolic subdomain G of R or R' and for a closed subset $F \subset G$, let $s_F{}^G$ denote the lower envelope of the family $\{s'\}$ of all positive superharmonic functions on G with $s' \geqq s$ quasi-everywhere on F. Let $\{R_n\}$ be a regular exhaustion of R, and let L: $a = L(t)$, $0 \leqq t < 1$, be a Jordan arc in R such that for every n there exists some $T(n)$ with $L(t) \subset R - \bar{R}_n$ for all $t \geqq T(n)$, where the bar denotes closure. The cluster set of f along L is defined by

$$C(f, L) = \bigcap_T \overline{f(L_T)},$$

where $L_T = \{L(t) | \; T \leqq t < 1, 0 \leqq T < 1\}$ and where the closure is taken on an arbitrary compactification of R'.

Henceforth let R be a hyperbolic Riemann surface, let R^* denote the Wiener compactification of R and let R'^* denote an arbitrary compactification of R'.

THEOREM. *Let f be a nonconstant analytic mapping from R into a hyperbolic Riemann surface R'. If there exists a family B of L such that $A' = \bigcup_{L \in B} C(f, L)$ is a polar set, then $A = \bigcup_{L \in B} \bar{L} \cap \Delta$ is of harmonic measure zero, where $\Delta = R^* - R$.*

Proof. Let s_1 be a positive superharmonic function on R' with

$$\lim_{b \to A'} s_1(b) = \infty.$$

Let V_k be an open set such that $A' \subset V_k$ and $s_1(b) > k$ on $D_k = V_k \cap R'$, $k = 1, 2, 3, \ldots$.

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For each $L \in B$, there exists some $T(k)$ with $f(L_{T(k)}) \subset D_k$. Indeed if $\overline{f(L_T)} \not\subset V_k$ for every T: $0 \leqq T < 1$, then for any finite number

$$\{T(j) | j = 1, 2, \ldots, m\}$$

of T it holds that

$$\bigcap_j (\overline{f(L_{T(j)})} - V_k) = \overline{f(L_{T(J)})} - V_k \neq \emptyset,$$

where $T(J) = \max_{1 \leqq j \leqq m} T(j)$. It hence follows that

$$\bigcap_T (\overline{f(L_T)} - V_k) \neq \emptyset.$$

This contradicts $C(f, L) \subset V_k$.

We have a sequence $\{L_{T(k)}\}_k$ such that $L_{T(k+1)} \subset L_{T(k)}$ and $f(L_{T(k)}) \subset D_k$. For simplicity we put $E(k) = \bigcup_{L \in B} L_{T(k)}$, $F(k) = \overline{E(k)} \cap R$ and $A(k) = \overline{E(k)} \cap \Delta$. Let $f_{A(k)}$ denote the characteristic function of $A(k)$ on Δ.

Let B' be the family of every bounded continuous function f' on Δ with $f_{A(k)} \leqq f'$, and put

$$\inf_{f' \in B'} \int_\Delta f' d\omega = \alpha,$$

where ω denotes the harmonic measure on Δ. It then holds that

$$\int_\Delta f_{A(k)} d\omega = \alpha.$$

Indeed if

$$\int_\Delta f_{A(k)} d\omega < \alpha,$$

then we take a sequence $\{f_n^*\}$ such that $f_n^* \in B'$, $f^*_{n+1} \leqq f_n^*$ and $\lim_n \int_\Delta f_n^* d\omega = \alpha$. We put $\lim_n f_n^* = f^*$. We then have

$$\int_\Delta f^* d\omega = \alpha$$

and

$$\int_{\Delta - A(k)} f^* d\omega > 0.$$

Therefore from the definitions of integrals and measures, there exists a compact set $K \subset \Delta - A(k)$ with $\int_K f^* d\omega > 0$. Let f_K be a continuous function such that f_K is 1 on $A(k)$ and is 0 on K, and which satisfies $0 \leqq f_K \leqq 1$ on Δ. We put $f^*_{K,n} = \min \{f_n^*, f_K\}$. It then follows that $f^*_{K,n} \in B'$ and

$$\inf_{f' \in B'} \int_\Delta f' d\omega < \alpha.$$

This is a contradiction.

We take a sequence $\{f_n'\}$ such that $f_n' \in B', f'_{n+1} \leqq f_n'$ and

$$\lim_n \int_\Delta f_n' d\omega = \int_\Delta f_{A(k)} d\omega.$$

For any point $p \in \Delta - A(k)$, let f_p' be a continuous function such that f_p' is 1 on $A(k)$ and is 0 at p, and which satisfies $0 \leqq f_p' \leqq 1$ on Δ. We put $f_{p,n} = \min\{f_n', f_p'\}$. Since R^* is a resolutive compactification, it follows that

$$\int_\Delta f_p d\omega = f_{p,H},$$

where $\lim_n f_{p,n} = f_p$ and where $f_{p,H}$ denotes the solution of the Dirichlet problem on R with f_p as boundary function. Let $u_{A(k)}$ denote the harmonic measure of $A(k)$. It then follows that

$$f_{p,H} = \int_\Delta f_{A(k)} d\omega = u_{A(k)}$$

and

$$f_{p,H} \leqq \int_\Delta f_p' d\omega = f_{p,H}'.$$

Since all the points of the harmonic boundary Γ of R^* are regular, it follows that for every point $p \in (\Delta - A(k)) \cap \Gamma$,

$$\lim_{R \ni a \to p} u_{A(k)}(a) \leqq \lim_{R \ni a \to p} f'_{p,H}(a) = 0,$$

and hence

$$\liminf_{Ra \ni \to p} (1_{F(k)}{}^R(a) - u_{A(k)}(a)) \geqq 0.$$

On the other hand, since each connected component of $F(k)$ is a nondegenerate continuum and since $1_{F(k)}{}^R$ can be extended continuously onto Δ, it follows that for every point $p \in A(k)$,

