ENGEL CONGRUENCES IN GROUPS OF PRIME-POWER EXPONENT

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1. Introduction. In this paper a simplified proof of a theorem of Sanov **(4)** is given. No mention is required of Lie elements or of the Baker–Hausdorff formula, both of which played central roles in Sanov's proof.

Let G be a group. Define, for all $x, y \in G$,

$$(y, x; 0) = y, \qquad (y, x; 1) = (y, x) = y^{-1}x^{-1}yx,$$

$$(y, x; n + 1) = ((y, x; n), x).$$

Define, as usual,

$$G_1 = G, \qquad G_{n+1} = (G_n, G) = \{(c, x)|c \in G_n, x \in G\}.$$

Sanov **(4)** proved that if G is a group of prime-power exponent, p^e, then

$$(1.1) \qquad (y_m, x; k(p^k - p^{k-1}))^{p^{e-k}} \in G_q, \qquad k = 1, 2, \ldots, e,$$

for all $y_m \in G_m$ and all $x \in G$, where

$$q = \min(pm + 1, m + k(p^k - p^{k-1}) + 1).$$

This is a non-trivial result only if

$$(1.2) \qquad m \geqslant kp^{k-1}.$$

In particular, if

$$y_m = (y, x; kp^{k-1} - 1),$$

then (1.1) becomes

$$(1.3) \qquad (y, x; kp^k - 1)^{p^{e-k}} \in G_{kp^k+1}.$$

In this paper (1.1) is proved with

$$(1.4) \qquad q = \min(pm + p - 1, m + k(p^k - p^{k-1}) + 1),$$

but even with this larger q, (1.1) is still non-trivial only if (1.2) holds.

In §2 we list some elementary results about commutators and binomial coefficients. In §3 we study the ideal in $Z[X]$ associated with the ring of endomorphisms of an abelian normal subgroup of a group of prime-power exponent. Properties of this ideal yield a commutator relation (3.29) between

Received January 25, 1965.

the subgroup and the group, which is strengthened in §4 and is shown in §5 to yield (1.1).

The authors wish to acknowledge their indebtedness to R. C. Lyndon for the material of §4, and to R. H. Bruck, whose conversations, lectures, and earlier work **(1)** provided techniques that permeate the present paper.

The first two authors wish to thank the National Science Foundation for support while writing this paper, and the third author wishes to thank Sigma Xi for a grant that made travel to Madison possible.

2. LEMMA 2.1. *Let G be a group. Then for all x, y, z ∈ G,*

$$(xy, z) = (x, z)((x, z), y)(y, z), \tag{2.1}$$

$$(x, yz) = (x, z)(x, y)((x, y), z). \tag{2.2}$$

Proof. See, for example **(2**, p. 150**)**.

Let

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

be the binomial coefficient, i.e., the coefficient of x^r in $(1 + x)^n$.

LEMMA 2.2. *Let A be an abelian normal subgroup of a group G; r, n, m be positive integers; a, $a_i \in A$; $x \in G$. Then*

$$\left(\prod_i a_i, x; r\right) = \prod_i (a_i, x; r), \tag{2.3}$$

$$(xa)^n = x^n a^n \prod_{i=1}^{n-1} (a, x; i)^{\binom{n}{i+1}}, \tag{2.4}$$

$$(a, x^m; r) = \prod_{w=r}^{mr} (a, x; w)^{E(r,m,w)}, \tag{2.5}$$

where

$$E(r, m, w) = \sum \binom{m}{s_1}\binom{m}{s_2}\cdots\binom{m}{s_r}$$

and the sum is taken over all distinct (ordered) r-tuples of positive integers $(s_1, s_2, \ldots, s_r)$ *such that* $s_1 + s_2 + \ldots + s_r = w$ *and* $s_i \leqslant m$.

