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STABLE EXTENSIONS AND FIELDS WITH THE GLOBAL DENSITY PROPERTY

MICHAEL FRIED AND MOSHE JARDEN

Introduction. For a field M we denote by M_s and $\tilde{M}$ respectively the separable closure and the algebraic closure of M. If V is a variety which is defined over M, then we denote by $V(M)$ the set of all M-rational points of V. M is said to be *pseudo-algebraically closed* (PAC) field, if $V(M) \neq \emptyset$ for every non-void abstract variety V defined over M. It can be shown that then $V(M)$ is dense in $V(\tilde{M})$ in the Zariski M-topology.

Suppose now that $\tilde{M}$ is equipped with an absolute value w. M is said to have the *density property with respect to* w, if $V(M)$ is w-dense in $V(\tilde{M}_w)$ for every abstract variety V defined over M. Here $\tilde{M}_w$ is the completion of $\tilde{M}$ with respect to w.

Let K be a field and let $\mathscr{G}(K_s/K)$ be the Galois group of K_s over K. Let e be a fixed positive integer and equip $\mathscr{G}(K_s/K)^e$ with the normalized Haar measure μ. For every $(\boldsymbol{\sigma}) \in \mathscr{G}(K_s/K)^e$ we denote by $K_s(\boldsymbol{\sigma})$ the fixed field of $\sigma_1, \ldots, \sigma_e$. The following theorem was proved in [**3**].

If K is a denumerable hilbertian field and if w is an absolute value of $\tilde{K}$, then $\tilde{K}(\boldsymbol{\sigma})$ has the density property with respect to w for almost all $(\boldsymbol{\sigma}) \in \mathscr{G}(K_s/K)^e$.

The aim of this work is to strengthen this result as follows:

If K is a denumerable hilbertian field and if v is an absolute value of K, then $\tilde{K}(\boldsymbol{\sigma})$ has the density property with respect to every extension w of v to $\tilde{K}$ for almost all $\boldsymbol{\sigma}$. In particular, if K is a global field, then for almost all $(\boldsymbol{\sigma}) \in \mathscr{G}(K_s/K)^e$, $\tilde{K}(\boldsymbol{\sigma})$ has the density property with respect to every absolute value w of $\tilde{K}$. (One can say that these $\tilde{K}(\boldsymbol{\sigma})$ have the *global density property*).

Thus we have solved Problem 2 of [**3**] affirmatively. In order to prove the theorem we show that every PAC field M has the following property:

For every finitely generated regular extension F of K of dimension r there exists a separating transcendence base $t_1, \ldots, t_r$ such that the Galois closure, $\hat{F}$, of $F/M(\mathbf{t})$ is regular over K.

This we prove in two steps, first for $r = 1$ by using the Riemann-Roch theorem and then by reducing the case $r \geqq 1$ to the case $r = 1$ and using the theory of simple points.

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1. Stable extensions. By the *Galois closure* of a separable algebraic field extension $F|E$ we mean the smallest extension $\hat{F}$ of F which is Galois over E.

A finitely generated regular field extension F/K of dimension r is said to be *stable*, if it has a separating transcendence base $t_1, \ldots, t_r$ such that the Galois closure $\hat{F}$ of the (separable) extension $F/K(\mathbf{t})$ is regular over K. The system $t_1, \ldots, t_r$ is said to be a *stabilizing* base for F/K.

Note that if $t_1, \ldots, t_r$ is a separating transcendence base for F/K, then a necessary and sufficient condition for $t_1, \ldots, t_r$ to stabilize F/K is that $\mathcal{G}(\hat{F}/K(\mathbf{t})) \cong \mathcal{G}(\hat{F} \cdot \tilde{K}(\mathbf{t})/\tilde{K}(\mathbf{t}))$. This follows, since $\hat{F}/K$ is regular if and only if $\hat{F}$ is linearly disjoint from $\tilde{K}$ over K.

A field K is said to be *stable*, if every finitely generated regular extension F of K is stable.

An abstract variety V is said to be *stable over a field* K, if V is defined over K and if the function field of V is stable over K.

One sees immediately that the stability of a variety over a field is preserved under birational transformations. Further, if an abstract variety V is stable over a field K, then it is stable over every algebraic extension of K.

An absolutely irreducible polynomial $f \in K[T_1, \ldots, T_r, X]$ is said to be *stable over K with respect to* $T_1, \ldots, T_r$ if $\partial f/\partial X \neq 0$ and if there exist elements $t_1, \ldots, t_r, x$ such that: (i) $f(\mathbf{t}, x) = 0$, (ii) $t_1, \ldots, t_r$ are algebraically independent over K and (iii) $t_1, \ldots, t_r$ is a stabilizing base for $K(\mathbf{t}, x)|K$.

The most common example of a stable polynomial is that of the general polynomial, $h(\mathbf{T}, X) = X^n + T_1X^{n-1} + \ldots + T_n$, of degree n. It is well known that the Galois group $\mathcal{G}(h(\mathbf{t}, X), K(\mathbf{t}))$, of $h(\mathbf{t}, X)$ over $K(\mathbf{t})$ is isomorphic to the symmetric group S_n for every field K (c.f. Lang [**8**], p. 201]). Hence f is stable over K with respect to $T_1, \ldots, T_r$ over every field K.

Note that it is possible that a polynomial is stable with respect to one system of variables but not with respect to another. Thus, for an odd prime p, $X^p - T$ is stable with respect to X over $\mathbf{Q}$ but not with respect to T.

Note also that questions concerning the stability of abstract varieties can be expressed as questions concerning the stability of absolutely irreducible polynomials, since every abstract variety is birationally equivalent to a hypersurface.

LEMMA 1.1. *Let M be an algebraic extension of a field K, let* $f \in M[T_1, \ldots, T_r, X]$ *be an absolutely irreducible polynomial which is stable over M with respect to* $T_1, \ldots, T_r$. *Then there exists a finite extension L of K which contains the coefficients of f and is contained in M such that f is stable over L with respect to* $T_1, \ldots, T_r$.

