A DUAL CHARACTERISATION OF THE EXISTENCE OF SMALL COMBINATIONS OF SLICES

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We characterise, by a property of roughness, the norms of a Banach space $X$ such that the dual unit ball has no small combination of $w^*$-slices. Among separable Banach spaces, the existence of an equivalent norm for this new property of roughness characterises spaces which contain an isomorphic copy of $c_0(N)$.

1 INTRODUCTION

Throughout this paper, $X$ denotes a Banach space, $B(X)$ its unit ball, $S(X)$ its unit sphere, $B(X^*)$ the unit ball of its dual and $S(X^*)$ the unit sphere of its dual. Let us first recall some basic definitions and introduce the notion of "average rough norm":

Definitions.

1. Let $C$ be a closed convex subset of $X$. We say that $C$ is $\varepsilon$-dentable (respectively $\varepsilon$-$w^*$-dentable if $X$ is a dual space) if there exists a slice $S$ (respectively $w^*$-slice $S$) with $\text{diam}(S) < \varepsilon$ ($\text{diam}(S)$ denotes the diameter of $S$).

We say that $C$ contains an $\varepsilon$-combination of slices (respectively an $\varepsilon$-combination of $w^*$-slices) if there are slices (respectively $w^*$-slices) $S_1, \ldots, S_n$ of $C$ with $\text{diam}\left(\frac{1}{n}(S_1 + \cdots + S_n)\right) < \varepsilon$.

2. A one-sided Gâteaux differential of the norm $\| \| \| X \| x \| X$ at $x \in X$ is a function $d^+\| x \| : X \to \mathbb{R}$ such that, for all $u \in X$, $d^+\| x \| (u) = \lim_{t \to 0^+} \frac{\| x + tu \| - \| x \|}{t}$.

3. We say that the norm of $X$, or merely $X$ where there is no ambiguity, is $\varepsilon$-rough if for all $x \in S(X)$ and for all $\eta > 0$ there exist $y, z \in S(X)$ and $u \in S(X)$ such that

(a) $\| y - x \| < \eta$ and $\| z - x \| < \eta$
(b) $(d^+\| y \| - d^+\| z \|)(u) > \varepsilon - \eta$.

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4. We say that the norm of \( X \), or merely \( X \) when there is no ambiguity, is \( \epsilon \)-average rough if, for all \( x_1, \ldots, x_n \in S(X) \) and for all \( \eta > 0 \), there exist \( y_1, \ldots, y_n, z_1, \ldots, z_n \in S(X) \) and \( u \in S(X) \) such that:
   
   \[ (a) \quad \text{for all } i, 1 \leq i \leq n, \quad \|y_i - x_i\| < \eta \quad \text{and} \quad \|z_i - z_i\| < \eta \]
   
   \[ (b) \quad \frac{1}{n} \sum_{i=1}^{n} \left( d^+\|y_i\| - d^+\|z_i\|(u) \right) > \epsilon - \eta. \]

   We refer the reader to [2] and [7] for a study of the small combination of slices property.

   An obvious observation is that, if \( X \) is \( \epsilon \)-average rough, then \( X \) is \( \epsilon \)-rough. More precisely, in the definition of \( \epsilon \)-average roughness, for any \( n \in \mathbb{N} \) and any \( x_1, \ldots, x_n \in S(X) \), there exists a common direction of roughness \( u \) for many of the \( x_i, 1 \leq i \leq n \) (but not necessarily for all of them).

   In [6], Leach and Whitfield introduced and studied rough norms. In [5], John and Zizler have shown that \( X \) is \( \epsilon \)-rough if and only if \( B(X^*) \) is not \( \epsilon \)-\( w^* \)-dentable. Moreover, it is shown, for instance in [3], that for separable Banach spaces, the existence of an equivalent rough norm characterises the separable Banach spaces with non-separable dual.

   We shall show that \( X \) is \( \epsilon \)-average rough if and only if \( B(X^*) \) does not contain any \( \epsilon \)-combination of \( w^* \)-slices. Moreover, for separable Banach spaces, the existence of an equivalent \( \epsilon \)-average rough norm, for some \( \epsilon > 0 \), characterises the spaces which contain \( \ell_1^1(\mathbb{N}) \).

