WEAK AND NORM SEQUENTIAL CONVERGENCE IN $M(S)$

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Let S be a compact Hausdorff space; let $C(S)$ be the algebra of all continuous complex valued functions on S; and let $M(S)$ be the dual space of (S) (the space of all regular Borel measures on S). In [2] Grothendieck gave a description of weak sequential convergence in $M(S)$ in terms of uniform convergence on sequences of disjoint open sets in S. In this note we give a condition on the carriers of measures to guarantee that weak zero convergent sequences are norm zero convergent. While this condition is interesting in its own right, it can also be used to obtain immediately some well-known results about compact operators from $C(S)$ to c_0.

We begin with a stronger version of Grothendieck's theorem which follows: (For a measure μ in $M(S)$, $|\mu|$ denotes its total variation.)

THEOREM 1. *Let $\{\mu_n\}$ be a sequence in $M(S)$. Then the following statements are equivalent:*

i) *$\{\mu_n\}$ is weakly sequentially compact.*

ii) *$\{|\mu_n|\}$ is weakly sequentially compact.*

iii) *For each sequence $\{E_m\}$ of pairwise disjoint open sets in S,*

$$\mu_n(E_m) \to 0 \text{ uniformly in } n.$$

iv) *For each sequence $\{F_m\}$ of pairwise disjoint Borel sets in S,*

$$|\mu_n|(F_m) \to 0 \text{ uniformly in } n.$$

v) *$\exists$ a constant K such that $\|\mu_n\| \leqq K$ for all n and*

$$\lim_{\mu(E)\to 0} |\mu_n|(E) = 0 \text{ uniformly in } n, \text{ where } \mu = \sum_{n=1}^{\infty} \frac{1}{2^n} |\mu_n|.$$

PROOF. The equivalence of i) and iii) is, as mentioned above, due to Grothendieck (see [2], Theorem 2, pg. 146). The implications i) $\Rightarrow$ v) and v) $\Rightarrow$ ii) are immediate consequences of well-known results applied to the space $L_1(\mu)$, where

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$$\mu = \sum_{n=1}^{\infty} \frac{1}{2^n} |\mu_n|.$$

(See [1], Corollaries 10 and 11, pp. 293–294). That v) $\Rightarrow$ iv) is clear since μ, as a finite measure, satisfies

$$\infty > \mu\left(\bigcup_{n=1}^{\infty} F_n\right) = \sum_{n=1}^{\infty} \mu(F_n).$$

Hence $\mu(F_n) \to 0$ and the convergence condition in v) yields the desired result. All that remains to show (see the diagram) is iv) $\Rightarrow$ iii) which is obvious.

$$\begin{array}{ccccc} \text{i)} & & & & \text{ii)} \\ & \searrow & & \swarrow\!\nearrow & \\ \Updownarrow & & \text{v)} & & \\ & & & \searrow & \\ \text{iii)} & & \Leftarrow & & \text{iv)} \end{array}$$

For our purposes the most important consequences of the theorem are conditions iv) and v) and the following corollary.

COROLLARY. *Let $\{\mu_n\}$ converge weakly in $M(S)$. Let $\{F_n\}$ be a family of Borel sets such that $\lim_n |\mu_n|(F_k) = 0$ for each k. If $F = \cup_{k=1}^{\infty} F_k$, then*

$$\lim |\mu_n| (F) = 0.$$

PROOF. Let $G_k = F_k \backslash (\bigcup_{i=1}^{k-1} F_i)$, $k = 2, 3, \cdots$ and let $G_1 = F_1$. Then $F = \bigcup_{k=1}^{\infty} G_k$ and $\{G_k\}$ is a sequence of pairwise disjoint Borel sets. Write $\mu = \sum 1/2^n |\mu_n|$. Then $\mu(F) = \sum_{k=1}^{\infty} \mu(G_k) < \infty$ so that, if $H_m = \bigcup_{k=m}^{\infty} G_k$ then $\mu(H_m) \to 0$. Let $\varepsilon > 0$. By condition v) of Theorem 1, $\exists\, N$ such that $|\mu_n|(H_N) < \varepsilon/2$ uniformly in n.

Since $\lim |\mu_n| (F_k) = 0$ for each K,

$$(\exists n_0)\ n \geqq n_0 \Rightarrow |\mu_n| (F_k) < \frac{\varepsilon}{2N},\ k = 1, 2, \cdots, N,$$

so that

$$n \geqq n_0 \Rightarrow |\mu_n| (F) \leqq |\mu_n|(F_1) + \cdots + |\mu_n| (F_N) + |\mu_n| (H_N) < \varepsilon$$

and hence $\lim |\mu_n|(F) = 0$.

For our main theorem we use the following notation: $\mathbb{Z}^+$ = positive integers and $\|\mu\|$ denotes the norm of $\mu \in M(S)$. By a *carrier* of μ we mean any Borel set off which $|\mu|$ is zero.