Proof. Let $t_1, \ldots, t_r$ be r algebraically independent elements over M and let x be an element such that $f(\mathbf{t}, x) = 0$. Then $M(\mathbf{t}, x)$ is a separable extension of $M(\mathbf{t})$. Let $\hat{M}$ be the Galois closure of $M(\mathbf{t}, x)/M(\mathbf{t})$. Then there exists an element $y \in \hat{M}$ such that $\hat{M} = M(\mathbf{t}, y)$. Let $g \in M[\mathbf{T}, Y]$ be an irreducible

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polynomial such that $g(\mathbf{t}, y) = 0$ and let $y = y_1, \ldots, y_d$ be all the roots of $g(\mathbf{t}, Y)$. Then x and $y_1, \ldots, y_d$ can be expressed as polynomials in y with coefficients $c(\mathbf{t})$ in $M(\mathbf{t})$. Extend K by adjoining to it all the elements of M appearing in the $c(\mathbf{t})$ and all the coefficients of f and g. Call this extension L. Then L is a finite extension of K which is contained in M. Write $\hat{L} = L(\mathbf{t}, y)$. Then $x, y_1, \ldots, y_d \in \hat{L}$ and hence $\hat{L}$ is the splitting field of $g(\mathbf{t}, Y)$ over $L(\mathbf{t})$, hence it is also Galois over $L(\mathbf{t})$. Moreover, L is linearly disjoint from $M(\mathbf{t})$ over $L(\mathbf{t})$, since $g(\mathbf{t}, Y)$ is certainly irreducible over $L(\mathbf{t})$. It follows that $\hat{L}$ is the Galois closure of $L(\mathbf{t}, x)/L(\mathbf{t})$, since any intermediate field $L(\mathbf{t}, x) \subset L' \subset \hat{L}$ which is Galois over $L(\mathbf{t})$ gives rise to a Galois extension $M' = M \cdot L'$ of $M(\mathbf{t})$ such that $M(\mathbf{t}, x) \subseteq M' \subset \hat{M}$.

Now $\hat{L}$ is also linearly disjoint from M over L and $\hat{M} = \hat{M} \cdot \hat{L}$ is, by assumption, linearly disjoint from $\tilde{K}$ over M. Hence $\hat{L}$ is linearly disjoint from $\tilde{K}$ over L. Thus $L(\mathbf{t}, x)$ is a stable extension of L.

If L is a hilbertian field and $f \in L[T_1, \ldots, T_r, X]$ is an absolutely irreducible stable polynomial over L with respect to $\mathbf{T}$, then one can construct, by induction, a sequence $(\mathbf{a}_1)$, $(\mathbf{a}_2)$, $(\mathbf{a}_3), \ldots$ of r-tuples of L such that the sequence $L_1, L_2, L_3, \ldots$ of the splitting fields over L of $f(\mathbf{a}_1, X), f(\mathbf{a}_2, X), f(\mathbf{a}_3, X), \ldots$ (respectively) is linearly disjoint over L. This is the crucial property of the stable polynomials and we shall use it later in an application to approximation theory. But first we have to worry about getting sufficiently many stable polynomials. This is done by proving that every *PAC* field M is stable. The above hilbertian field L will be obtained from M by using Lemma 1.1.

A sufficient condition for a separating transcendence base to be a stabilizing base is given by the following lemma:

LEMMA 1.2. *Let $t_1, \ldots, t_r$ be a separating transcendence base of a finitely generated regular extension F/K, let $n = [F : K(\mathbf{t})]$ and denote by $\hat{F}$ the Galois closure of $F/K(t)$. If $\mathscr{G}(\hat{F} \cdot \tilde{K}/\tilde{K}(\mathbf{t}))$ is isomorphic to the symmetric group S_n, then $t_1, \ldots, t_r$ stabilizes F/K, and hence F/K is stable.*

Proof. By Galois theory

$$\mathscr{G}(\hat{F} \cdot \tilde{K}/\tilde{K}(\mathbf{t})) \cong \mathscr{G}(\hat{F}/\hat{F} \cap \tilde{K}(\mathbf{t})).$$

Hence

$$n! = [\hat{F} : \hat{F} \cap \tilde{K}(\mathbf{t})] \leqq [\hat{F} : K(\mathbf{t})] \leqq n!$$

since $[F : K(t)] = n$. It follows that $\hat{F} \cap \tilde{K}(\mathbf{t}) = K(\mathbf{t})$ and hence

$$\mathscr{G}(\hat{F} \cdot \tilde{K}/\tilde{K}(\mathbf{t})) \cong \mathscr{G}(\hat{F}/K(\mathbf{t})),$$

i.e. $t_1, \ldots, t_r$ is a stabilizing basis for $F/K(\mathbf{t})$.

2. Function field of one variable. The theory of divisors of function fields of one variable makes it possible to realize the conditions of Lemma 1.2 and thus to construct stable extensions.

LEMMA 2.1. *Let L be an algebraically closed field, let t be a transcendental element over L, let E be a separable extension of $L(t)$ of prime degree l and let $\hat{E}$ be its Galois closure. Suppose that there exists a prime divisor $\mathfrak{p}$ of $L(t)/L$ which decomposes in E as*

$$\mathfrak{p} = \mathfrak{P}_1 + \ldots + \mathfrak{P}_{l-2} + 2\mathfrak{P},$$

where $\mathfrak{P}_1, \ldots, \mathfrak{P}_{l-2}$ are distinct prime divisors of E/L. Then $\mathscr{G}(\hat{E}/L(t)) \cong S_l$.

Proof. Write $G = \mathscr{G}(\hat{E}/L(t))$ and $H = \mathscr{G}(\hat{E}/E)$, and let $\Sigma = \{\sigma H | \sigma \in G\}$ be the set of all left cosets of H in G. The order of Σ is obviously l. Note that $G_1 = \{\tau \in G | \tau\sigma H = \sigma H \text{ for all } \sigma \in G\}$ is a normal subgroup of G which is contained in H. Its fixed field E_1 is normal over $L(t)$ and contains E, hence $E_1 = \hat{E}$. It follows that $G_1 = 1$ or, in other words, G acts faithfully on Σ. We can therefore consider G as a subgroup of the group $S(\Sigma)$ of all permutations of Σ. Choose now, for every $1 \leqq i \leqq l - 2$, an extension $\mathfrak{Q}_i$ of $\mathfrak{P}_i$ to $\hat{E}$ and let $\mathfrak{Q}$ be an extension of $\mathfrak{P}$ to $\hat{E}$. Then $\mathfrak{Q}_i$ is conjugate to $\mathfrak{Q}$ over $L(t)$, i.e. there exists a $\sigma_i \in G$ such that $\mathfrak{P}^{\sigma_i} = \mathfrak{P}_i$. If, for $1 \leqq i, j \leqq l - 2$, $\sigma_i H = \sigma_j H$, then $\sigma_i | E = \sigma_j | E$, hence $\mathfrak{P}_i = \mathfrak{P}_j$, hence $i = j$. It follows that $\sigma_1 H, \ldots, \sigma_{l-2} H$ are $l - 2$ distinct elements of Σ. Let $\sigma_{l-1} H$ and $\sigma_l H$ be the remaining two.