2 CHARACTERISATION OF AVERAGE ROUGH NORMS.

THEOREM 1. Let \( 0 < \epsilon < 1 \) and let \( X \) be a Banach space. The following conditions are equivalent:

1. \( X \) is \( \epsilon \)-average rough;
2. For each \( x_1, \ldots, x_n \in S(X) \),
   \[ \limsup_{\|y\| \to 0} \frac{1}{n} \sum_{i=1}^{n} \left( \frac{\|x_i + y\| + \|x_i - y\| - 2}{\|y\|} \right) \geq \epsilon; \]
3. \( B(X^*) \) does not contain any \( \epsilon \)-combination of \( w^* \)-slices.

Dually, an analogous result holds:

THEOREM 2. The following conditions are equivalent:

1. \( X^* \) is \( \epsilon \)-average rough;
2. For each \( x_1^*, \ldots, x_n^* \in S(X^*) \),
   \[ \limsup_{y^* \in X^*, \|y^*\| \to 0} \frac{1}{n} \sum_{i=1}^{n} \left( \frac{\|x_i^* + y^*\| + \|x_i^* - y^*\| - 2}{\|y^*\|} \right) \geq \epsilon; \]
3. \( B(X) \) does not contain any \( \epsilon \)-combination of slices.

\textbf{Remark:} These two results are isometric, and give dual characterisation of the existence of small combinations of slices \( B(X) \) (respectively small combinations of \( w^* \)-slices of \( B(X^*) \)).

\textbf{Proof of Theorem 1:} Some of the arguments are refinements of ideas in [5] and [8].

(1) \( \Rightarrow \) (3): Suppose \( X \) is \( \epsilon \)-average rough and let \( S_1, \ldots, S_n \) be \( w^* \)-slices of \( B(X^*) \). Replacing \( S_1, \ldots, S_n \) by smaller \( w^* \)-slices, we can assume that for all \( i \in \{1, \ldots, n\} \), \( S_i = S_i(x_i, B(X^*), \eta) = \{ f \in B(X^*) ; f(x_i) > 1 - \eta \} \), with \( x_i \in S(X) \) and \( \eta > 0 \).

By hypothesis there exists \( y_1, \ldots, y_n, z_1, \ldots, z_n \in S(X) \) and \( u \in S(X) \) satisfying:

\begin{itemize}
  \item[a.] for all \( i \), \( \| y_i - x_i \| < \eta \) and \( \| z_i - x_i \| < \eta \);
  \item[b.] \( \frac{1}{n} \sum_{i=1}^{n} (d^+ \| y_i \| - d^+ \| z_i \|)(u) > \epsilon - \eta \).
\end{itemize}

Using the Ascoli-Mazur theorem ([4]), choose \( f_i, g_i \in S(X^*) \) satisfying \( f_i(y_i) = 1 \), \( f_i(u) = d^+ \| y_i \|(u) \), \( g_i(z) = 1 \) and \( g_i(u) = d^+ \| z_i \|(u) \). Condition (b) implies that \( \frac{1}{n} \sum_{i=1}^{n} (f_i - g_i)(u) > \epsilon - \eta \) hence \( \| \frac{1}{n} \sum_{i=1}^{n} (f_i - g_i) \| > \epsilon - \eta \). On the other hand, \( f_i(x_i) \geq f_i(y_i) - \| x_i - y_i \| > 1 - \eta \) and so \( f_i \in S_i \); an analogous calculation shows that \( g_i \in S_i \) and we have shown that \( \text{diam} \left( \frac{1}{n} \left( \sum_{i=1}^{n} S_i \right) \right) > \epsilon - \eta \).

Since if we replace \( \eta \) by \( \eta' \in (0, \eta) \), then the \( S_i \) are replaced by \( S_i' \subset S_i \) and

\[
\text{diam} \left( \frac{1}{n} \left( \sum_{i=1}^{n} S_i \right) \right) \geq \text{diam} \left( \frac{1}{n} \left( \sum_{i=1}^{n} S_i' \right) \right) > \epsilon - \eta'
\]

this shows that \( \text{diam} \left( \frac{1}{n} \left( \sum_{i=1}^{n} S_i \right) \right) \geq \epsilon \).