THEOREM 2. *Let $\{\mu_n\}$ be a weakly zero convergent sequence in $M(S)$. Let S_n denote a carrier of μ_n. Then the following statements are equivalent.*

i) $\|\mu_n\| \to 0$.

ii) *For each infinite set $I \subset \mathbb{Z}^+$, there is an integer N_I such that*

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$$\lim_{n\to\infty} |\mu_n| \Big(\bigcap_{\substack{i \leqq NI \\ i \in I}} S_i \Big) = 0$$

iii) *For each infinite set $I \subset \mathbb{Z}^+$, there is an integer N_I such that*

$$\lim_{n\in I} |\mu_n| \Big(\bigcap_{\substack{i \leqq N_I \\ i \in I}} S_i \Big) = 0$$

PROOF. That i) $\Rightarrow$ ii) $\Rightarrow$ iii) is obvious. We show iii) $\Rightarrow$ ii) $\Rightarrow$ i).

Proof of ii) $\Rightarrow$ i): Let $R = \{\text{functions } \phi\colon \mathbb{Z}^+ \to \{0,1\}\}$. Let

$$T_\phi(i) = \begin{cases} S_i & \text{if } \phi(i) = 0 \\ S \backslash S_i & \text{if } \phi(i) = 1. \end{cases}$$

Then

$$S = \bigcup_{\phi\in R} \bigcap_{i\in \mathbb{Z}^+} T_\phi(i).$$

Let $R_0 = \{\phi \in R\colon \phi(i) = 0 \text{ infinitely often}\}$.

Then

$$S = \Big[\bigcup_{\phi\in R_0} \bigcap_{i\in\mathbb{Z}} T_\phi(i) \Big] \cup \Big[\bigcup_{\phi\in R\backslash R_0} \bigcap_{i\in \mathbb{Z}^+} T_\phi(i) \Big] \equiv A_0 \cup A_1.$$

Now if $\phi \in R_0$ then $T_\phi(i) = S_i$ infinitely often.
Let $I = \{i\colon \phi(i) = 0\}$. By the condition ii)

$$\exists N_I\colon \lim_{n\to\infty} |\mu_n| \Big(\bigcap_{\substack{i \leqq N_I \\ i \in I}} S_i \Big) = 0.$$

Also

$$\bigcap_{i\in\mathbb{Z}^+} T_\phi(i) \subset \bigcap_{\substack{i \leqq N_I \\ i \in I}} S_i.$$

Hence for each $\phi \in R_0$ there corresponds an integer N_I such that the above inclusion is valid. Since there are at most countably many such sets of the form $\bigcap_{i\leqq N_I / i\in I} S_i$, if we enumerate them $T_1, T_2, \cdots$, then $A_0 \subset \bigcup_{n=1}^\infty T_n$ and $\lim_n |\mu_n| (T_k) = 0$ for each k. Also, $R\backslash R_0$ contains only countably many ϕ, say $\{\phi_n\}_{n=1}^\infty$. Let $U_n = \bigcap_{i\in\mathbb{Z}^+} T_{\phi_n}(i)$. Since $\phi_n \notin R_0$, $\phi_n(i) = 0$ only finitely often, i.e. $T_{\phi_n}(i) = S \backslash S_i$ for all i sufficiently large. Thus $|\mu_n| (U_k) = 0$ for all sufficiently large n and $A_1 = \bigcup_{k=1}^\infty U_k$ with $\lim_n |\mu_n| (U_k) = 0$ for each k. By the Corollary to Theorem 1, we have

$$\lim |\mu_n|(S) = \lim |\mu_n| (A_0) + \lim |\mu_n| (A_1) = 0.$$

But $|\mu_n| (S) = \|\mu_n\|$ and so $\|\mu_n\| \to 0$.

Proof of iii) ⇒ ii). Suppose by way contradiction that iii) holds, but for some infinite set $I \subset \mathbb{Z}^+$

$$\lim_n \sup |\mu_n| \left(\bigcap_{\substack{i \leqq N \\ i \in I}} S_i \right) > 0, \text{ for each positive integer } N.$$

In particular, $\exists \varepsilon_1 > 0$ such that

$$\lim_n \sup |\mu_n| \left(\bigcap_{\substack{i \leqq N_I \\ i \in I}} S_i \right) > \varepsilon_1,$$

where N_I is the integer given by iii). Choose J_1 to be an infinite subset of $\mathbb{Z}^+$ such that $n \in J_1 \Rightarrow |\mu_n| (S_0) > \varepsilon_1$, where

$$S_0 = \bigcap_{\substack{i \leqq N_I \\ i \in I}} S_i.$$

We consider two cases.