Our assumption that L is algebraically closed leads to the conclusion that $I(\mathfrak{Q}) = \{\tau \in G | \mathfrak{Q}^\tau = \mathfrak{Q}\}$ and $I(\mathfrak{Q}_i) = \{\tau \in G | \mathfrak{Q}_i^\tau = \mathfrak{Q}_i\}$ are the inertia groups of $\mathfrak{Q}$ and $\mathfrak{Q}_i$, $i = 1, \ldots, l - 2$ respectively. The $\mathfrak{P}_i$ are, by assumption, unramified over $L(t)$, hence the inertia fields of $\mathfrak{Q}_i$ contain E, which means that $I(\mathfrak{Q}_i) \subseteq H$ for $i = 1, \ldots, l - 2$. On the other hand $\mathfrak{Q}$ is certainly ramified over $L(t)$. Therefore there exists a $\tau \in I(\mathfrak{Q})$ such that $\tau \neq 1$. For every $1 \leqq i \leqq l - 2$ we have $\sigma_i^{-1}\tau\sigma_i \in I(\mathfrak{Q}_i)$, hence $\tau\sigma_i H = \sigma_i H$. It follows that $\tau\sigma_{l-1} H = \sigma_l H$ and $\tau\sigma_l H = \sigma_{l-1} H$, i.e. τ is a transposition. In addition, the order of G is a multiple of l, hence, by Sylow's theorem, G contains an element ρ of order l, which is necessarily a cycle of length l. It follows that G must coincide with $S(\Sigma)$, since obviously G acts transitively on Σ (c.f. van der Waerden [**10**, p. 201]).

LEMMA 2.2. *Let F be a function field of one variable with genus g over an infinite field of constants K. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be prime divisors of F/K and let $a_1, \ldots, a_n$ be positive integers such that*

$$(1) \quad \sum_{i=1}^{n} a_i \deg \mathfrak{p}_i > 3g - 2.$$

Then there exists an $x \in F$ whose pole divisor $(x)_\infty$, is equal to $\sum_{i=1}^n a_i\mathfrak{p}_i$ and hence $[F : K(x)] = \sum_{i=1}^n a_i \deg \mathfrak{p}_i$.

Proof. Write $\mathfrak{a} = \sum_{i=1}^n a_i\mathfrak{p}_i$ and $\mathfrak{a}_j = \mathfrak{a} - \mathfrak{p}_j$ for $j = 1, \ldots, n$. The set $\mathscr{L}(\mathfrak{a}) = \{x \in F | (x) \geqq -\mathfrak{a}\}$ is a finite dimensional K-vector space ((x) is the divisor of x). We denote its dimension by $\dim \mathfrak{a}$. By (1) we have $\deg \mathfrak{a} > 2g - 2$. Hence, by the Riemann-Roch theorem

$$(2) \quad \dim \mathfrak{a} = \deg \mathfrak{a} + 1 - g.$$

If $\deg \mathfrak{a}_j > 2g - 2$, then by the Riemann-Roch theorem and by (2)

$$\dim \mathfrak{a}_j = \deg \mathfrak{a} - \deg \mathfrak{p}_j + 1 - g < \dim \mathfrak{a}.$$

If $\deg \mathfrak{a}_j \leqq 2g - 2$, then by the Riemann-Roch theorem, by (2) and by (1)

$$\begin{aligned}\dim \mathfrak{a}_j &= \deg \mathfrak{a}_j + 1 - g + \dim(\mathfrak{c} - \mathfrak{a}_j) \\ &\leqq 2g - 2 + 1 - g + g < \dim \mathfrak{a},\end{aligned}$$

where $\mathfrak{c}$ is a canonical divisor, since

$$\dim(\mathfrak{c} - \mathfrak{a}_j) \leqq \dim \mathfrak{c} = g$$

(c.f. Lang [**6**, p. 241]). Thus in both cases we have $\dim \mathfrak{a}_j < \dim \mathfrak{a}$, hence $\mathscr{L}(\mathfrak{a}_j)$ is properly contained in $\mathscr{L}(\mathfrak{a})$ for $j = 1, \ldots, n$. It follows that $\bigcup_{j=1}^n \mathscr{L}(\mathfrak{a}_j)$ is also properly contained in $\mathscr{L}(\mathfrak{a})$, since K is infinite. Every $x \in \mathscr{L}(\mathfrak{a}) - \bigcup_{j=1}^n \mathscr{L}(\mathfrak{a}_j)$ will satisfy the requirements of the lemma (c.f. Lang [**6** p. 237]).

THEOREM 2.3. *Let F be a function field of one variable over an infinite field of constants K. If F has a prime divisor $\mathfrak{P}$ of degree* 1, *then F/K is a stable extension.*

Proof. Let g be the genus of K, let $p = \text{char}(K)$ and choose a prime l such that

(3) $\quad 1 > 3g \quad$ and $\quad p \nmid l(l-2)$

By Lemma 2.2 there exists an $s \in F$ such that $(s)_\infty = (l-2)\mathfrak{P}$ and $[F : K(s)] = l - 2$, since $\deg \mathfrak{P} = 1$. There are infinitely many prime divisors of $K(s)/K$ of degree 1, since K is infinite. By (3), F is a finite separable extension of $K(s)$, hence we can find a prime divisor $\mathfrak{q}$ of degree 1 of $K(s)/K$ which is unramified in F and such that if $\mathfrak{Q}_1, \ldots, \mathfrak{Q}_m$ are the distinct prime divisors of F which lie over $\mathfrak{q}$, then their residue fields $\mathfrak{Q}_i F$ are separable over $K = \mathfrak{q}K(s)$ (c.f. Chevalley [**2**, p. 72]). Furthermore we have $\sum_{i=1}^m \deg \mathfrak{Q}_i = l - 2$ (c.f. [**2**, p. 52]).

Consider now the divisor $\mathfrak{A} = \mathfrak{Q}_1 + \ldots + \mathfrak{Q}_m + 2\mathfrak{P}$. Its degree l is, by (3), greater than $3g - 2$. Hence, by Lemma 2.2 there exists a $t \in F$ such that $(t)_\infty = \mathfrak{A}$ and $[F : K(t)] = l$.

Extend the field of constants from K to $\tilde{K}$ and write $E = \tilde{K} \cdot F$. Then $[E : \tilde{K}(t)] = [F : K(t)] = l$, since F is linearly disjoint from $\tilde{K}$ over K. Hence $E/\tilde{K}(t)$ is separable. Denote by $\hat{E}$ the Galois closure of $E/\tilde{K}(t)$. If $\mathfrak{Q}$ is any prime divisor of F/K such that $\mathfrak{Q}F$ is a separable extension of K, then there exists exactly $\deg \mathfrak{Q} = [\mathfrak{Q}F : K]$ distinct prime divisors of $E/\tilde{K}$ which lie over $\mathfrak{Q}$ (c.f. [**2**, p. 95]). In particular exactly one prime divisor, $\mathfrak{P}'$, of $E/\tilde{K}$ lies over $\mathfrak{P}$ and exactly $l - 2$ prime divisors, $\mathfrak{P}_1', \ldots, \mathfrak{P}_{l-2}'$, of $E/\tilde{K}$ lie over $\mathfrak{Q}_1, \ldots, \mathfrak{Q}_m$. It follows that the pole divisor $(t)_\infty'$ of t in $\tilde{K}(t)$, which is a prime divisor there, factors in E as $(t)_\infty' = \mathfrak{P}_1' + \ldots + \mathfrak{P}_{l-2}' + 2\mathfrak{P}$. By Lemma 2.1 $\mathscr{G}(\hat{E}/\tilde{K}(t)) \cong S_l$. Hence, by Lemma 1.2, F is a stable extension of K.

