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# MULTIPLICATION INVARIANT SUBSPACES OF HARDY SPACES

T. L. LANCE AND M. I. STESSIN

ABSTRACT. This paper studies closed subspaces $L$ of the Hardy spaces $H^p$ which are $g$-invariant (*i.e.*, $g\cdot L \subseteq L$) where $g$ is inner, $g \neq 1$. If $p = 2$, the Wold decomposition theorem implies that there is a countable "$g$-basis" $f_1, f_2, \ldots$ of $L$ in the sense that $L$ is a direct sum of spaces $f_j \cdot H^2[g]$ where $H^2[g] = \{f \circ g \mid f \in H^2\}$. The basis elements $f_j$ satisfy the additional property that $\int_T |f_j|^2 g^k = 0$, $k = 1, 2, \ldots$ . We call such functions $g$-2-inner. It also follows that any $f \in H^2$ can be factored $f = h_{f,2} \cdot (F_2 \circ g)$ where $h_{f,2}$ is $g$-2-inner and $F$ is outer, generalizing the classical Riesz factorization. Using $L^p$ estimates for the canonical decomposition of $H^2$, we find a factorization $f = h_{f,p} \cdot (F_p \circ g)$ for $f \in H^p$. If $p \geq 1$ and $g$ is a finite Blaschke product we obtain, for any $g$-invariant $L \subseteq H^p$, a finite $g$-basis of $g$-$p$-inner functions.

1. **Introduction.** Let $X$ be a Hilbert space and $V\colon X \longrightarrow X$ be an isometry. The well-known Wold decomposition theorem states that

$$X = X_0 \bigoplus_{n=0}^{\infty} V^n X_1 \tag{1}$$

where $X_1 = X \ominus VX$ is a wandering subspace and $X_0 = \bigcap_{n=0}^{\infty} V^n X$ ([6], [4, p. 3]). If $X = H^2$ and $V$ is the operator of multiplication by an inner function $g$ the decomposition (1) implies that any function $f \in H^2$ can be written as

$$f(z) = \sum_{n=0}^{\infty} s_i(z) f_i\big(g(z)\big) \tag{2}$$

where $f_i \in H^2$, and $s_1, s_2, \ldots$ form an orthonormal basis of $H^2 \ominus gH^2$ (in this case $X_0 = \{0\}$). In the case when $g$ is a finite Blaschke product, $H^2 \ominus gH^2$ is finite dimensional with dimension equal to the order of $g$.

Any closed subspace $M \subset H^2$ which is invariant under multiplication by $g$ could be considered as $X$. Then (1) implies that any $f \in M$ can be written in the way similar to (2):

$$f(z) = \sum_{i=0}^{\infty} t_i(z) f_i\big(g(z)\big) \tag{3}$$

where $t_i$ form an orthonormal basis of $M \ominus gM$. It is easily seen that functions $t_i(z)$ (and $s_i(z)$) satisfy

$$\int_{\mathbf{T}} |t_i(z)|^2 g^k(z)\, dm(z) = 0 \tag{4}$$

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where **T** stands for the unit circle and $dm(z)$ is the normalized Lebesgue measure on **T**. We call a function that satisfies (4) $g$-2-inner. Thus, any $g$-invariant subspace of $H^2$ has a $g$-basis consisting of $g$-2-inner functions.

It is natural to ask which of these results could be extended to the case $p \neq 2$. Of course, if we are interested in a generating system such that its linear combinations are dense in the subspace, then the existence of such a system is easily obtainable from Hilbert space results. But in this paper we shall deal with the following question.

Let $M \subset H^p$ be a $g$-invariant subspace. By analogy with (4) we call a function $\varphi(z)$ $g$-$p$-inner if

$$\int_{\mathbf{T}} |\varphi(z)|^p g^k(z)\, dm(z) = 0, \quad k = 1, 2, \ldots. \tag{5}$$

We investigate whether $M$ has a $g$-basis consisting of $g$-$p$-inner functions. Our main result is

THEOREM. *If $g$ is a finite Blaschke product of order $n$ and $p \geq 1$ then any $g$-invariant subspace $M$ has a $g$-basis consisting of $g$-$p$-inner functions. That is, any $\varphi \in M$ can be written as*

$$\varphi(z) = \sum_{i=1}^{k} h_{i,p}(z)\varphi_i\big(g(z)\big)$$

*where the functions $h_{i,p}$ are $g$-$p$-inner, $i = 1, \ldots, k$ $k \leq n$ and $\varphi_i \in H^p$.*

The proof of this theorem is based on $g$-$p$-factorization of $H^p$ functions which generalizes the classical canonical factorization (if $g(z) = z$ they are the same) and on some estimates which give additional information about the decomposition (2).

The paper is organized as follows. In Section 2 we consider properties of $g$-2-inner functions and obtain $g$-2-factorization. Section 3 is devoted to $L^p$ estimates, which are used in Section 4 to prove the basis theorem. R. Douglas noted that the estimates of Section 3 should lead to another proof of the result of V. Mascioni [8] about operators similar to a contraction. We sketch these ideas in Section 5.

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2. $g$-2-**factorization.** Let $g$ be an inner function, $g \neq 1$. We denote by $H^2[g]$ the subspace of $H^2$ given by

$$H^2[g] = \{h(z) = \psi \circ g(z) : \psi \in H^2\}$$

and $P[g]$ the (non-closed) subspace of all polynomials in $g$. Note that if $g(0) = 0$, then $\|\psi \circ g\|_{H_2} = \|\psi\|_{H_2}$. Therefore, if $g(0) = 0$ then $H^2[g]$ is closed in $H^2$. Since $H^2[g] = H^2[\frac{g-g(0)}{1-\overline{g(0)}g}]$ we conclude that $H^2[g]$ is closed in $H^2$ for any inner function $g$.

For any subset $A \subset H^2$ we denote by $[A]_g$ the minimal closed $g$-invariant subspace of $H^2$ which contains $A$. If $L$ is a $g$-invariant subspace of $H^2$ then we define $L \ominus gL = (gL)_L^{\perp}$ to be the orthogonal complement in $L$ of $gL$ (note that $gL$ is closed).

Let $B$ be a Blaschke product with zeros $a_0, a_1, \dots$, whose multiplicities are $k_0, k_1, \dots$ respectively. Denote by $M$ the following subspace of $H^2$.

$$M = \overline{\text{span}}\left\{ \frac{z^{\ell-1}}{(1-\bar{a}_i z)^{\ell}}; \left\{ \begin{array}{l} i = 0, 1, 2, \dots \\ \ell = 1, 2, \dots, k_i \end{array} \right\} \right\}.$$

We arrange the generators of $M$ in the following order

$$\begin{gathered} \varphi_0 = \frac{1}{(1-\bar{a}_0 z)}, \varphi_1 = \frac{z}{(1-\bar{a}_0 z)^2}, \dots, \varphi_{k_0-1} = \frac{z^{k_0-1}}{(1-\bar{a}_0 z)^{k_0}}, \\ \varphi_{k_0} = \frac{1}{1-\bar{a}_1 z}, \varphi_{k_0+1} = \frac{z}{(1-\bar{a}_1 z)^2}, \dots, \varphi_{k_0+k_1-1} = \frac{z^{k_1-1}}{(1-\bar{a}_1 z)^{k_1}} \\ \varphi_{k_0+k_1} = \frac{1}{1-\bar{a}_2 z}, \dots . \end{gathered} \tag{6}$$

There is an orthonormal basis of $M$, $s_0, s_1, \dots$, such that $s_0, \dots, s_m$ form an orthonormal basis of $\text{span}\{\varphi_0, \dots, \varphi_m\}$ (such a basis might be obtained by the Gram-Schmidt process).