(3) \( \Rightarrow \) (2): Let \( x_1, \ldots, x_n \in S(X) \), let \( \lambda \) and \( \alpha \) be two non-negative real numbers, and let \( \delta \in (0, \alpha \lambda \epsilon) \) be fixed. By hypothesis, \( B(X^*) \) does not contain any \( \epsilon \)-combination of \( w^* \)-slices, so there exist \( f_1, \ldots, f_n, g_1, \ldots, g_n \) so that

\begin{itemize}
  \item[a.] for all \( i \), \( f_i \in S_i \) and \( g_i \in S_i \), where \( S_i = S(x_i, B(X^*), \delta) \);
  \item[b.] \( \left\| \frac{1}{n} \sum_{i=1}^{n} f_i - \frac{1}{n} \sum_{i=1}^{n} g_i \right\| > \epsilon (1 - \alpha) \).
\end{itemize}
So there exists \( u \in S(X) \) such that \( \frac{1}{n} \sum_{i=1}^{n} (f_i - g_i)(u) > \varepsilon(1 - \alpha) \), hence

\[
\frac{1}{n} \sum_{i=1}^{n} (\|x_i + \lambda u\| + \|x_i - \lambda u\|) \geq \frac{1}{n} \sum_{i=1}^{n} (f_i(x_i + \lambda u) + g_i(x_i - \lambda u))
\]

\[
\geq \frac{1}{n} \sum_{i=1}^{n} f_i(x_i) + \frac{1}{n} \sum_{i=1}^{n} g_i(x_i) + \lambda \left( \frac{1}{n} \sum_{i=1}^{n} (f_i - g_i)(u) \right)
\]

\[
\geq 1 - \delta + 1 - \delta + \lambda \varepsilon(1 - \alpha)
\]

\[
\geq 2 - 2\lambda \varepsilon + \lambda \varepsilon(1 - \alpha)
\]

\[
\geq 2 + \lambda \varepsilon(1 - 3\alpha).
\]

So we have:

\[
\frac{1}{n} \sum_{i=1}^{n} \left( \frac{\|x_i + \lambda u\| + \|x_i - \lambda u\| - 2}{\lambda} \right) \geq \varepsilon(1 - 3\alpha).
\]

This shows that, for every \( \lambda > 0 \):

\[
\sup_{y \in X, \|y\| = \lambda} \frac{1}{n} \sum_{i=1}^{n} \left( \frac{\|x_i + y\| + \|x_i - y\| - 2}{\|y\|} \right) \geq \varepsilon
\]

whence

\[
\limsup_{y \in X, \|y\| \to 0} \frac{1}{n} \sum_{i=1}^{n} \left( \frac{\|x_i + y\| + \|x_i - y\| - 2}{\|y\|} \right) \geq \varepsilon.
\]

(2) \( \Rightarrow \) (1): Let \( x_1, \ldots, x_n \in S(X) \) and \( \eta \in (0,1) \). By hypothesis, there exists \( u \in S(X) \) and \( t \in (0, \eta/3) \) such that

\[
\frac{1}{n} \sum_{i=1}^{n} \left( \frac{\|x_i + tu\| + \|x_i - tu\| - 2}{t} \right) \geq \varepsilon - \frac{\eta}{3}
\]

Since the real functions \( \varphi_i : t \to \|x_i + tu\| \) are convex, if \( \varphi_i' \) denotes the right derivative of \( \varphi_i \), we have, for \( t > 0 \):

\[
\varphi_i'(t) \geq \frac{1}{t} (\varphi_i(t) - \varphi_i(0)) \quad \text{and} \quad \varphi_i'(-t) \leq \frac{1}{t} (\varphi_i(0) - \varphi_i(-t)).
\]

So

\[
d^+ \|x_i + tu\|(u) \geq \frac{1}{t} (\|x_i + tu\| - 1)
\]

and

\[
-d^+ \|x_i - tu\|(u) \geq \frac{1}{t} (\|x_i - tu\| - 1).
\]
Summing these inequalities for $1 \leq i \leq n$:

$$
\sum_{i=1}^{n} \left( d^{+} \| x_{i} + tu \| - d^{+} \| x_{i} - tu \| \right)(u) \leq \frac{\sum_{i=1}^{n} \| x_{i} + tu \| + \| x_{i} - tu \| - 2}{t}.
$$

Therefore, using (1):

$$
\frac{1}{n} \sum_{i=1}^{n} \left( d^{+} \| x_{i} + tu \| - d^{+} \| x_{i} - tu \| \right)(u) \geq \varepsilon - \frac{\eta}{3}.
$$

Putting $y_{i} = \frac{x_{i} + tu}{\| x_{i} + tu \|}$, $z_{i} = \frac{x_{i} - tu}{\| x_{i} - tu \|}$ we have that

(a) For all $i$, $y_{i}, z_{i} \in S(X)$.

(b) For all $i$, $\| x_{i} - y_{i} \| \leq \left( \left\| x_{i} + tu \| - 1 \right\| + \left\| tu \right\| \right) \leq \frac{2t}{\| x_{i} + tu \|} \leq 3t$ so $\| x_{i} - y_{i} \| < \eta$ and similarly $\| x_{i} - z_{i} \| < \eta$.