3. Function fields of several variables. In this section we apply Theorem 2.3 to develop a certain condition under which a finitely generated regular extension F/K is stable.

LEMMA 3.1. *Let F/K be an arbitrary field extension. If F has a K-rational K-place φ, then F is a regular extension of K.*

Proof. Extend φ to a $\tilde{K}$-place of $\tilde{K} \cdot F$ and denote this place also by φ. Then the restriction of φ to $\tilde{K}$ is an isomorphism (c.f.: Lang [**6**, p. 8]). Consider now n elements $a_1, \ldots, a_n$ of $\tilde{K}$ which are linearly independent over K. Then $\varphi(a_1), \ldots, \varphi(a_n)$ are also linearly independent over K. Assume that they become linearly dependent over F, i.e. that there exists $x_1, \ldots, x_n \in F$, not all zero, such that

(1) $x_1a_1 + \ldots + x_na_n = 0.$

It is known that for one of the x_i's, say x_1, all the quotients $x_j/x_1, j = 1, \ldots, n$, are finite under φ and hence the $\varphi(x_j/x_1)$ belong to K. In particular $x_1 \neq 0$. From (1) we therefore get that

$$\varphi(a_1) + \varphi(x_2/x_1)\varphi(a_1) + \ldots + \varphi(x_n/x_1)\varphi(a_n) = 0,$$

which is a contradiction. It follows that $\tilde{K}$ is linearly disjoint from F over K.

By a *model* of a finitely generated regular field-extension F/K we mean an absolutely irreducible affine variety V defined over K, the function field of which is F.

LEMMA 3.2. *Let F/K be a finitely generated regular field extension of dimension $r \geqq 1$. Suppose that*

(*) *F/K has a model V with a K-rational simple point P.*

Then:

a) *there exists an intermediate field $K \subseteq L \subset F$ such that L/K is a purely transcendental extension of dimension $r - 1$ and F/L is a regular extension which has a prime divisor $\mathfrak{p}$ of degree 1;*

b) *if K is an infinite field, then F is a stable extension of K.*

Proof. a) The local ring R of P in F is regular. This means that its maximal ideal M is generated by r elements, say $s_1, \ldots, s_r$ (c.f. Lang [**6**, p. 201]). R can be imbedded into the ring of formal power series $\hat{R} = K[[s_1, \ldots, s_r]]$. $\hat{R}$ has a discrete valuation v which can be described as follows: Each element $f \in \hat{R}$ can be written in a unique way as $f = f_0 + f_1 + f_2 + \ldots,$ where f_i is a homogenous polynomial in $s_1, \ldots, s_r$ of degree i. The value, $v(f)$, of v at f is the smallest integer n such that $f_n \neq 0$ (c.f. Zariski-Samuel [**12**, p. 130]). This valuation can be restricted to R and then extended in the usual way to the quotient field F of R, where it keeps the same notation v. The ring R is certainly contained in (but probably not identical with) the valuation ring S of v in F. Let N be the maximal ideal of S and let φ be the K-place of F into $\bar{F} = S/N$ induced by v.

Now write $t_i = s_i/s_r$, $i = 1, \ldots, r - 1$. Obviously $t_1, \ldots, t_{r-1}$ are in S and if we write $\bar{t}_i = \varphi(t_i)$ for $i = 1, \ldots, r - 1$, then $\bar{t}_1, \ldots, \bar{t}_{r-1}$ are algebraically independent over K and $\bar{F} = K(\bar{t}_1, \ldots, \bar{t}_{r-1})$ (c.f. [**12**, 132]). It follows that $t_1, \ldots, t_{r-1}$ are also algebraically independent over K and that if we put $L = K(t_1, \ldots, t_{r-1})$, then the map $t_i \rightarrow \bar{t}_i$, $i = 1, \ldots, r - 1$, induces a K-isomorphism of L onto F. L is therefore a purely transcendental extension of K of dimension $r - 1$. If we identify $\bar{F}$ with L under the above isomorphism, we get that φ is an L-rational L-place of F. This gives us the desired prime divisor $\mathfrak{p}$ of F/L of degree 1 and also implies, by Lemma 3.1, that F is a regular extension of L.

b) By Theorem 2.3 there exists a t_r in F which serves as a stabilizing basis for the regular extension F/L of dimension 1. The system $t_1, \ldots, t_r$ will therefore be a stabilizing basis for F/K.

LEMMA 3.3. *Let V be an abstract variety defined over a PAC field L. Then the set $V(L)$ of all L-rational points of V is dense in V in the Zariski L-topology.*

Proof. Without loss of generality we can assume that V is affine and hence contained in a certain affine space S^n. Let A be an L-closed subset of S^n which does not contain V. We have to show that $V(L) - A \neq \emptyset$. Indeed, let $(\mathbf{x}) = (x_1, \ldots, x_n)$ be a generic point of V over L. Then $L(x)$ is a regular extension of L and there exists a polynomial $g \in L[X_1, \ldots, X_n]$ which vanishes on A but not at $(\mathbf{x})$. Write $y = g(x)^{-1}$. Then $L(\mathbf{x}, y) = L(\mathbf{x})$ and hence $(\mathbf{x}, y)$ generates an absolutely irreducible variety W over L. W has an L-rational point $(\mathbf{a}, b)$. It is a specialization of $(\mathbf{x}, y)$ over L. The point $(\mathbf{a})$ belongs to $V(L)$ and satisfies $g(\mathbf{a})b = 1$, hence $(\mathbf{a}) \notin A$.

THEOREM 3.4. *Every PAC field L is stable.*

Proof. Let F be a finitely generated regular extension of L and let V be an affine model of F/L. By Lemma 3.3, V has an L-rational simple point P, since the set V_{sim} of all simple points of V is L-open in V (c.f. Lang [**2**, p. 199]). In addition L must be infinite, since if L contains only q elements, then the absolutely irreducible polynomial $(X^q - X)(Y^q - Y) + 1$ has no zeros in L. It follows, by Lemma 3.2, that F is stable over L.

4. Valued fields. Let (K, v, Γ) be a valued field of one of the following two types:

I. The archimedean type: K is a subfield of the field $\mathbf{C}$ of complex numbers, v is the usual absolute value and $\Gamma \subseteq R$.

II. The non-archimedean type: K is an arbitrary field and v is a non-trivial multiplicative valuation of K with values in the ordered, multiplicative, divisible, abelian group Γ.