Then each of $s_0, \dots, s_m, \dots$ is a finite linear combination of the generators (6).

PROPOSITION 1. *The functions $s_0, s_1, \dots$ form an orthonormal $B$-basis of $H^2$, that is any function $f \in H^2$ is uniquely represented as an orthogonal sum*

$$f(z) = \sum_{i=0}^{\infty} s_i(z) f_i\big(B(z)\big)$$

*where $f_i \in H^2$, $i = 1, \dots$ and if*

$$f(z) = \sum_{i=0}^{\infty} s_i(z) f_i\big(B(z)\big) \text{ and } h(z) = \sum_{i=0}^{\infty} s_i(z) h_i\big(B(z)\big),$$

*then*

$$\langle f, h \rangle_{H^2} = \sum_{i=0}^{\infty} \langle f_i, h_i \rangle_{H^2} = \sum_{i=0}^{\infty} \int_{\mathbf{T}} f_i(z) \overline{h_i(z)}\, dm(z). \tag{7}$$

PROOF. The basis property is straightforward since any function which is orthogonal to $M$ is in $BH^2$. This implies that any function orthogonal to $\text{span}\{s_j(z)B^l(z) : j, l = 0, \dots, \}$ is divisible by all powers of $B$ and, therefore, vanishes identically.

To prove (7) it suffices to prove it in the case $f = s_i B^k$, $h = s_j B^l$ but in this case it is obvious. ∎

COROLLARY 1. *Let $g$ be any inner function. Then there is a $g$-basis of $H^2$, $s_0, \dots$, consisting of rational functions holomorphic in the closed disk and such that $s_i g^k \perp s_j g^l$ if $i \neq j$, for $i, j, k, l = 0, 1, \dots$ .*

PROOF. By Frostman's Theorem [5, p. 79] there is $\varepsilon \in \Delta$ such that

$$B = \frac{g - \varepsilon}{1 - \bar{\varepsilon} g}$$

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is a Blaschke product. Since $H^2[B] = H^2[g]$, the result follows from Proposition 1. ∎

DEFINITION. A function $\varphi \in H^p (p > 0)$ is called $g$-$p$-inner if $\|\varphi\|_p = 1$ and $\int_{\mathbf{T}} |\varphi(z)|^p g(z)^k \, dm(z) = 0$, $k = 1, 2, \ldots$ .

REMARK. We use the terminology similar to the classical one because, first, in case $g(z) = z$, $z$-$p$-inner functions are classical inner functions and, second, we shall see soon that a $g$-$p$-inner function satisfies some properties similar to a classical one.

REMARK. It follows directly from the definition that if $\varphi(z)$ is inner and $\psi(z)$ is $g$-$p$-inner, then $\chi = \varphi\psi$ is $g$-$p$-inner.

COROLLARY 2. *Let* $f(z) = \sum_{k=0}^{\infty} s_k(z) f_k\big(g(z)\big) \in H^2$. *Then* $f$ *is* $g$-2-*inner if and only if*

$$\sum_{i=0}^{\infty} |f_i(z)|^2 \Big|_{\mathbf{T}} = 1 \tag{8}$$

*where the equality (8) for boundary values of* $\{f_i\}$ *holds almost everywhere on* $\mathbf{T}$.

PROOF. We have by (7)

$$\begin{aligned} 0 &= \int_{\mathbf{T}} |f(z)|^2 g(z)^k \, dm(z) = \langle f(z) \cdot g(z)^k, f(z) \rangle_{H^2} \\ &= \sum_{i=0}^{\infty} \langle f_i(z) \cdot z^k, f_i(z) \rangle_{H^2} = \sum_{i=0}^{\infty} \int_{\mathbf{T}} |f_i(z)|^2 z^k \, dm(z) \\ &= \int_{\mathbf{T}} \Big( \sum_{i=0}^{\infty} |f_i(z)|^2 \Big) z^k \, dm(z). \end{aligned}$$

This equality holds for $k = \pm 1, \pm 2, \ldots$ . The Uniqueness Theorem implies that $\sum_{i=0}^{\infty} |f_i(z)|^2 \big|_{\mathbf{T}} \underset{a.e.}{=}$ constant. Since $\|f\|_2 = 1$, (8) holds a.e. ∎

REMARK. If $g$ is a finite Blaschke product of order $n$ then all the basis functions $s_0, s_1, s_2, \ldots, s_{n-1}$ are analytic in the closed disk $\overline{\Delta}$, and Corollary 2 implies that any $g$-2-inner function is in $H^\infty$. In the general case, this is not true. For example, let $a_n = 1 - \frac{1}{n^{3/2}}$. Then $\{a_n\}_{n=1}^{\infty}$ satisfies the Blaschke condition. Put

$$g(z) = B(z) = \prod_{n=1}^{\infty} \frac{a_n - z}{1 - \overline{a_n} z}.$$

Then it is easy to verify that

$$s_0 = 1, \; s_m(z) = \Big( \prod_{k=1}^{m} \frac{z - a_k}{1 - \overline{a_k} z} \Big) \frac{\sqrt{1 - |a_{m+1}|^2}}{1 - \overline{a_{m+1}} z}, \; m > 0.$$

By Corollary 2,

$$f(z) = \lambda \sum_{n=1}^{\infty} \frac{1}{n^{5/8}} s_n(z), \text{ where } \lambda = \Big( \sum_{n=1}^{\infty} \frac{1}{n^{5/4}} \Big)^{-1/2}$$

is $g$-2-inner. It is easily seen that $f(z)$ is unbounded as $z \longrightarrow 1$.

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PROPOSITION 2. *Every function $f \in H^2$ is uniquely (up to unimodular factor) represented as a product*

$$f(z) = h_{f,2}(z) \cdot F_2\big(g(z)\big) \tag{9}$$

*where $h_{f,2}$ is $g$-2-inner and $F_2(z) \in H^2$ is outer.*

REMARK. If $g(z) = z$ then the factorization (9) coincides with the classical canonical factorization.

REMARK. In the proof that follows we use Proposition 8 from Section IV which considers norm properties of products involving $g$-$p$-inner functions for arbitrary $p$. This result, which does not depend on any intervening work, is placed there for convenience.

PROOF OF PROPOSITION 2. Let $f \in H^2$. Denote by $M_f^2$ the $g$-invariant subspace generated by $f$:

$$M_f^2 = \overline{f \cdot P[g]}.$$

(Recall that $P[g]$ stands for the set of polynomials in $g$). Since

$$\dim(M_f^2 \ominus gM_f^2) = 1,$$

$M_f^2 \ominus gM_f^2$ is generated by a $g$-2-inner function $h$. We have $M_f^2 = \overline{h \cdot P[g]}$. By Proposition 8, $\overline{h \cdot P[g]} = h \cdot \overline{P[g]} = h \cdot H^2[g]$. In particular,

$$f = h \cdot \varphi\big(g(z)\big)$$

for some $\varphi \in H^2$. If $\varphi(z) = \hat{\varphi}(z) \cdot F(z)$, where $\hat{\varphi}(z)$ is inner and $F$ is outer, we write

$$h_{f,2}(z) = h(z) \cdot \hat{\varphi}\big(g(z)\big).$$

To prove the uniqueness let us suppose that there are two $g$-2-factorizations of $f \in H^2$, $f = h_1\,(F_1 \circ g) = h_2\,(F_2 \circ g)$, where $h_i$ is $g$-2-inner, $F_i$ is outer, $i = 1, 2$. If $P_n$ is a sequence of polynomials such that $F_1 P_n \underset{H^2}{\longrightarrow} 1$ then by Proposition 8

$$\|h_1 - f \cdot P_n(g)\|_2 = \Big\|h_1\big(1 - F_1(g)\,P_n(g)\big)\Big\|_2 = \|1 - F_1\,P_n\|_2 \longrightarrow 0$$