(c) $\frac{1}{n} \sum_{i=1}^{n} \left( d^{+} \| y_{i} \| - d^{+} \| z_{i} \| \right)(u) \geq \varepsilon - \eta$.

Let us check condition (c). Indeed, for each $i$,

$$
\| x_{i} + tu \| d^{+} \| y_{i} \| (u) = d^{+} \| x_{i} + tu \| (u)
$$

and

$$
\| x_{i} - tu \| d^{+} \| z_{i} \| (u) = d^{+} \| x_{i} - tu \| (u)
$$

therefore

$$
\begin{align*}
d^{+} \| y_{i} \| (u) - d^{+} \| z_{i} \| (u) &\geq d^{+} \| x_{i} + tu \| (u) - d^{+} \| x_{i} - tu \| (u) \\
&\quad - \left( \| x_{i} + tu \| - 1 \right) d^{+} \| x_{i} + tu \| (u) \\
&\quad - \left( \| x_{i} - tu \| - 1 \right) d^{+} \| x_{i} - tu \| (u) \\
&\geq d^{+} \| x_{i} + tu \| (u) - d^{+} \| x_{i} - tu \| (u) - 2t.
\end{align*}
$$

Condition (c) is obtained by summing these inequalities and applying (2). This shows that the norm of $X$ is $\varepsilon$-average rough and completes the proof of Theorem 1. □

The proof of Theorem 2 is similar and left to the interested reader. Note that in the proof of Theorem 2, (1) ⇒ (3), it is enough to choose $f_{i}, g_{i} \in S(X)$ satisfying $f_{i}(y_{i}) > 1 - (\eta - \| x_{i} - y_{i} \|)$, $g_{i}(z_{i}) > 1 - (\eta - \| z_{i} - x_{i} \|)$ and analogous conditions for $f_{i}(u)$ and $g_{i}(u)$, and to apply the local reflexivity principle ([10, Theorem 3.1, p.33]).
3 Examples and Applications

In [3], Godefroy and Maurey define a norm \( \| \cdot \| \) on \( X \) to be everywhere octahedral if, for every finite dimensional subspace \( Y \) of \( X \) and every \( \varepsilon > 0 \), there is an \( x \in X \setminus \{0\} \) (depending on \( Y \) and \( \varepsilon \)) such that for all \( t \in Y \), \( \| t + x \| \geq (1 - \varepsilon)(\| t \| + \| x \|) \).

**Example:** The usual norm in \( \ell_1(N) \) is everywhere octahedral. Indeed, let \( Y \) be a finite dimensional subspace of \( \ell_1(N) \) and \( \varepsilon > 0 \). By compactness of \( S(Y) \), we can find an \( n \in \mathbb{N} \) such that, if \( Z \) is the subspace of \( \ell_1(N) \) whose elements are supported by the \( n \) first coordinates, then for all \( y \in S(Y) \),

\[
d(y, Z) = \inf\{\| y - z \|; z \in Z \} < \varepsilon/2.
\]

Let \( x \in \ell^1(N) \setminus \{0\} \) have its first \( n \) coordinates equal to 0. If \( y \in Y \), choose \( z \in Z \) such that \( \| y - z \| < \varepsilon \| y \| \), then

\[
\| y + x \| \geq \| x \| - \| y - z \| \\
\geq \| x \| - \| y \| + \| z \| - \| y - z \| \\
\geq \| y \| + \| z \| - 2\| y - z \| \\
\geq (1 - \varepsilon)(\| y \| + \| z \|)
\]

as required.

**Proposition 3.** An everywhere octahedral norm is 2-average rough.

**Proof:** Observe that by homogeneity, a norm on \( X \) is everywhere octahedral if and only if, for every finite dimensional subspace \( Y \) of \( X \) and every \( \varepsilon > 0 \), there is an \( x \in X \), \( \| x \| = 1 \) such that, for all \( t \in Y \) and \( \alpha \in \mathbb{R} \), \( \| t + \alpha x \| \geq (1 - \varepsilon)(\| t \| + |\alpha|) \).