We shall use the notations $|a|_v$ instead of $v(a)$ for elements a of K and reserve the notation $v(A)$ for the value set of a subset A of K. Γ is assumed to be the divisible closure of $v(K^{\times})$. We shall also denote by K_v the completion of K under v and always assume that v has been extended to K_v.

Along with v we shall consider also the set Ω of all extensions of v to $\tilde{K}$. All the completions $\tilde{K}_\omega$, $\omega \in \Omega$, are assumed to be contained in some universal field and thus contain the same copy of $\tilde{K}$. Every $\omega \in \Omega$ defines a field topology on $\tilde{K}_\omega$, the basis sets of which are $\{x \in \tilde{K}_\omega | \, |x - a|_\omega < \epsilon\}$, where $a \in \tilde{K}_\omega$ and $\epsilon \in \Gamma$. We shall refer to it as the ω-topology.

LEMMA 4.1. *Let*

$$f(\mathbf{T}, x) = f_n(\mathbf{T})X^n + \ldots + f_k(\mathbf{T})X^k + \ldots + f_0(\mathbf{T})$$

be a polynomial with coefficients in K *in the variables* $(\mathbf{T}, X) = (T_1, \ldots, T_r, X)$. *Let* $(\mathbf{a}, b)$ *be a* $\tilde{K}$*-rational zero of* f *for which there exists a* k, $0 \leqq k \leqq n$, *such that* $f_k(\mathbf{a}) \neq 0$. *Then for every* $\epsilon \in \Gamma$ *there exists a* $\delta \in \Gamma$ *such that for every* $\omega \in \Omega$ *and for every* $a_1', \ldots, a_k' \in \tilde{K}$ *which satisfy*

$$|a_i' - a_i|_\omega < \delta \quad i = 1, \ldots, r$$

there exists a $b' \in \tilde{K}$ *such that*

$$f(\mathbf{a}', b') = 0, \quad f_k(\mathbf{a}') \neq 0 \quad \textit{and} \quad |b' - b|_\omega < \epsilon.$$

Proof. Without loss of generality we can assume that $(\mathbf{a}, b) = (\mathbf{0}, 0)$ and $\epsilon < 1$. Then $f_0(\mathbf{0}) = 0$ and $1 \leqq k \leqq n$. Let L be a finite extension of K which contains all the coefficients of f and let $v_1, \ldots, v_l$ be all the extensions of v to L. For every $1 \leqq j \leqq l$ we choose an extension ω_j of v_j to $\tilde{K}$. Then there exists a $\delta \in \Gamma$ such that for $j = 1, \ldots, l$,

(1) $|a_i'|_{\omega_j} < \delta, \quad i = 1, \ldots, r \quad$ implies $\quad f_k(a') \neq 0 \quad$ and

$$\left|\frac{f_0(\mathbf{a}')}{f_k(\mathbf{a}')}\right|_{\omega_j} < \begin{cases} \dfrac{\epsilon^n}{n} & \text{in Case I} \\ \epsilon^n & \text{in Case II} \end{cases}$$

since f_0 and f_k are ω_j-continuous.

Let $\omega \in \Omega$. Then there exists a $1 \leqq j \leqq 1$ and an automorphism σ of $\tilde{K}$ over L such that $\omega = \omega_j \circ \sigma$ (c.f. Lang [8, p. 293]). Suppose that $a_1', \ldots, a_r'$ are elements of $\tilde{K}$ such that $|a_i'|_\omega < \delta$ for $i = 1, \ldots, r$. Then $|\sigma a_i'|_{\omega_j} < \delta$ for $i = 1, \ldots, r$ and hence by (1), $\sigma f_k(\mathbf{a}') \neq 0$, hence $f_k(\mathbf{a}') \neq 0$ and

$$(2) \quad \left|\frac{f_0(\mathbf{a}')}{f_k(\mathbf{a}')}\right|_\omega = \left|\frac{f_0(\sigma\mathbf{a}')}{f_k(\sigma\mathbf{a}')}\right|_{\omega_j} < \begin{cases} \dfrac{\epsilon^n}{n!} & \text{in Case I} \\ \epsilon^n & \text{in Case II} \end{cases}$$

Let m be the greatest integer for which $f_m(\mathbf{a}') \neq 0$. Then $k \leqq m \leqq n$ and

$$f(\mathbf{a}', X) = f_m(\mathbf{a}')X^m + \ldots + f_k(\mathbf{a}')X^k + \ldots + f_0(\mathbf{a}')$$

$$= f_m(\mathbf{a}') \sum_{i=1}^{m} (X - b_i'),$$

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where $b_1', \ldots, b_m' \in \tilde{K}$. Then

$$\frac{f_0(\mathbf{a}')}{f_m(\mathbf{a}')} = (-1)^m b_1' \ldots b_m' \quad \text{and} \quad \frac{f_k(\mathbf{a}')}{f_m(\mathbf{a}')} = (-1)^{m-k} \sum_{\pi} b'_{\pi(1)} \ldots b'_{\pi(m-k)}$$

where π runs over all the injective maps of the set $\{1, \ldots, m - k\}$ into the set $\{1, \ldots, m\}$. If $f_0(\mathbf{a}') = 0$, then $b_i' = 0$ for at least one i between 1 and m. If $f_0(\mathbf{a}') \neq 0$ then we extend each of the above π to a permutation of $\{1, \ldots, m\}$. Then

$$\frac{f_k(\mathbf{a}')}{f_0(\mathbf{a}')} = (-1)^k \sum_{\pi} \frac{1}{b'_{\pi(m-k+1)} \ldots b'_{\pi(m)}}$$

and by (2) we deduce that there must be a b_i' which satisfies $|b_i'|_\omega < \epsilon$.

LEMMA 4.2. *Let $\omega \in \Omega$ and let M be a PAC field which is algebraic over K. Suppose that for every polynomial $f \in M[T_1, \ldots, T_r, X]$ which is stable over M with respect to $T_1, \ldots, T_r$ the set*

$$\left\{(\mathbf{a}, b) \in M^{r+1} \middle| f(\mathbf{a}, b) = 0 \quad \textit{and} \quad \frac{\partial f}{\partial X}(\mathbf{a}, b) \neq 0\right\}$$

is ω-dense in the set

$$\left\{(\mathbf{a}, b) \in \tilde{K}_\omega^{\ r+1} \middle| f(\mathbf{a}, b) = 0 \quad \textit{and} \quad \frac{\partial f}{\partial X}(\mathbf{a}, b) \neq 0\right\}$$

Then M has the density property with respect to ω, i.e. $V(M)$ is ω-dense in $V(\tilde{K}_\omega)$ for every abstract variety V which is defined over M.