as $n \longrightarrow \infty$. This shows that the sequence $\{h_2(F_2 \circ g)(P_n \circ g)\}_{n=1}^{\infty}$ converges to $h_1$ in $H^2$. By the same Proposition 8, $\{F_2 P_n\}$ converges in $H^2$ to some function $\varphi$ and $h_2(z)\varphi\big(g(z)\big) = h_1(z)$. Write $\varphi(z) = \sum_{k=0}^{\infty} c_k z^k$. Since both $h_1$ and $h_2$ are $g$-2-inner we have

$$\begin{aligned} 0 &= \int_{\mathbf{T}} |h_1(z)|^2\, g(z)^k\, dm(z) = \int_{\mathbf{T}} |h_2(z)|^2 \left|\varphi\big(g(z)\big)\right|^2 g(z)^k\, dm(z) \\ &= \Big(\sum_{m=0}^{\infty} c_m \bar{c}_{m+k}\Big) \int_{\mathbf{T}} |h_2(z)|^2\, dm(z) = \int_{\mathbf{T}} |\varphi(z)|^2\, z^k\, dm(z). \end{aligned}$$

This implies that $|\varphi(z)| = 1$ almost everywhere on $\mathbf{T}$, that is $\varphi$ is inner. Since both $F_1$ and $F_2$ are outer, the $z$-invariant subspaces of $H^2$ generated by $h_1$ and $h_2$ are the same as the $z$-invariant subspace of $H^2$ generated by $f$. This yields $\varphi$ is a unimodular constant. ∎

## 3. $L^p$-estimates.

PROPOSITION 3. *Let $g$ be an inner function and $f \in H^\infty$, $f(z) = \sum_{k=0}^{\infty} s_k(z) f_k\big(g(z)\big)$, where $s_i$ are rational functions holomorphic in $\bar{\Delta}$ satisfying Corollary 1 and $1 \le p \le \infty$. Then there are constants $C_{k,p}$ such that*

$$\|f_k\|_p \le C_{k,p} \|f\|_p. \tag{10}$$

PROOF. Let us denote by $P_g$ the orthogonal projection $P_g : H^2 \longrightarrow \overline{\operatorname{span}}\{g^k, k = 0, 1, 2, \ldots\}$. This projection coincides with the restriction to $H^2$ of the conditional expectation operator associated with the $\sigma$-algebra determined by $g$. Therefore, ([3, p. 184])

$$\|f\|_p \ge \|P_g f\|_p \tag{11}$$

holds for all $p \ge 1$. This implies that $P_g$ may be extended to $H^p$ as a linear operator $H^p \longrightarrow H^p$ with norm 1. We use the same notation, $P_g$, for this extension. Obviously $P_g$ maps $H^p$ into the closure in $H^p$ of $\operatorname{span}\{g^k, k \ge 0\}$. It is easily seen that

$$f_k \circ g = P_g(T_{\bar{s}_k} f) \tag{12}$$

where $T_{\bar{s}_k}$ stands for the Toeplitz operator with symbol $\bar{s}_k$. Write

$$s_k = \sum_{l=1}^{m} \sum_{r=1}^{n_l} \frac{\lambda_{lr} z^{r-1}}{(1 - \overline{a_l} z)^r}.$$

It is easy to verify that

$$\begin{aligned} T_{\bar{s}_k} f(z) = \sum_{l=1}^{m} \sum_{r=1}^{n_l} \frac{\lambda_{lr}}{(z - a_l)^r} \Big\{ z\Big( f(z) - \sum_{t=0}^{r-2} \frac{1}{t!} f^{(t)}(a_l)(z - a_l)^t \Big) \\ - \frac{1}{(r-1)!} a_l f^{(r-1)}(a_l)(z - a_l)^{r-1} \Big\}. \end{aligned} \tag{13}$$

Since $|z - a_l|$, $l = 1, \ldots, m$ are separated from zero when $|z| = 1$, (13) implies that there are constants $C_{k,p}$ such that

$$\|T_{\bar{s}_k} f\|_p \le C_{k,p} \|f\|_p.$$

Now, (10) follows from (11). ■

Let $f \in H^\infty$, $f(z) = \sum_{k=0}^{\infty} s_k(z) f_k\big(g(z)\big)$. Denote by $Q_g^k$ the operator

$$Q_g^k(f) = f_k. \tag{14}$$

The following results are immediate corollaries of the previous proposition.

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COROLLARY 3. *The operator $Q_g^k$ may be extended to $H^p$ as a bounded linear operator $Q_g^k: H^p \longrightarrow H^p$.*

COROLLARY 4. *If $g$ is a finite Blaschke product of order $n$ then for all $1 \leq p \leq \infty$ and $f \in H^p$ we have the unique representation*

$$f(z) = \sum_{k=0}^{n} s_k(z) f_k\big(g(z)\big) \tag{15}$$

*where $f_k \in H^p$.*

PROPOSITION 4. *Let $g$ be a finite Blaschke product of order $n$, $f \in H^\infty$ and $f(z) = h_{f,2}(z) \cdot F_2\big(g(z)\big)$ be the $g$-2-factorization (9). Then $F_2 \in H^\infty$.*

PROOF. Let $g = \frac{z-a_0}{1-\overline{a_0}z} \cdots \frac{z-a_{n-1}}{1-\overline{a}_{n-1}z}$, where $a_1, \ldots, a_n \in \Delta$. Write

$$s_0 = \frac{\sqrt{1-|a_0|^2}}{1-\overline{a_0}z}, s_1 = \frac{z-a_0}{1-\overline{a_0}z} \cdot \frac{\sqrt{1-|a_1|^2}}{1-\overline{a_1}z}, \ldots,$$
$$s_k = \frac{z-a_0}{1-\overline{a_0}z} \cdots \frac{z-a_{k-1}}{1-\overline{a}_{k-1}z} \cdot \frac{\sqrt{1-|a_k|^2}}{1-\overline{a}_k z}, \ldots .$$

(This is the orthonormal basis associated to (6) in this case). Let

$$h_{f,2}(z) = \sum_{k=0}^{n-1} s_k(z) \cdot \hat{h}_k\big(g(z)\big) \text{ and } f(z) = \sum_{k=0}^{n-1} s_k(z) f_k\big(g(z)\big).$$

Then

$$f_k\big(g(z)\big) = \hat{h}_k\big(g(z)\big) \cdot F_2\big(g(z)\big)$$

and, by (8),

$$|F_2(w)|^2 = \sum_{k=0}^{n-1} |f_k(w)|^2$$

for almost all $w \in \mathbf{T}$. Now the result follows from Proposition 3. ∎

The following result establishes the estimate similar to (10) for an arbitrary $g$-basis in the case when $g$ is a finite Blaschke product.

