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ON A PROBLEM OF KLEE

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Let E be a Hausdorff topological vector space. A subset A of E is a *polytope* iff A is the convex hull of a finite number of points. In this note a necessary condition for every maximal convex subset of a subset B of E to be a polytope is given. This is related to a problem first posed by Klee [1] for compact three-cells in Euclidean 3 space.

If A is a convex subset of E, then a point x in A is an *extreme point* of A iff $A\sim\{x\}$ is convex. Let B be any subset of E. A point x in B is a *local extreme point* of B iff there exists an open neighborhood V of x such that $(V\cap B)\sim\{x\}$ is convex. The local extreme points of a set B are denoted by $1\times B$.

LEMMA. *Let A be a convex subset of a set B contained in E. If $x\in 1\times B$ and $x\in A$, then x is an extreme point of A.*

Proof. Let V be an open neighborhood of x such that $(V\cap B)\sim\{x\}$ is convex. Then $(A\cap V)\sim\{x\}=A\cap((V\cap B)\sim\{x\})$ is convex. If x is not an extreme point of A, then there are points a and b of $A\sim\{x\}$ with $x\in[a, b]$, the closed line segment from a to b. Since V is open and contains x, $V\cap[a, b]$ contains two distinct points a' and b' with $x=\frac{1}{2}(a'+b')$. (For a proof of this see [2].) But a' and b' belong to $A\cap V$ and are distinct from x, so $(A\cap V)\sim\{x\}$ cannot be convex. This contradiction forces x to be an extreme point of A.

In particular, the lemma implies that the local extreme points of a convex set coincide with the extreme points.

THEOREM. *Let B be a subset of E. If every maximal convex subset of B is a polytope, then no point of $1\times B$ is a limit point of $1\times B$.*

Proof. Suppose that there exists an x in $1\times B$ such that x is a limit point of $1\times B$. Select an open neighborhood V of x such that $(V\cap B)\sim\{x\}$ is convex. Since E is Hausdorff and x is a limit point of $1\times B$, $V\cap 1\times B$ is infinite. Since $(V\cap B)\sim\{x\}$ is convex, a standard Zorn's lemma argument proves that there is a maximal convex subset M of B containing $(V\cap B)\sim\{x\}$. Thus $M\cap 1\times B$ is also infinite, so the lemma implies that M has infinitely many extreme points. Since a polytope has only a finite number of extreme points, M is not a polytope, a contradiction.

COROLLARY. *Suppose E is strongly Lindelöf (i.e. every open subspace has the Lindelöf property). If B is a subset of E and every maximal convex subset of B is a polytope, then $1\times B$ is at most countable.*

Proof. By the theorem, no point of $1\times B$ is a limit point of $1\times B$. Thus for each

x in $1 \times B$, there is an open set V_x such that $V_x \cap 1 \times B = \{x\}$. Since every open subset of E is Lindelöf, there is a countable subfamily $\{V_i\}_{i=1}^{\infty}$ of $\{V_x \mid x \in 1 \times B\}$ such that $\bigcup_{i=1}^{\infty} V_i = \bigcup \{V_x \mid x \in 1 \times B\} \supseteq 1 \times B$. Thus since each V_i contains exactly one point of $1 \times B$, $1 \times B$ is countable.

An example may be constructed in the plane to show that the converse of the corollary is false.

References

1. V. L. Klee, *Some characterizations of convex polyhedra*, Acta Math. **102** (1959), 79–107.
2. F. A. Valentine, *Convex sets*, McGraw-Hill, New York (1964), 6–7.

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