Proof. Let V be an abstract variety of dimension r which is defined over M. Then V is stable over M, by Theorem 3.4. Let P be a generic point of V over M and let $t_1, \ldots, t_r$ be a stabilizing base for $M(P)/M$. Then $M(P)/M(\mathbf{t})$ is finite separable extension and hence there exists an element x such that $M(P) = M(\mathbf{t}, x)$. There exists an absolutely irreducible polynomial $f \in M[\mathbf{T}, X]$ such that $f(\mathbf{t}, x) = 0$ and $\partial f/\partial X(\mathbf{t}, x) \neq 0$. By definition f is stable over M with respect to $(\mathbf{T})$. Denote by W the hypersurface defined by f over M. It has $(\mathbf{t}, x)$ as a generic point and the map $(\mathbf{t}, x) \mapsto P$ defines a birational correspondence φ between W and V over M. The set $W' = \{(\mathbf{a}, b) \in W | \partial f/\partial X(\mathbf{a}, b) \neq 0\}$ is a non-void open subset of W in the Zariski M-topology. W' contains a non-void subset W_0' which is open in the Zariski M-topology and V contains a subset V_0' which is open in the Zariski-M-topology such that φ is biregular at W_0' and has V_0' as a set theoretic image of W_0' (c.f. Lang [**6**, p. 94]). The correspondence φ induces therefore ω-homeomorphisms of $W_0'(M)$, $W_0'(\tilde{K}_\omega)$ onto $V_0'(M)$, $V_0'(\tilde{K}_\omega)$, respectively (c.f. Weil [**11**, p. 352]).

By assumption $W'(M)$ is ω-dense in $W'(\tilde{K}_\omega)$, hence $W_0'(M)$ is ω-dense in $W_0'(\tilde{K}_\omega)$. Hence $V_0'(M)$ is ω-dense in $V_0'(\tilde{K}_\omega)$. By Lemma 2.2 in [**3**], $V_0'(\tilde{K}_\omega)$ is ω-dense in $V(\tilde{K}_\omega)$, since $\tilde{K}_\omega$ is algebraically closed (c.f. Lemma 1.4 in [**3**]). Hence $V(M)$ is ω-dense in $V(\tilde{K}_\omega)$.

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5. Hilbertian valued field. Let K be a field. A *hilbertian subset* H of K^r is a set of the form

$$H = \{(\mathbf{a}) \in K^r | f_j(\mathbf{a}, \mathbf{X}) \text{ is defined and irreducible in } K[\mathbf{X}] \text{ for } j = 1, \ldots, m\},$$

where $f_1, \ldots, f_m$ are irreducible polynomials in $K(T_1, \ldots, T_r)[X_1, \ldots, X_n]$. The field K is said to be *hilbertian* if for every $n \geqq 1$, all hilbertian subsets of K^n are non-empty.

LEMMA 5.1. *Let K be a hilbertian field and let $f \in K[T_1, \ldots, T_r, X]$ be a stable polynomial over K with respect to $T_1, \ldots, T_r$. Let $G = \mathscr{G}(f(\mathbf{t}, X), K(\mathbf{t}))$ be the Galois group of $f(\mathbf{t}, X)$ over $K(\mathbf{t})$ ($t_1, \ldots, t_r$ are algebraically independent elements over K), and let L be a finite separable extension of K. Then there exists a hilbertian set $H \subseteq K^r$ such that every $(\mathbf{a}) \in H$, $\mathscr{G}(f(\mathbf{a}, X), K) \cong \mathscr{G}(f(\mathbf{a}, X), L) \cong G$, hence the splitting field of $f(\mathbf{a}, X)$ over K is linearly disjoint from L over K.*

Proof. There exists a hilbertian set $H_1 \subseteq K^r$ such that $\mathscr{G}(f(\mathbf{a}, X), K) \cong G$ for every $(\mathbf{a}) \in H_1$ (c.f. Kuyk [**5**, p. 396]). There exists also a hilbertian set $H_2' \subseteq L^r$ such that $\mathscr{G}(f(\mathbf{a}, X), L) \cong G$ for every $(\mathbf{a}) \in H_2'$, since f is stable and hence $\mathscr{G}(f(\mathbf{t}, X), L(\mathbf{t})) \cong G$. H_2' contains a hilbertian set $H_2 \subseteq K^r$ (c.f. Lang [**7**, p. 152]). $H = H_1 \cap H_2$ will satisfy the requirements of the lemma.

From now on we shall suppose that K is a hilbertian valued field and we shall keep all the notations of § 4. We refer to § 5 of [**3**] for the introduction of the Haar measure to $\mathscr{G}(K_s/K)^e$. Here e is a fixed positive integer.

LEMMA 5.2. *Let L be a finite separable extension of K and let $f \in L[T_1, \ldots, T_r, X]$ be a stable polynomial over L with respect to $T_1, \ldots, T_r$. Let $c_1, \ldots, c_r, d \in K_s$ be such that $f(\mathbf{c}, d) = 0$ and $\partial f/\partial X(\mathbf{c}, d) \neq 0$. Let $\delta < \epsilon$ be two elements of Γ such that for every $\omega \in \Omega$ and for every $c_1', \ldots, c_r' \in K_s$ which satisfy $|c_i' - c_i|_\omega < \delta$ for $i = 1, \ldots, r$, there exists a $d' \in K_s$ such that $f(\mathbf{c}', d') = 0$, $\partial f/\partial X(\mathbf{c}', d') \neq 0$ and $|d' - d|_\omega < \epsilon$. Suppose also that for every $\omega \in \Omega$ there exist $c_1'', \ldots, c_r'' \in L$ such that $|c_i'' - c_i|_\omega < \delta/2$ in Case I and $|c_i'' - c_i|_\omega < \delta$ in Case II, for $i = 1, \ldots, r$. Then, for almost all $(\boldsymbol{\sigma}) \in \mathscr{G}(K_s/L)^e$ and for every $\omega \in \Omega$ there exist $a_1, \ldots, a_r, b \in K_2(\boldsymbol{\sigma})$ such that*

$$(1) \quad f(\mathbf{a}, b) = 0, \quad \frac{\partial f}{\partial X}(\mathbf{a}, b) = 0,$$
$$|a_i - c_i|_\omega < \epsilon \quad \textit{for } i = 1, \ldots, r \quad \textit{and} \quad |b - d|_\omega < \epsilon.$$

Proof. Let n be the order of $G = \mathscr{G}(f(\mathbf{t}, X), L(\mathbf{t}))$ and let $v_1, \ldots, v_l$ be all the extensions of v to L. We construct by induction a linearly disjoint sequence $\{L_j/L\}_{j=1}^\infty$ of Galois extensions of degree nl such that for every $j \geqq 1$ and for every $\omega \in \Omega$ there exist $a_1, \ldots, a_r, b \in L_j$ which satisfy (1). Suppose that $L_1, \ldots, L_{j-1}$ have already been constructed. Then $L' = L_1 \ldots L_{j-1}$ is a finite Galois extension of L. For every $1 \leqq \lambda \leqq l$ there exist, by assumption $c_{\lambda 1}', \ldots, c_{\lambda r}' \in L$ such that $|c_{\lambda i}' - c_i|_{v_\lambda} < \delta/2$ in Case I and $|c_{\lambda i}' - c_i|_{v_\lambda} < \delta$ in