PROPOSITION 5. *Let $g$ be a finite Blaschke product of order $n$, and let $\varphi_1, \ldots, \varphi_k (k \leq n)$ be $g$-2-inner functions such that*

$$\varphi_i g^\ell \perp \varphi_j g^m \quad i,j = 1, \ldots, k, \quad i \neq j, \quad m, \ell = 0, 1, 2, \ldots . \tag{16}$$

*Then there are constants $D_{\ell,p} (1 \leq p \leq \infty)$, $\ell = 1, 2, \ldots, k$ such that for any $f \in H^\infty$,*

$$f(z) = \sum_{i=1}^{k} \varphi_i(z) f_i\big(g(z)\big)$$

*we have the estimate*

$$\|f_i\|_p \leq D_{i,p} \|f\|_p, \quad i = 1, \ldots, k. \tag{17}$$

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PROOF. Write

$$\varphi_i(z) = \sum_{m=0}^{n-1} s_m(z)\hat{\varphi}^i_m\big(g(z)\big).$$

By Corollary 2 we have

$$\sum_{m=0}^{n-1} |\hat{\varphi}^i_m(w)|^2\Big|_{\mathbf{T}} \underset{a.e.}{=} 1. \tag{18}$$

The orthogonality condition (16) yields

$$\chi(z) = \int_{\mathbf{T}} \frac{\varphi_i(w)\overline{\varphi_j(w)}}{1 - z\bar{w}}\, dm(w) \in (H^2_0[g])^{\perp}, \quad i \neq j. \tag{19}$$

A proof similar to the one of Corollary 2 and (16) show that (19) yields

$$\sum_{m=0}^{n-1} \hat{\varphi}^i_m(w)\overline{\hat{\varphi}^j_m(w)}\Big|_{\mathbf{T}} \underset{a.e.}{=} 0, \quad i \neq j. \tag{20}$$

Denote by $A(w)$ the following $n \times k$ matrix

$$A(w) = \begin{bmatrix} \hat{\varphi}^1_0(w) & \cdots & \hat{\varphi}^k_0(w) \\ \cdots & \cdots & \cdots \\ \hat{\varphi}^1_{n-1}(w) & \cdots & \hat{\varphi}^k_{n-1}(w) \end{bmatrix}.$$

Then (18), (20) imply

$$A^*(w)A(w) = I \tag{21}$$

a.e. on $\mathbf{T}$ (where $A^*(w) = \overline{A(w)^T}$ is the adjoined matrix). If we denote by $A_{j_1\cdots j_k}(w)$ the $k \times k$ minor of $A(w)$ which is formed by rows $j_1, \ldots, j_k$ of $A(w)$, then (21) and the Binet-Cauchy formula [7, p. 35] imply

$$\sum_{(j_1,\ldots,j_k)} |\det\big(A_{j_1,\ldots,j_k}(w)\big)|^2 = 1$$

a.e. on $\mathbf{T}$. Hence, for almost every $w \in \mathbf{T}$

$$\max_{(j_1,\ldots,j_k)} \big|\det\big(A_{j_1,\ldots,j_k}(w)\big)\big| \geq \frac{1}{\sqrt{\binom{n}{k}}} = \sqrt{\frac{k!\,(n-k)!}{n!}}. \tag{22}$$

Denote by $B_{j_1,\ldots,j_k}$ the following subset of the circle $\mathbf{T}$.

$$B_{j_1\cdots j_k} = \left\{ w \in \mathbf{T} : \big|\det\big(A_{j_1\ldots j_k}(w)\big)\big| \geq \sqrt{\frac{k!\,(n-k)!}{n!}} \right\}.$$

Then (22) implies that

$$m(\mathbf{T}) = m\Big(\bigcup_{(j_1\cdots j_k)} B_{j_1\cdots j_k}\Big). \tag{23}$$

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where $m$ stands for the normalized Lebesgue measure on $\mathbf{T}$. But (22) and (23) imply the existence of at least one measurable step-function $N$, which maps the unit circle $\mathbf{T}$ into the set of $k$-tuples $(j_1, \ldots j_k)$, $0 \le j_\ell \le n-1$, $\ell = 1, \ldots, k$, $j_\ell \ne j_m$ if $\ell \ne m$,

$$N : w \longmapsto \big(j_1(w), \ldots, j_k(w)\big),$$

such that

$$\Big|\det\big(A_{N(w)}(w)\big)\Big| \ge \sqrt{\frac{k!\,(n-k)!}{n!}} \tag{24}$$

a.e. on $\mathbf{T}$.

Let

$$f(z) = \sum_{m=0}^{n-1} s_m(z)\hat{f}_m\big(g(z)\big) = \sum_{i=1}^{k} \varphi_i(z) f_i\big(g(z)\big).$$

Then

$$\begin{aligned} f(z) &= \sum_{m=0}^{n-1} s_m(z)\hat{f}_m\big(g(z)\big) = \sum_{i=1}^{k}\sum_{m=1}^{n-1} s_m(z)\hat{\varphi}^i_m\big(g(z)\big) f_i\big(g(z)\big) \\ &= \sum_{m=0}^{n-1} s_m(z) \sum_{i=1}^{k} \hat{\varphi}^i_m\big(g(z)\big) f_i\big(g(z)\big). \end{aligned}$$

This yields

$$\sum_{i=1}^{k} \hat{\varphi}^i_m(w) f_i(w) = \hat{f}_m(w), \quad m = 0, \ldots, n-1, \quad w \in \mathbf{T}.$$

In particular,

$$\sum_{i=1}^{k} \hat{\varphi}^i_m(w) f_i(w) = \hat{f}_m(w), \quad m = j_1(w), \ldots, j_k(w).$$

By Cramer's rule,

$$\begin{aligned} f_i(w) &= \frac{\det\begin{vmatrix} \hat{\varphi}^1_{j_1(w)}(w) & \cdots & \hat{f}_{j_1(w)}(w) & \cdots & \hat{\varphi}^k_{j_1(w)}(w) \\ \vdots & & \vdots & & \vdots \\ \hat{\varphi}^1_{j_k(w)}(w) & \cdots & \hat{f}_{j_k(w)}(w) & \cdots & \hat{\varphi}^k_{j_k(w)}(w) \end{vmatrix}}{\det\big(A_{N(w)}(w)\big)} \\ &= \lambda_1(w)\hat{f}_{j_1(w)}(w) + \lambda_2(w)\hat{f}_{j_2(w)}(w) + \cdots + \lambda_k(w)\hat{f}_{j_k(w)}(w). \end{aligned}$$

By (18), $\|\hat{\varphi}^l_j\|_\infty \le 1$, so we conclude by (24) that $\lambda_j(w) \in L^\infty(\mathbf{T})$ and $\|\lambda_j(w)\|_\infty \le \frac{(k-1)!\,\sqrt{n!}}{\sqrt{k!}\,\sqrt{(n-k)!}}$. Now (17) follows from (10). ∎

4. **The Case $p > 1$.** In this section we extend previous results to the case $p \ne 2$.

PROPOSITION 6. *Let $p > 0$. Any $H^p$-function $f$ is uniquely (up to a unimodular factor) written as a product*

$$f(z) = h_{f,p}(z) F_p\big(g(z)\big) \tag{25}$$

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*where $h_{f,p}$ is $g$-$p$-inner and $F_p$ is an outer $H^p$-function.*

PROOF. Let $f(z) = \varphi(z) \cdot F(z)$ be the classical factorization of $f$, where $\varphi$ is inner and $F$ is outer. Then $F^{p/2} \in H^2$ and by (9)

$$F^{p/2}(z) = h(z) \cdot F_2\big(g(z)\big)$$

where $h$ is $g$-2-inner and $F_2$ is outer. Then $h$ is zero free in the unit disk and, therefore, $h^{2/p}$ is $g$-$p$-inner.