Let \( \varepsilon > 0 \) and let \( x_1, \ldots, x_n \in X \) be of norm 1. Denote by \( Y \) the linear space spanned by \( x_1, \ldots, x_n \). By the previous remark, there is an \( x \in S(X) \) such that for all \( y \in Y \),

\[
\| t + \sqrt{\varepsilon} x \| \geq (1 - \varepsilon)(\| t \| + \sqrt{\varepsilon})
\]

and

\[
\| t - \sqrt{\varepsilon} x \| \geq (1 - \varepsilon)(\| t \| + \sqrt{\varepsilon}).
\]

So

\[
\frac{\| t + \sqrt{\varepsilon} x \| + \| t - \sqrt{\varepsilon} x \| - 2\| t \|}{\sqrt{\varepsilon}} \geq 2(1 - \varepsilon - \sqrt{\varepsilon}\| t \|).
\]

Applying this inequality successively for \( t = x_i \), \( 1 \leq i \leq n \), and summing:

\[
\frac{1}{n} \sum_{i=1}^{n} \frac{\| x_i + h_\varepsilon \| + \| x_i - h_\varepsilon \| - 2}{\| h_\varepsilon \|} \geq 2(1 - \varepsilon - \sqrt{\varepsilon}),
\]

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where \( h_\varepsilon = \sqrt{\varepsilon} z \). This inequality holds for arbitrary \( \varepsilon > 0 \), so the proposition follows from Theorem 2(2).

REMARKS: (a) The usual norm in \( \ell_1(N) \) is everywhere octahedral, hence, by Proposition 3, it is 2-average rough.

(b) Remark (a) is false if we replace the usual norm of \( \ell^1(N) \) by an equivalent norm. Indeed, there exists on \( c_0(N) \) an equivalent locally uniformly rotund norm, hence its unit ball is dentable and the dual norm in \( \ell^1(N) \) is not even rough.

(c) The author does not know if the converse of Proposition 3 holds.

PROPOSITION 4. Let \( X \) be a separable Banach space. The following are equivalent:

1. \( X \) has an equivalent \( \varepsilon \)-average rough norm for some \( \varepsilon > 0 \);
2. \( X \) has an equivalent 2-average rough norm;
3. \( X \) contains \( \ell^1(N) \).

PROOF: (1) \( \Rightarrow \) (3): If there exists an \( \varepsilon \)-average norm on \( X \), then, by Theorem 1, \( B(X^*) \) does not contain any \( \varepsilon \)-combination of \( w^* \)-slices, and so, by a result of Bourgain ([1, lemme 3-7]), \( X \) contains \( \ell^1(N) \).

(3) \( \Rightarrow \) (2): If \( X \) is separable and contains \( \ell^1(N) \), then Godefroy and Maurey ([3, corollaire III.13]) showed that there exists on \( X \) an everywhere octahedral norm and by Proposition 3, this norm is 2-average rough.

(2) \( \Rightarrow \) (1) is obvious.

PROPOSITION 5. If a Banach space \( X \) is \( \varepsilon \)-average rough, then there is a separable closed subspace \( Y \subset X \) such that \( Y \), with the restricted norm of \( X \), is also average rough.

PROOF: Let \( Y_0 \) be a one-dimensional subspace of \( X \) and let us define a sequence of finite dimensional subspaces of \( X \) in the following way: suppose \( Y_0, Y_1, \ldots, Y_p \) have been defined and choose a \( \frac{1}{p} \)-net \( A_p = \{ x_1, x_2, \ldots, x_{k_p} \} \) in \( S(Y_p) \) and for each subset \( A \) of \( A_p \), if we denote \( A = \{ t_1, \ldots, t_n \} \), choose \( y^A_1, \ldots, y^A_n, z^A_1, \ldots, z^A_n \) and \( u^A \in S(X) \) satisfying:

- \( \text{a. for all } i \in \{ 1, \ldots, n \}, \quad \| y^A_i - t_i \| < \frac{1}{p} \text{ and } \| z^A_i - t_i \| < \frac{1}{p} ; \)
- \( \text{b. } \frac{1}{n} \sum_{i=1}^{n} (d^+\| y^A_i \| - d^+\| z^A_i \| ) (u^A) > \varepsilon - \frac{1}{p} . \)

Let \( Y_{p+1} \) be the subspace of \( X \) spanned by \( Y_p \) and all the \( y_i^A, z_i^A, u^A \) (for all \( A \subset A_p \)). Let \( Y = \bigcup_{p \in \mathbb{N}} Y_p \). Then \( Y \) is a separable subspace of \( X \) and \( Y \) is \( \varepsilon \)-average rough.
COROLLARY 6. Let $X$ be a Banach space (not necessarily separable) which is $\varepsilon$-average rough for some $\varepsilon > 0$. Then $X$ contains $\ell^1(N)$.

PROOF: By Proposition 5, there exists a separable subspace $Y$ of $X$ which is $\varepsilon$-average rough. By Proposition 4, $Y$ contains $\ell^1(N)$ and the corollary follows.

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