Case 1. Suppose that for each $m \in J_1$, $\lim_{n \in J_1} |\mu_n| (S_0 \cap S_m) = 0$. Fix $n_1 \in J_1$. Then $\exists\ m_2$ such that

$$n \geqq m_2,\ n \in J_1 \Rightarrow |\mu_n| (S_0 \backslash S_2) > \varepsilon_1.$$

Let $n_2 \in J_1$, $n_2 > \max\{m_2, n_1\}$. Then since $\lim_{n \in J_1} |\mu_n| (S_0 \cap S_{n_2}) = 0$ there is m_3 such that

$$n \geqq m_3,\ n \in J_1, \Rightarrow |\mu_n| [S_0 \backslash (S_{n_1} \cap S_{n_2})] > \varepsilon_1.$$

Inductively we can define a sequence $\{n_k\} \subset J_1$ satisfying

$$|\mu_{n_{k+1}}| \left[S_0 / \bigcup_1^k S_{n_k} \right] > \varepsilon_1,\ k = 1, 2, \cdots.$$

But if we consider the Borel sets

$$E_1 = (S_0 \cap S_{n_1}),$$

$$E_2 = (S_0 \cap S_{n_2}) \backslash S_{n_1}, \cdots, E_k = (S_0 \cap S_{n_k}) \backslash (S_{n_1} \cup \cdots \cup S_{n_{k-1}}),$$

then $|\mu_{n_k}| (E_k) > \varepsilon_1$ and the $\{E_k\}$ are pairwise disjoint Borel sets. By Theorem 1, the sequence $\{\mu_{n_k}\}$ cannot converge weakly to zero. Hence Case 1 cannot obtain.

Case 2. For some $n_1 \in J_1$, $\lim_{n \in J_1} \sup |\mu_n| (S_0 \cap S_{n_1}) > 0$.

Choose J_2 to be infinite subset of J_1 and $\varepsilon_2 > 0$ such that

$$n \in J_2 \Rightarrow |\mu_n| (S_0 \cap S_{n_1}) \geqq \varepsilon_2 > 0.$$

Now, either

a) $\lim_{n \in J_2} |\mu_n| (S_0 \cap S_{n_1} \cap S_m) = 0$ for all $m \in J_2$ *or*

b) $\exists\ n_2 \in J_2$ such that $\lim_{n \in J_2} \sup |\mu_n| (S_0 \cap S_{n_2} \cap S_{n_2}) > 0$.

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But a) is merely a repetition of Case 1 above and cannot obtain. Therefore it is possible to inductively obtain sequences $\{J_k\}$, $\{\varepsilon_k\}$, and $\{n_k\}$ satisfying

$$J_1 \supset J_2 \supset \cdots \supset J_k \supset \cdots,$$

each J_k is infinite, $\varepsilon_k > 0$, $n_k \in J_k$, $n_{k+1} > n_k$ for each k, and

$$|\mu_n| \, (S_0 \cap S_{n_1} \cap \cdots \cap S_{n_k}) \geqq \varepsilon_k > 0$$

if $n \in J_{k+1}$. Let $J_0 = \{n_k\}$. Then for any positive integer M, $\exists\ \delta > 0$ such that

$$|\mu_n| \Big(\bigcap_{\substack{i \leqq M \\ i \in J_0}} S_i \Big) \geqq \delta$$

for all n sufficiently large in J_0. This contradicts iii), and we have proven iii) $\Rightarrow$ ii).

Remark. Note that in condition (iii), the limit is taken only over the integers $n \in I$ so that (iii) is actually a statement about subsequences of $\{\mu_n\}$.

Some simple examples of the theorem are the following: Let $\{\mu_n\}$ be a weakly zero convergent sequence in $M(S)$.

I. *If* $S_n \cap S_m = \varnothing$ *for* $n \neq m$, *then* $\|\mu_n\| \to 0$.

II. *If for each* m, $\lim_n |\mu_n| \, (S_m) = 0$, *then* $\|\mu_n\| \to 0$.

III. *If each* μ_n *is atomic, then* $\|\mu_n\| \to 0$.

To indicate the connection with compact operators, recall that if T is a bounded operator from $C(S)$ to c_0, then T is represented by a sequence of measures $\{\mu_n\}$ so that

i) $T(f) = (\int f d\mu_1, \int f d\mu_2, \cdots), f \in C(S)$,

ii) $\mu_n \to 0$, weak*.

Assume, in addition, that in $M(S)$ weak* sequential convergence implies weak sequential convergence. This is the case, for example, when S is extremely disconnected or a closed subset of an extremely disconnected space. Then we can reformulate I, II, and III above in terms of known results or easy consequences of known results about operators (by replacing the phrase "then $\|\mu_n\| \to 0$" with "then T is compact"). In particular, when $C(S) = m$ (where m denotes the space of bounded sequences), any matrix operator is compact. For it is easy to check that such operators correspond to measures carried on the integers.

The precise reformulation of Theorem 2 for operators becomes: *T is noncompact if and only if there is a sequence of Borel sets R_i which are carriers for a subsequence $\{\nu_i\}$ of $\{\mu_n\}$ such that for each positive integer N,*

$$\limsup |v_n| \left(\bigcap_{i=1}^{N} R_i \right) > 0.$$

This reformulation characterizes non-compact operators from $C(S)$ to c_0 and is useful for constructing such operators.

References

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[2] A. Grothendieck, 'Sur les applications linéaires faiblement compactes d'espaces du type C (K)'. *Canadian Journal of Mathematics* 5 (1953), 129–173.

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