Now we define $h_{f,p}$ and $F_p$ by

$$h_{f,p}(z) = \varphi(z) \cdot h(z)^{2/p},$$
$$F_p\big(g(z)\big) = \Big(F_2\big(g(z)\big)\Big)^{2/p}.$$

To prove uniqueness of factorization (25) let us suppose that

$$h^1_{f,p}(z) \cdot F^1_p\big(g(z)\big) = f(z) = \varphi(z) \cdot F(z) = h^2_{f,p}(z) \cdot F^2_p\big(g(z)\big)$$

are two factorizations. Since both $F^1_p$ and $F^2_p$ are outer we have

$$h^1_{f,p}(z) = \varphi(z) \cdot \hat{h}^1_{f,p}(z)$$
$$h^2_{f,p}(z) = \varphi(z) \cdot \hat{h}^2_{f,p}(z)$$

and both $\hat{h}^1_{f,p}$, $\hat{h}^2_{f,p}$ are $g$-$p$-inner and zero-free in $\Delta$. Then

$$\big(\hat{h}^1_{f,p}(z)\big)^{p/2}\Big(F^1_p\big(g(z)\big)\Big)^{p/2} = F(z)^{p/2} = \big(\hat{h}^2_{f,p}(z)\big)^{p/2}\Big(F^2_p\big(g(z)\big)\Big)^{p/2}$$

are two factorization of the $H^2$-function $F^{2/p}$. By Proposition 2 they are the same up to unimodular factors. ∎

COROLLARY 5. *Let $g$ be a finite Blaschke product, $f \in H^\infty$ and*

$$f(z) = h_{f,p}(z)F_p\big(g(z)\big)$$

*the $g$-$p$-factorization of $f$. Then $F_p \in H^\infty$.*

PROOF. Write the canonical factorization $f = h \cdot F$ where $h$ is inner, $F$ is outer. As we saw in the Proof of Proposition 6.

$$F_p = (\hat{F}_2)^{2/p}$$

where

$$F(z)^{p/2} = \hat{h}(z) \cdot \hat{F}_2\big(g(z)\big)$$

is the $g$-2-factorization of $F^{p/2}$. Since $F^{p/2} \in H^\infty$ we conclude by Proposition 4 that $\hat{F}_2$ is bounded. ∎

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Like classical inner functions, $g$-$p$-inner functions have some extremal properties. Let $f$ be an $H^{p'}$-function which annihilates $gH^p$ (we use the usual notation $\frac{1}{p} + \frac{1}{p'} = 1$). For a subspace $M \subset H^p$ we define the number $S_k^f(M)$ ($k \geq 0$ is an integer) by

$$S_k^f(M) = \sup\Big\{ |\ell_k^f(h)| = \Big| \int_{\mathbf{T}} h(z)\overline{f(z)\big(g(z)\big)^k}\, dm(z) \Big| : h \in M,\ \|h\|_p \leq 1 \Big\}. \tag{26}$$

We say that $M$ has $f$-rank $k$ if $S_k^f(M) \neq 0$, but $S_m^f(M) = 0$ for all $0 \leq m < k$.

If $M$ has $f$-rank $k$, then we call the extremal function of the problem (26) an $f$-extremal function of $M$. If $p > 1$ and $M$ is closed, the existence and uniqueness (up to unimodular factor) of the extremal element of the problem (26) follows from the following standard argument. Given a maximizing sequence $h_n \in M$ we find a subsequence $h_{n_m}$ that is weak-$*$ convergent (the unit ball of $H^p$ is weak-$*$ compact). Let $\psi$ be the weak-$*$ limit. Then $\psi \in M$, $\ell_k^f(\psi) = \lim_{m\to\infty} \ell_k^f(h_{n_m})$ and $\|\psi\| \leq 1$. This implies that $|\ell_k^f(\psi)| = S_k^f(M)$. The uniqueness follows from the strict convexity of the $H^p$-sphere.

Obviously, any $f$-extremal function has norm 1.

Note that if $M$ has $f$-rank $k$, then for any $h \in M$, $\ell_k^f(hg^m) = 0$ for all $m \geq 1$. Indeed, if $m \leq k$, then $\ell_k^f(hg^m) = \ell_{k-m}^f(h) = 0$ by definition of $f$-rank. If $m > k$, then

$$\ell_k^f(hg^m) = \int_{\mathbf{T}} h(z)g^{m-k}(z)\overline{f(z)}\, dm(z) = 0$$

since $f$ annihilates the ideal generated by $g$.

PROPOSITION 7. *Let $M \subset H^p$ be a closed $g$-invariant supspace of $f$-rank $k$, where $f \in (gH^p)^\perp$. Then an $f$-extremal function of $M$ is $g$-$p$-inner.*

PROOF. Let $h$ be the extremal function for (26). Without loss of generality we may assume that $\ell_k^f(h) > 0$. Let $r \geq 1$. Consider the function

$$F(\varepsilon) = \frac{\ell_k^f\big(h(1+\varepsilon g^r)\big)}{\|h(1+\varepsilon g^r)\|_p} = \frac{\ell_k^f(h)}{\|h(1+\varepsilon g^r)\|_p}$$

(the second equality follows from the above note) where $\varepsilon \in \mathbf{C}$. The extremality of $h$ implies that $F$ has local maximum at the origin. A direct computation shows that

$$\frac{\partial F}{\partial \varepsilon}\Big|_{\varepsilon=0} = \frac{-\frac{1}{2}\ell_k^f(h) \int_{\mathbf{T}} |h(z)|^p \big(g(z)\big)^r\, dm(z)}{\|h\|_p^2}.$$

Now the condition $\frac{\partial F}{\partial \varepsilon}\Big|_{\varepsilon=0} = 0$ yields

$$\int_{\mathbf{T}} |h(z)|^p g^r(z)\, dm(z) = 0. \qquad \blacksquare$$

PROPOSITION 8. *A function $h$ is $g$-$p$-inner if and only if for every polynomial $Q$ the following equality holds*

$$\|h(z)\cdot Q\big(g(z)\big)\|_p = \|h(z)\|_p \cdot \|Q(z)\|_p = \|Q(z)\|_p.$$

PROOF. Let $h = h_1\hat{h}$ be the canonical factorization of $h$, where $h_1$ is inner, $\hat{h}$ is outer. If $h$ is $g$-$p$-inner then the same is true for $\hat{h}$ and, therefore, $\hat{h}^{p/2}$ is $g$-2-inner. Write the representation of $\hat{h}^{p/2}$

$$\hat{h}^{p/2}(z) = \sum_{k=0}^{\infty} s_k(z)\hat{h}_k\big(g(z)\big).$$

By Corollary 2 we have

$$\Big(\sum_{k=0}^{\infty} |\hat{h}_k(z)|^2\Big)\Big|_{\mathbf{T}} \underset{a.e}{=} 1.$$

Then by (7)

$$\|h\|_p^p = \|\hat{h}\|_p^p = \|\hat{h}^{p/2}\|_2^2 = 1.$$

Let $Q = q \cdot \hat{Q}$ be the Riesz factorization of $Q$, where $q$ is inner, $\hat{Q}$ is outer. Now the relation (7) yields

$$\begin{aligned}\Big\|h(z)Q\big(g(z)\big)\Big\|_p^p = \Big\|\hat{h}(z)\hat{Q}\big(g(z)\big)\Big\|_p^p = \Big\|\hat{h}(z)^{p/2}\hat{Q}\big(g(z)\big)^{p/2}\Big\|_2^2 \\ = \|\hat{Q}(z)^{p/2}\|_2^2 = \|\hat{Q}\|_p^p = \|Q\|_p^p.\end{aligned}$$

Conversely, let $\|h\|_p = 1$ and

$$\Big\|h(z)Q\big(g(z)\big)\Big\|_p = \|Q(z)\|_p,$$

for all $Q$. In particular,

$$\Big\|h(z)\big(1+\varepsilon g^k(z)\big)\Big\|_p^p = \|1+\varepsilon z^k\|_p^p \tag{27}$$

for all $k \geq 1$, $\varepsilon \in \mathbf{C}$. Differentiate both sides of (27) with respect to $\varepsilon$ at $\varepsilon = 0$. We obtain

$$\frac{p}{2}\int_{\mathbf{T}} |h(z)|^p g(z)^k \, dm(z) = \frac{\partial}{\partial\varepsilon}\Big(\int_{\mathbf{T}} |1+\varepsilon z^k|^p \, dm(z)\Big)\Big|_{\epsilon=0} = 0. \qquad \blacksquare$$