Case II, for $i = 1, \ldots, r$. By Lemma 4.1 of [**3**] every hilbertian set of L^r is v_1-dense in L^r, hence, by Lemma 5.1, there exist $a_{11}, \ldots, a_{1r} \in L$ which satisfy. $|a_{1i} - c_{1i}'|_{v_1} < \delta/2$ in Case I and $|a_{1i} - c_{1i}'|_{v_1} < \delta$ in Case II, for $i = 1, \ldots, r$, such that $\mathscr{G}(f(\mathbf{a}, X), L) \cong \mathscr{G}(f(\mathbf{a}, X), L') \cong G$. Let L_{j1} be the splitting field of $f(\mathbf{a}, X)$ over L. Then L_{j1} is a Galois extension of L of degree n which is linearly disjoint from L' over L.

If $\omega \in \Omega$ is an absolute value whose restriction to L coincides with v_1, then $|a_{1i} - c_i|_\omega < \delta/2$ in Case I and $|a_{1i} - c_i|_\omega < \delta$ in Case II for $i = 1, \ldots, r$; hence, by assumption, there exists a $b_1 \in K_s$ such that $f(\mathbf{a}_1, b_1) = 0$, $\partial f/\partial X(\mathbf{a}_1, b_1) \neq 0$ and $|b_1 - d|_\omega < \epsilon$. In particular b_1 is a root of $f(\mathbf{a}_1, X)$ and hence belongs to L_j'.

In the same way we can construct, step by step, for every $1 \leqq \lambda \leqq l$, a Galois extension $L_{j\lambda}$ of L of degree r which is linearly disjoint from $L'L_{j1} \ldots L_{j,\lambda-1}$ and elements $a_{\lambda 1}, \ldots, a_{\lambda r}, b_\lambda \in L_{j\lambda}$ such that (1) is satisfied for every $\omega \in \Omega$ whose restriction to L coincides with v_λ.

The field $L_j = L_{j1} \ldots L_{jl}$ is a Galois extension of L of degree nl which is linearly disjoint from L' over L which satisfies the requirements, since the restriction of each of the $\omega \in \Omega$ to L coincides with one of the v_λ.

By Lemma 5.1 of [**3**] the union $\bigcup_{j=1}^{\infty} \mathscr{G}(K_s/L_j)^e$ is almost equal to $\mathscr{G}(K_s/L)^e$. If $(\boldsymbol{\sigma}) \in \bigcup_{j=1}^{\infty} \mathscr{G}(K_s/L_j)^e$ then $L_j \subseteq K_s(\boldsymbol{\sigma})$ for at least one j. Hence for every $\omega \in \Omega$ there exist $a_1, \ldots, a_r, b \in K_s(\boldsymbol{\sigma})$ such that (1) is satisfied.

LEMMA 5.3. *If K is a denumerable hilbertian field, then almost all $(\boldsymbol{\sigma}) \in \mathscr{G}(K_s/K)^e$ have the following property:*

For every $d \in \tilde{K}$ and for every $\epsilon \in \Gamma$ there exists a finite subset B of $K_s(\boldsymbol{\sigma})$ such that for every $\omega \in \Omega$ there exists a $b \in B$ such that $|b - d|_\omega < \epsilon$.

In particular, $K_s(\boldsymbol{\sigma})$ is ω-dense in $\tilde{K}$ for every $\omega \in \Omega$.

Proof. Let $d \in K$, let $f(X) = X^n + c_1X^{n-1} + \ldots + c_n$ be a polynomial with coefficients in K such that $f(d) = 0$ and let $\epsilon > 0$. We shall construct by induction a linearly disjoint sequence $\{K_i/K\}_{i=1}^{\infty}$, of Galois extensions of degree n and in every K_i a subset B_i with n elements such that for every $\omega \in \Omega$ there exists a $b \in B_i$ such that $|b - d|_\omega < \epsilon$.

Suppose that $K_1, \ldots, K_{i-1}$ have already been constructed. Then $K' = K_1 \ldots K_{i-1}$ is a finite separable extension of K. The general polynomial, $h(\mathbf{T}, X) = X^n + T_1X^{n-1} + \ldots + T_n$, of degree n is, as we already noted in § 1, stable over K with respect to $T_1, \ldots, T_n$ and it has the Galois group S_n over $K(\mathbf{T})$. Hence, we can find, as in the proof of Lemma 5.2, $a_1, \ldots, a_n \in K$ which are v-close to $c_1, \ldots, c_n$ such that $h(\mathbf{a}, X)$ is a separable polynomial with S_n as a Galois group both over K and over K'. Let K_i be the splitting field of $h(\mathbf{a}, X)$ over K. Then K_i is a Galois extension of K of degree $n!$ Let $b_1, \ldots, b_n$ be all the roots of $h(a, X)$ and write $B_i = \{b_1, \ldots, b_n\}$. Then, by Lemma 4.1, for every $\omega \in \Omega$ there exists a $b \in B_i$ such that $|b - d|_\omega < \epsilon$.

Write now $S(d, \epsilon)$ for the set of all $(\boldsymbol{\sigma}) \in \mathscr{G}(K_s/K)^e$ for which there exists a finite subset B of $K_s(\boldsymbol{\sigma})$ such that for every $\omega \in \Omega$ there exists a $b \in B$ for which $|b - d|_\omega < \epsilon$. Clearly $\bigcup_{i=1}^{\infty} \mathscr{G}(K_s/K_i)^e \subseteq S(d, \epsilon)$. The right hand side of this inclusion has, by Lemma 5.1 of [**3**], the measure 1, hence $\mu(S(d, \epsilon)) = 1$. Since K is denumerable field, there are only countably many $d \in \tilde{K}$ and $\epsilon \in \Gamma$. Hence the intersection of all the sets $S(d, \epsilon)$ has the measure 1. This concludes the proof of the lemma.

THEOREM 5.4. *If K is a denumerable hilbertian field, then $K_s(\boldsymbol{\sigma})$ has the density property with respect to every $\omega \in \Omega$ for almost all $(\boldsymbol{\sigma}) \in \mathscr{G}(K_s/K)^e$.*

Proof. Let S be the set of all $(\boldsymbol{\sigma}) \in \mathscr{G}(K_s/K)^e$ such that $K_s(\boldsymbol{\sigma})$ is a PAC field and which have the following property:

For every polynomial $f(T_1, \ldots, T_r, X)$ which is stable over $K_s(\boldsymbol{\sigma})$ with respect to $T_1, \ldots, T_r$ and for every $\omega \in \Omega$ the set

$$W_f'(K_s(\boldsymbol{\sigma})) = \{(\mathbf{a}, b) \in K_s(\boldsymbol{\sigma})^{r+1} | f(\mathbf{a}, b) = 0, \partial f/\partial X(\mathbf{a}, b) \neq 0\}$$

is ω-dense in the set $W_{f(}(K_s) = \{(\mathbf{a}, b) \in K_s^{r+1} | f(\mathbf{a}, b) = 0, \partial f/\partial X(\mathbf{a}, b) \neq 0\}$.