$$\liminf_{R \ni a \to p} (1_{F(k)}{}^R(a) - u_{A(k)}(a)) \geqq 0.$$

Therefore it follows from the minimum principle that

$$u_{A(k)} \leqq 1_{F(k)}{}^R \text{ on } R.$$

Further since

$$1_{F(k)}{}^R \leqq 1_{F'(k)}{}^{R'} \circ f \text{ on } R,$$

where $F'(k) = \overline{f(F(k))} \cap R'$, it follows that

$$u_{A(k)} \leqq 1_{F'(k)}{}^{R'} \circ f \text{ on } R.$$

Next since s_1 is quasi-continuous on R', from [**2**, p. 51] we can find a positive superharmonic function s_2 such that the restriction of s_1 to the closed set $\{b \in R' | s_2(b) \leqq k\}$ is continuous. It hence holds that

$$s^* = s_1 + s_2 \geqq k \text{ on } \bar{D}_k \cap R'.$$

Since $A(k+1) \subset A(k)$, we have

$$u_{A(k+1)} \leqq u_{A(k)} \text{ on } R.$$

Hence $u = \lim_k u_{A(k)}$ is harmonic on R. Further,

$$f(F(k)) \subset \overline{f(E(k))} \cap R' \subset \bar{D}_k \cap R',$$

that is,

$$F'(k) \subset \bar{D}_k \cap R'.$$

Therefore

$$1_{F'(k)}{}^{R'} \circ f \leqq (1/k)s^* \circ f \text{ on } R.$$

It follows that

$$u(c) = \lim_k u_{A(k)}(c) \leqq \lim_k (1/k)s^* \circ f(c) = 0$$

at a point c on R except for the polar set of $s^* \circ f$. Therefore $\bigcap_k A(k)$ is of harmonic measure zero.

Thus since $A \subset A(k)$, A is of harmonic measure zero.

COROLLARY 1. *Let f be a Lindelöfian mapping from R into a closed Riemann surface R'. If there exists a family B of L such that $\bigcup_{L \in B} C(f, L)$ is a single point b_0, then $\bigcup_{L \in B} \bar{L} \cap \Delta$ is of harmonic measure zero.*

Proof. Let b_0' be a point of R' distinct from b_0. There exists a function h which is harmonic on R' save at b_0 and b_0', and which has a positive normalized logarithmic singularity at b_0 and a negative normalized logarithmic singularity at b_0' (see [**3**, p. 213]). Further there exist two positive superharmonic functions s_1' and s_2' on R such that

$$h \circ f = s_1' - s_2'$$

(see [**2**, p. 113]). Let D_k' be a parametric disc about b_0 with $h > k$ on D_k', $k = 1, 2, 3, \ldots$.

It is now easy to see from the proof of the theorem that the assertion of Corollary 1 is proved. Indeed by using the notations in the proof of the theorem, we have

$$k \leqq h \circ f \leqq s_1' + s_2' \text{ on } F(k).$$

Therefore

$$u_{A(k)} \leqq 1_{F(k)}{}^R \leqq (s_1' + s_2')/k \text{ on } R.$$

It hence follows that

$$u(c') \leqq \lim_k (s_1'(c') + s_2'(c'))/k = 0$$

at a point c' on R except for the polar set of $s_1' + s_2'$. Thus $\bigcup_{L \in B} \bar{L} \cap \Delta$ is of harmonic measure zero.

Let $\{C_n\}$ be a sequence of Jordan arcs such that each C_n is compact on R. We say that $\{C_n\}$ converges to Δ, if for every compact set $K \subset R$ there exists an $n(K)$ such that $C_n \subset R - K$ for all $n \geqq n(K)$. We put $E^*(k) = \bigcup_{n \geqq k} C_n$ and $F^*(k) = \overline{E^*(k)} \cap R$. Since $1_{F^*(k)}{}^R$ can be extended continuously onto Δ, it is easy to see from the proofs of the theorem and Corollary 1 that the assertions of the following Corollaries 2 and 3 are proved.

COROLLARY 2. *Let f be a nonconstant analytic mapping from R into a hyperbolic Riemann surface R'. If $\{C_n\}$ converges to Δ and if $\bigcap_k f(\bigcup_{n \geqq k} C_n)$ is a polar set, then $\overline{\bigcup C_n} \cap \Delta$ is of harmonic measure zero.*

COROLLARY 3. *Let f be a Lindelöfian mapping from R into a closed Riemann surface R'. If $\{C_n\}$ converges to Δ and if $\bigcap_k f(\bigcup_{n \geqq k} C_n)$ is a single point, then $\overline{\bigcup C_n} \cap \Delta$ is of harmonic measure zero.*

Let E_0 be a subset of R with $\bar{E}_0 \cap \Delta \neq \emptyset$ and let $\{R_n'\}$ be an exhaustion of R. We put $E_0(k) = E_0 \cap (R - \bar{R}_k')$ and

$$A^* = \bigcap_k \overline{f(E_0(k))}.$$

A^* does not depend on the choice of exhaustions of R. It is easy to see that the assertions of the following Corollaries 4 and 5 are proved.

COROLLARY 4. *Let f be a nonconstant analytic mapping from R into a hyperbolic Riemann surface R'. If A^* is a polar set, then $\bar{E}_0 \cap \Delta$ is of harmonic measure zero.*

COROLLARY 5. *Let f be a Lindelöfian mapping from R into a closed Riemann surface R'. If A^* is a single point, then $\bar{E}_0 \cap \Delta$ is of harmonic measure zero.*

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