As in the case $p \neq 2$, we denote by $M_f^p$ the closed $g$-invariant subspace of $H^p$ generated by $f$:

$$M_f^p = \overline{\text{span}}\{f \cdot g^k,\ k \geq 0\}.$$

COROLLARY 6. *Let $\psi \in (gH^p)^\perp$ and $f(z) = h_{f,p}(z). F_p\big(g(z)\big)$ be the $g$-$p$-factorization (25) of an $H^p$-function $f$. Then $h_{f,p}$ is the $\psi$-extremal function of $M_f^p$.*

PROOF. Suppose that the $\psi$-rank of $M_f^p$ is $k$. Since $F_p$ is outer, we have

$$M_f^p = M_{h_{f,p}}^p.$$

Now, if $\varphi(z) = h_{f,p}(z) \cdot Q\big(g(z)\big) \in M_f^p$, $\|\varphi\|_p = 1$ then, by Proposition 8,

$$\|Q(z)\|_p = 1$$

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and, therefore,

$$|Q(0)| \leq 1.$$

Write $Q(z) = \sum_{i=0}^{\infty} c_i z^i$. The note preceding Proposition 8 implies

$$\ell_k^{\psi}\Big(h_{f,p}(z)Q\big(g(z)\big)\Big) = c_0\ell_k^{\psi}(h_{f,p}) = Q(0)\ell_k^{\psi}(h_{f,p}).$$

Therefore,

$$|\ell_k^{\psi}(\varphi)| \leq |\ell_k^{\psi}(h_{f,p})|. \qquad \blacksquare$$

As in the case $p = 2$ for a subset $A \subset H^p$ we denote by $[A]_g$ the minimal closed $g$-invariant subspace of $H^p$ which contains $A$.

COROLLARY 7. *If $M \subset H^p$ is $g$-invariant and $M_I$ is the collection of all $g$-$p$-inner functions of $M$, then*

$$M = [M_I]_g.$$

PROOF. Let $f \in M$. By Proposition 6

$$f(z) = h_{f,p}(z) \cdot F_p\big(g(z)\big)$$

where $h_{f,p}$ is $g$-$p$-inner and $F_p$ is outer in $H^p$. Let $P_n$ be a sequence of polynomials such that $F_p \cdot P_n$ converges to 1 in $H^p$. By Proposition 8

$$\Big\|h_{f,p}(z) - h_{f,p}(z)F_p\big(g(z)\big) \cdot P_n\big(g(z)\big)\Big\|_p = \|1 - F_p(z)P_n(z)\|_p \longrightarrow 0$$

as $n \longrightarrow \infty$. This implies

$$h_{f,p}(z) \in M_I. \qquad \blacksquare$$

THEOREM. *If $g$ is a finite Blaschke product of order $n$ and $p > 0$ then any $g$-$p$-invariant subspace $M$ has a set of $g$-$p$-inner generators consisting of at most $n$ elements. If $p \geq 1$ then these generators form a $g$-basis: that is, every $\varphi \in M$ is uniquely written as*

$$\varphi(z) = \sum_{i=1}^{k} h_{i,p}(z)\varphi_i\big(g(z)\big)$$

*where the $g$-$p$-inner functions $h_{i,p}$, $i = 1, \ldots, k$, $k \leq n$ are the generators and $\varphi_i \in H^p$.*

PROOF. First, we note that if $g$ is a finite Blaschke product then any $g$-$p$-inner function is in $H^\infty$. Indeed, if $f$ is $g$-$p$-inner, $f = \varphi F$, where $\varphi$ is inner, $F$ is outer, then $F$ is $g$-$p$-inner and $F^{p/2}$ is $g$-2-inner. By Corollary 2, $F^{p/2} \in H^\infty$ and so is $F$. By Corollary 7, $\tilde{M} = M \cap H_\infty$ is dense in $M$. Obviously, $\tilde{M}$ is $g$-invariant. Let $\hat{M}$ be the closure of $\tilde{M}$ in $H^2$. Then $\hat{M}$ is a $g$-invariant subspace of $H^2$ and by (3) and (4) there are $g$-2-inner functions $\tilde{\varphi}_1, \ldots, \tilde{\varphi}_k$, $k \leq n$ which form a $g$-basis of $\hat{M}$. Let

$$\tilde{\varphi}_i(z) = h_{i,p}(z) \cdot F_{i,p}\big(g(z)\big), \quad i = 1, \ldots, k \tag{28}$$

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be the factorization (25). Then $h_{i,p} \in H^\infty$, $i = 1, \ldots, k$. By Corollary 7, $h_{i,p} \in M$ and $h_{i,p}$, $i = 1, \ldots, k$, generate $\tilde{M}$. Let $f \in M$ and

$$\sum_{i=1}^{k} h_{i,p}(z) R_i^n\big(g(z)\big) \xrightarrow[n \longrightarrow \infty]{H^p} f(z).$$

We must prove that $R_i^n$ converges in $H^p$ as $n \longrightarrow \infty$, $i = 1, \ldots, n$. By the Wold decomposition theorem we might choose $\tilde{\varphi}_i$, $i = 1, \ldots, k$ such that

$$\tilde{\varphi}_i g^\ell \perp \tilde{\varphi}_j g^m, \quad i \neq j, \quad \ell, m = 0, 1, \ldots . \tag{29}$$

Since $\tilde{\varphi}_1, \ldots, \tilde{\varphi}_k$ form a $g$-basis of $\hat{M}$, (29) implies

$$h_{i,p}(z) = \tilde{\varphi}_i(z) \Phi_{i,p}\big(g(z)\big). \tag{30}$$

Since $h_{i,p} \in H^\infty$, Proposition 5 yields

$$\Phi_{i,p} \in H^\infty$$
$$F_{i,p} \Phi_{i,p} \equiv 1.$$

Since both $F_{i,p}$ and $\Phi_{i,p}$ are bounded, this implies

$$\operatorname*{ess\,inf}_{z \in \Delta}(|F_{i,p}|) > 0, \text{ and } \operatorname*{ess\,inf}_{z \in \Delta}(|\Phi_{i,p}|) > 0. \tag{31}$$

We have

$$f_n(z) = \sum_{i=1}^{k} h_{i,p}(z) R_i^n\big(g(z)\big) = \sum_{i=1}^{k} \tilde{\varphi}_i(z) \Phi_{i,p}\big(g(z)\big) R_i^n\big(g(z)\big).$$

By (29) $\tilde{\varphi}_1, \ldots, \tilde{\varphi}_k$ satisfy Proposition 5 and, since $f_n \longrightarrow f$ in $H^p$ as $n \longrightarrow \infty$, we conclude by this Proposition that $\Phi_{i,p} R_i^n$ converge in $H^p$ as $n \longrightarrow \infty$. Because of (31) this implies that $R_i^n$ converges in $H^p$, $i = 1, \ldots, k$. ∎

5. **Application to operators similar to a contraction.** Let $A: X \longrightarrow X$ be a bounded operator in a Hilbert space $X$. In accordance with the standard notation we denote by $\mathrm{Sp}(A)$ the spectrum of $A$. Let $f$ be a holomorphic function in an open neighborhood $U$ of $\mathrm{Sp}(A)$, and $V$ be another open neighborhood of $\mathrm{Sp}(A)$, which is compact in $U$. If $\partial V = \Gamma$ is a smooth manifold in $\mathbf{R}^2$, then, as usual,

$$f(A) = \frac{1}{2\pi i} \int_\Gamma f(z)(z - A)^{-1}\, dz. \tag{32}$$

In particular, if $g$ is an inner function, $g = B \cdot S$, where

$$B(z) = z^\ell \prod_{k=1}^{\infty} \frac{\bar{a}_k}{|a_k|} \frac{a_k - z}{1 - \bar{a}_k z}$$

is a Blaschke product and

$$S(z) = \exp\Big\{ -\int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\mu(\theta) \Big\}, \quad \mu \geq 0$$