By Lemma 2.4 in [**3**], $W_f'(K_s)$ is ω-dense in the set

$$W_f'(\tilde{K}_\omega) = \{(\mathbf{a}, b) = \tilde{K}_\omega^{r+1} | f(\mathbf{a}, b) = 0, \partial f/\partial X(\mathbf{a}, b) \neq 0\},$$

hence $W_f'(K_s(\boldsymbol{\sigma}))$ is ω-dense in $W_f'(\tilde{K}_\omega)$. It follows, by Lemma 4.2, that $K_s(\boldsymbol{\sigma})$ has the density property with respect to ω.

It suffices therefore to prove that $\mu(S) = 1$.

Let L be a finite separable extension of K and let $f \in L[T_1, \ldots, T_r, X]$ be a stable polynomial over L with respect to $T_1, \ldots, T_r$. Let $c_1, \ldots, c_r, d \in K_s$ such that $f(\mathbf{c}, d) = 0$ and $\partial f/\partial X(\mathbf{c}, d) = 0$. Let $\epsilon \in \Gamma$. Then by Lemma 4.1 there exists a $\delta \in \Gamma$ which satisfies

(2) $\delta < \epsilon$

such that for every $\omega \in \Omega$ and for every $c_1', \ldots, c_r' \in \tilde{K}$ which satisfy $|c_i' - c_i|_\omega < \delta$ for $i = 1, \ldots, r$, there exists a $d' \in \tilde{K}$ such that $f(\mathbf{c}', d') = 0$, $\partial f/\partial X(\mathbf{c}', d') \neq 0$ and $|d' - d|_\omega < \epsilon$. In particular, if $c_1', \ldots, c_r' \in K_s$ then $d' \in K_s$. Suppose further that for every $\omega \in \Omega$ there exist $c_1'', \ldots, c_r'' \in L$ such that $|c_i'' - c_i|_\omega < \delta/2$ in case I and $|c_i'' - c_i| < \delta$ in Case II, for $i = 1, \ldots, r$.

Denote by $S(L, f(\mathbf{c}, d), \epsilon)$ the set of all $(\boldsymbol{\sigma}) \in \mathscr{G}(K_s/K)^e$ for which there exist $a_1, \ldots, a_r, b \in K_s(\boldsymbol{\sigma})$ such that

(3) $f(\mathbf{a}, b) = 0, \quad \dfrac{\partial f}{\partial X}(\mathbf{a}, b) \neq 0,$

$|a_i - c_i|_\omega < \epsilon \quad i = 1, \ldots, r \quad \text{and} \quad |b - d|_\omega < \epsilon.$

By Lemma 5.2:

(4) $\mu(\mathscr{G}(K_s/L)^e - S(L, f, (\mathbf{c}, d), \epsilon)) = 0.$

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Denote by T the set of all $(\boldsymbol{\sigma}) \in \mathcal{G}(K_s/K)^e$ such that $K_s(\boldsymbol{\sigma})$ is a PAC field and which satisfy:

For every $c \in \tilde{K}$ and for every $\delta \in \Gamma$ there exists a finite extension L of K which is contained in $K_s(\boldsymbol{\sigma})$ such that for every $\omega \in \Omega$ there exists an $a \in L$ for which $|a - c|_\omega < \delta$.

By Theorem 2.5 of [**4**] and by Lemma 5.3

(5) $\mu(T) = 1$.

We prove that

(6) $T - S \subseteq \bigcup[\mathcal{G}(K_s/L)^e - S(L, f, (\mathbf{c}, d), \epsilon)]$

where the union runs over all possible $L, f, (\mathbf{c}, d)$ and ϵ.

Let $(\boldsymbol{\sigma}) \in T - S$. Then $K_s(\boldsymbol{\sigma})$ is a PAC field and there exists a polynomial $f(T_1, \ldots, T_r, X)$ which is stable over $K_s(\boldsymbol{\sigma})$ with respect to $T_1, \ldots, T_r$, and there exists a $\omega_0 \in \Omega$ for which the set $W_f'(K_s(\boldsymbol{\sigma}))$ is not ω_0-dense in $W_f'(K_s)$, i.e. such that there exists $c_1, \ldots, c_r, d \in K_s$ which satisfy $f(\mathbf{c}, d) = 0$ and $\partial f/\partial X(\mathbf{c}, d) \neq 0$ and there exists an $\epsilon \in \Gamma$ for which there do not exist $a_1, \ldots, a_r, b \in K_s(\boldsymbol{\sigma})$ which satisfy (3). Let δ be an element of Γ which satisfies (2). By Lemma 1.1 and since $(\boldsymbol{\sigma}) \in T$ there exists a finite extension L of K which is contained in $K_s(\boldsymbol{\sigma})$ such that f is stable over L with respect to $T_1, \ldots, T_r$ and such that for every $\omega \in \Omega$ there exist $c_1'', \ldots, c_r'' \in L$ which satisfy $|c_i'' - c_i|_\omega < \delta/2$ in Case I and $|c_i'' - c_i|_\omega < \delta$ in Case II, for $i = 1, \ldots, r$. It follows that $(\boldsymbol{\sigma}) \in \mathcal{G}(K_s/L)^e - S(L, f, (\mathbf{c}, d), \epsilon)$.

Now, there are only countably many summands on the right hand side of (6), since K is denumerable. Each one of them is, by (4), of measure zero. Hence the right hand side of (6) has measure zero and hence, by (5), $\mu(S) = 1$.

Let K now be a global field, i.e. a number field or a function field of one variable over a finite field. Then K is denumerable, hilbertian (c.f. Lang [**7**, p. 15]) and it has only countably many absolute values (c.f. Cassels and Frohlich [**1**, pp. 45, 46]). The intersection of countably many subsets of $\mathcal{G}(K_s/K)^e$ of measure 1 is again a set of measure 1. Theorem 5.4 therefore implies:

THEOREM 5.5. *If K is a global field, then for almost all $(\boldsymbol{\sigma}) \in \mathcal{G}(K_s/K)^e$, the field $K_s(\boldsymbol{\sigma})$ has the density property with respect to every absolute value of $\tilde{K}$.*

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University of California at Irvine,
Irvine, California;
Tel-Aviv University,
Ramat-Aviv, Tel-Aviv, Israel