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is a singular function, and $\overline{(\operatorname{supp}(\mu) \cup \{\frac{1}{\bar{a}_k}\}_{k=1}^{\infty})} \cap \operatorname{Sp}(A) = \phi$, then the relation (32) determines $g(A) = B(A)S(A)$. It is easy to show that

$$B(A) = A^{\ell} \prod_{k=1}^{\infty} \frac{\bar{a}_k}{|a_k|}(a_k - A)(1 - \bar{a}_k A)^{-1}.$$

Consider the following problem:

Let $g = B \cdot S$ be an inner function satisfying the above condition

$$\overline{\left(\operatorname{supp}(\mu) \cup \left\{\frac{1}{\bar{a}_k}\right\}_{k=1}^{\infty}\right)} \cap \operatorname{Sp}(A) = \phi. \tag{33}$$

Given that $g(A)$ is similar to a contraction, does this imply that $A$ is similar to a contraction?

The answer in general is unknown. To the best of our knowledge the only published result related to this problem is the following theorem by V. Mascioni [8].

THEOREM (V. MASCIONI). *If $B$ is a finite Blaschke product satisfying (33), and $B(A)$ is similar to a contraction, then $A$ is similar to a contraction.*

As we mentioned before, R. Douglas suggested that there must be a proof of this theorem different from the one of [8] and based on the estimate (10). Below we sketch this proof.

We denote by $H_n^{p,2}$ the space of $n$-dimensional vector-functions $F(z) = \big(f_1(z), \ldots, f_n(z)\big)$, $z \in \Delta$, $f_i \in H^p$, with the norm

$$\begin{aligned} \|F\|_{n,p,2} &= \left(\int_{\mathbf{T}} \Big(\sum_{i=1}^{n} |f_i(z)|^2\Big)^{p/2} dm(z)\right)^{1/p}, \quad 1 \le p < \infty \\ \|F\|_{n,\infty,2} &= \sup_{z \in \Delta} \Big(\sum_{i=1}^{n} |f_i(z)|^2\Big)^{1/2}. \end{aligned} \tag{34}$$

It is clear that $H_n^{p,2}$ is a Banach space and if $1 < p < \infty$ its dual consists of $n$-dimensional vector-functions $\Phi = (\varphi_1, \ldots, \varphi_n) \in H_n^{p',2}$ (of course, the dual norm is different from the $H_n^{p',2}$-norm) with the duality given by

$$\langle F, \Phi \rangle = \int_{\mathbf{T}} \sum_{i=1}^{n} f_i(z) \overline{\varphi_i(z)}\, dm(z).$$

Let $g$ be an inner function. We denote by $H_n^{p,2}[g]$ the subspace of $H_n^{p,2}$ consisting of vector-functions whose components are in $H^p[g]$. As in the case $n = 1$, we use the similar notation $P_g^n$ for the operator

$$\begin{aligned} P_g^n &: H_n^{p,2} \longrightarrow H_n^{p,2}[g], \\ P_g^n F &= (P_g f_1, \ldots, P_g f_n) \end{aligned}$$

where $P_g$ is the projection used in Proposition 3.

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PROPOSITION 9. *The projection $P^n_g$ has norm 1 as an operator $P^n_g: H^{p,2}_n \longrightarrow H^{p,2}_n$ for all* $1 < p \leq \infty$.

REMARK. Unfortunately the definition (34) of the norm in $H^{p,2}_n$ does not allow us to use conditional expectation (as in Proposition 3) to prove this result. Instead we use the technique based on invariant minimal interpolation ([10]).

PROOF OF PROPOSITION 9. Let $1 < p < \infty$, $F \in H^{p,2}_n$. Consider the following extremal problem. Find

$$\delta_{F,p} = \inf\{\|G\|_{n,p,2} : \langle G, \Phi\rangle = \langle F, \Phi\rangle \text{ for all } \Phi \in H^{p',2}_n[g]\}. \tag{35}$$

The following standard argument shows that there is a unique extremal function of this problem. Let $\{\Phi_k\}_{k=1}^{\infty}$ be a minimizing sequence. It is bounded in $H^{p,2}_n$ and, therefore, it is weak-$*$ compact, so without loss of generality we may assume that $\Phi_k \xrightarrow{w^*} F^* \in H^{p,2}_n$. Then for any $\Phi \in H^{p,2}_n[g]$

$$\langle F^*, \Phi\rangle = \lim_{k\to\infty} \langle \Phi_k, \Phi\rangle = \langle F, \Phi\rangle$$

and $\|F^*\|_{n,p,2} \leq \lim_{k\to\infty} \|\Phi_k\|_{n,p,2} = \delta_{F,p}$. This implies $\|F^*\|_{n,p,2} = \delta_{F,p}$. The uniqueness follows from strict convexity.

Further, the application of the variational principle similar to [2] shows that $F^*_p = (f^*_{1,p}, \ldots, f^*_{n,p})$ is the extremal function of the problem (35) if and only if

(i) $\langle F^*_p, \Phi\rangle = \langle F, \Phi\rangle$ for all $\Phi \in H^{p',2}_n[g]$

(ii) For any $\Psi \in H^{p,2}_n$ such that $\langle \Psi, \Phi\rangle = 0$ for all $\Phi \in H^{p',2}_n$ the following equality holds

$$\int_{\mathbf{T}} \Big(\sum_{i=1}^{n} |f_{i,p}(z)|^2\Big)^{\frac{p}{2}-1} \sum_{i=1}^{n} f^*_{i,p}(z)\overline{\psi_i(z)}\, dm(z) = 0. \tag{36}$$

The rest of the proof is based on the following result.

LEMMA. *Let $F \in H^{\infty,2}_n$. Then the extremal function $F^*_p$ of the problem (35) is the same for all* $1 < p < \infty$.

PROOF. Let $(H^{p'}[g])^{\perp}$ be the annihilator of $H^{p'}[g]$, and $\chi \in (H^{p'}[g])^{\perp}$. Then for any polynomial $P = c_0 + c_1 z + \cdots + c_k z^k = c_0 + zP_1(z)$ we have

$$\begin{aligned}\int_{\mathbf{T}} \overline{g(z)\chi\big((z)\big)} P\big(g(z)\big)\, dm(z) &= c_0 \int_{\mathbf{T}} \overline{g(z)\chi\big((z)\big)}\, dm(z) + \int_{\mathbf{T}} \overline{\chi(z)} P_1\big(g(z)\big)\, dm(z)\\ &= c_0 \overline{g(0)\chi\big((0)\big)} = 0,\end{aligned}$$

since $\chi$ is orthogonal to 1 and, therefore, vanishes at the origin. Thus, $\chi \in (H^{p'}[g])^{\perp} \Rightarrow g\chi \in (H^{p'}[g])^{\perp}$ and, therefore, for any $\psi \in H^{\infty}$ we have

$$\chi \in (H^{p'}[g])^{\perp} \Longrightarrow (\psi \circ g) \cdot \chi \in (H^{p'}[g])^{\perp}. \tag{37}$$

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Further, it is obvious that the annihilator of $H_n^{p',2}[g]$ consists of all vector-functions

$$\Psi = (\psi_1, \dots, \psi_n), \text{ where } \psi_j \in (H^{p'}[g])^{\perp}.$$

Now, let $\Phi \in H_n^{\infty,2}[g]$, $\Phi = (\varphi_1, \dots, \varphi_n)$. Without loss of generality we may assume that $\sup_{z\in\Delta} \sum_{i=1}^{n} |\varphi_i(z)|^2 < 1$. Fix $\Psi \in (H_n^{p',2}[g])^{\perp}$ and consider the function

$$F(\alpha) = \int_{\mathbf{T}} \Big( \sum_{i=1}^{n} |\varphi_i(z)|^2 \Big)^{\alpha} \sum_{i=1}^{n} \varphi_i(z)\overline{\psi_i(z)}\, dm(z).$$

This function is analytic and bounded in the halfplane $\{\operatorname{Re}\alpha > -1\}$. If $\alpha = k$ (a positive integer), we have by (37)

$$\begin{aligned} F(k) &= \sum_{i=1}^{n} \sum_{\ell_1+\cdots+\ell_n=k} \int_{\mathbf{T}} |\varphi_1(z)|^{2\ell_1} \cdots |\varphi_n(z)|^{2\ell_n} \varphi_i(z)\overline{\psi_i(z)}\, dm(z) \\ &= \sum_{i=1}^{n} \sum_{\ell_1+\cdots+\ell_n=k} \langle \varphi_1^{\ell_1} \cdots \varphi_n^{\ell_n+1} \cdots \varphi_n^{\ell_n}, (\varphi_1^{\ell_1} \cdots \varphi_n^{\ell_n})\psi_i \rangle = 0. \end{aligned}$$

Since the sequence of positive integers does not satisfy the Blaschke condition, this implies $F(\alpha) \equiv 0$ in $\{\operatorname{Re}\alpha > -1\}$. Since $\Psi$ was an arbitrary element of $(H_n^{p',2}[g])^{\perp}$ we conclude by (36) that for any pair $\Phi \in H_n^{\infty,2}[g]$, $\Psi \in H_n^{\infty,2} \cap (H_n^{p',2}[g])^{\perp}$, we have

$$(\Phi + \Psi)_p^* = \Phi, \text{ for } 1 < p < \infty. \tag{38}$$

It is easy to show that $H_n^{\infty,2}[g] \oplus \big(H_n^{\infty,2} \cap (H_n^{p',2}[g])^{\perp}\big)$ is dense on $H_n^{p,2}$ and then to deduce the result of the Lemma from this and (38). ■

Now we are ready to finish the Proof of Proposition 9. Since for $p = 2$ we obviously have

$$F_2^* = P_g^n F,$$

we conclude by the Lemma that

$$F_p^* = P_g^n F, \quad 1 < p < \infty. \tag{39}$$

In particular, (39) implies

$$\|P_g^n F\|_{n,p,2} \le \|F\|_{n,p,2} \le \|F\|_{n,\infty,2}$$

and

$$\|P_g^n F\|_{n,\infty,2} = \sup_{p>1} \|P_g^n F\|_{n,p,2} \le \|F\|_{n,\infty,2}.$$

The proof is complete. ■

Now, let $A(z)$ be a holomorphic polynomial $(n \times n)$-matrix function in $\Delta$. Put

$$\|A(z)\|_{\infty} = \sup_{|z|<1} \Big( \sup_{|\xi|<1} |A(z)(\xi)| \Big)$$

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where $\xi = (\xi_1, \ldots, \xi_n) \in \mathbf{C}^n$ (as usual $|(b_1, \ldots, b_n)| = \left(\sum |b_i|^2\right)^{1/2}\right)$. Write

$$A(z) = [a_{ij}(z)]_{i,j=1}^n$$

where $a_{ij}(z)$ are polynomials.

For an inner function $g$ let $s_0, s_1, \ldots$ be a rational $g$-basis of $H^2$ which satisfies the condition of Corollary 1. Write each entry $a_{ij}(z)$ in the form

$$a_{ij}(z) = \sum_{k=0}^{\infty} s_k(z) a_{ij}^k\big(g(z)\big).$$

This decomposition leads to the following decomposition of $A(z)$

$$A(z) = \sum_{k=0} s_k(z) A_k\big(g(z)\big), \tag{40}$$

where, by Proposition 3,

$$A_k(z) = \left[a_{ij}^k\big((z)\big)\right]_{i,j=1}^n, \quad k = 0, 1, \ldots$$

are $H^\infty$-matrix functions in $\Delta$.

The following result is the matrix-function version of the estimate (10).

PROPOSITION 10. *There are constants $D_k$, $k = 0, 1, \ldots$, depending only on $g$ such that for any $H^\infty$-matrix function*

$$A(z) = \sum_{k=1}^{\infty} s_k(z) A_k\big(g(z)\big)$$

*the estimate $\|A_k\|_\infty \le D_k \|A\|_\infty$ holds.*

PROOF. Let as above $T_{\bar{s}_k}$ stands for the Toeplitz operator with symbol $\bar{s}_k$. We extend the action of $T_{\bar{s}_k}$ to $H_n^{\infty,2}$ by componentwise action. Now (13) and the usual estimate which uses the Cauchy formula shows that there are constants $D_k$, depending only on $g$ such that for any $F \in H_n^{\infty,2}$

$$\|T_{\bar{s}_k} F\|_{n,\infty,2} \le D_k \|F\|_{n,\infty,2}. \tag{41}$$

For any $z \in \Delta$, $\xi \in \mathbf{C}^n$ we have by Proposition 9 and (41)

$$\begin{aligned} |A_k(z)\xi| &= |P_g^n T_{\bar{s}_k} A(z)\xi| \le \|P_g^n T_{\bar{s}_k} A(z)\xi\|_{n,\infty,2} \\ &\le \|T_{\bar{s}_k} A(z)\xi\|_{n,\infty,2} \\ &\le D_k \|A(z)\xi\|_{n,\infty,2} \le D_k \|A(z)\|_\infty \cdot |\xi|. \end{aligned}$$

∎

Let $B$ be a Blaschke product of order $m$ and $G$ an operator on a Hilbert space $X$ whose spectrum is off the poles of $B$ and such that

$$B(G) = C^{-1} R C,$$

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where $\|R\| \leq 1$. For any holomorphic polynomial $n \times n$-matrix function, $F(G)$, in $G$ write the representation (40) for $F(G)$

$$F(G) = s_0(G)F^0\big(B(G)\big) + \cdots + s_{m-1}(G)F^{m-1}\big(B(G)\big)$$
$$= s_0(G)C^{-1}F^0(R)C + \cdots + s_{m-1}(G)C^{-1}F^{m-1}(R)C.$$

Since $G$ is bounded, $s_0(G), \ldots, s_{m-1}(G)$ are bounded (recall that the spectrum $G$ is off the poles of $B$). Say

$$\|s_i(G)\| \leq M, \quad i = 0, \ldots, m-1.$$

Further, we have

$$\|F^j(R)\| \leq \|F^j(z)\|_\infty, \quad j = 0, \ldots, m-1$$

([1, Proposition 3.6.1]). Finally, Proposition 10 yields

$$\|F(G)\| \leq \sum_{i=0}^{m-1} \|s_i(G)\| \cdot \|C\| \cdot \|C^{-1}\| \cdot \|F^i(R)\|$$
$$\leq M \cdot \|C\| \cdot \|C^{-1}\| \sum_{i=0}^{m-1} \|F^i(z)\|_\infty$$
$$\leq M \cdot \|C\| \cdot \|C^{-1}\| \Big(\sum_{i=0}^{m-1} D_i\Big) \|F(z)\|_\infty.$$

Thus, $G$ is completely polynomialy bounded. The theorem of Mascioni now follows from the theorem of V. Paulsen [9].

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*Department of Mathematics and Statistics*
*University at Albany*
*State University of New York*
*Albany, NY*
*USA 12222*
*e-mail: lance@math.albany.edu*
*stessin@math.albany.edu*


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