Proof. To establish (2.3) use induction and (2.1). To establish (2.4) use induction and the following argument:

$$(xa)^{n+1} = (xa)^n xa = x^n a^n \prod_{i=1}^{n-1} (a, x; i)^{\binom{n}{i+1}} xa.$$

Now move the right-most x to the far left by repeated use of the identity

$$ux = xu(u, x),$$

to obtain

$$(xa)^{n+1} = x^{n+1}a^{n+1}\prod_{i=1}^{n-1}(a, x; i)^{\binom{n}{i+1}}\prod_{i=0}^{n-1}(a, x; i+1)^{\binom{n}{i+1}}$$

$$= x^{n+1}a^{n+1}\prod_{i=1}^{n}(a, x; i)^{\binom{n}{i+1}+\binom{n}{i}}.$$

Finally observe that

$$\binom{n}{i+1} + \binom{n}{i} = \binom{n+1}{i+1}.$$

One also proves (2.5) by induction; see (**3**, Lemma 4.1).

Note. If $m = p$ in (2.5), then

$$p \left| \binom{p}{s_i} \right. \qquad \text{for all } s_i < p$$

and hence $p|E(r, p, w)$ for all $w < rp$.

The following lemma is well known.

LEMMA 2.3. *If p is a prime and e, r, k are positive integers such that $e \geqslant r$, $kp^r \leqslant p^e$ and $(k, p) = 1$, then*

$$(2.6) \qquad p^{e-r} \left| \binom{p^e}{kp^r} \right. \quad but \quad p^{e-r+1} \nmid \binom{p^e}{kp^r}.$$

3. Let G be a group of exponent p^e, let A be an abelian normal subgroup, let R denote the ring of endomorphisms of A, and let x be a fixed but arbitrary element of G.

Using (2.3) one verifies that the mapping from A to A defined by

$$(3.1) \qquad a \to \prod_{i=0}^{k}(a, x; i)^{r_i} \qquad (\text{all } a \in A),$$

where k and the r_i are fixed integers, k non-negative, is an endomorphism of A. One can also verify that the mapping from $Z[X]$ to R, which sends

$$(3.2) \qquad f(X) = \sum_{i=0}^{k} r_i X^i$$

into the endomorphism, (3.1), is a homomorphism. Denote by J_x the kernel of this homomorphism; denote by J the intersection of all the J_x ($x \in G$).

If we denote by a^X the image of a under the endomorphism assigned to X, namely $a^X = (a, x)$; and if we denote by a^r the image of a under the endomorphism assigned to r, namely $a^r = a^r$; then it follows that if $f(X)$ is given by (3.2),

$$(3.3) \qquad a^{f(X)} = \prod_{i=0}^{k}(a, x; i)^{r_i}.$$

Hence J can be described as

$$J = \{f(X) \in Z[X] | a^{f(X)} = 1 \text{ for all } a \in A \text{ and all } x \in G\}.$$

For example, note that $p^e \in J$ and that by (2.4)

$$\sum_{i=1}^{p^e-1} \binom{p^e}{i+1} X^i \in J.$$

THEOREM 3.1. *Let*

(3.4) *G be a group of exponent p^e,*
(3.5) *A be an abelian normal subgroup of G,*
(3.6) *$J = \{f(X) \in Z[X] | a^{f(X)} = 1$ for all $a \in A$ and all $x \in G\}$.*

Then, for each $i = 1, 2, \ldots, e$, J contains a polynomial $f_i(X)$ given by

$$f_i(X) = \sum_{j=1}^{e} p^{e-j} X^{p^{i+j-1}-p^{i-1}} a_{ij}(X), \tag{3.7}$$

where the $a_{ij}(X)$ have constant coefficients prime to p, and $a_{ie}(X) = 1$.

Proof. Put $n = p^e$ in (2.4) to obtain

$$f_1(X) = \sum_{i=1}^{p^e-1} \binom{p^e}{i+1} X^i \in J. \tag{3.8}$$

Use Lemma 2.3 to put (3.8) into the form of (3.7) with $i = 1$. Assume now that $f_i(X) \in J$; i.e.,

$$1 = \prod_r (a, x; r)^{s_r} \tag{3.9}$$

where

$$f_i(X) = \sum_r s_r X^r.$$

In (3.9) replace x by x^p and use (2.5) with $m = p$. Then

$$1 = \prod_r (a, x^p; r)^{s_r} = \prod_w (a, x; w)^{t_w}.$$

We wish to show that

$$\sum_w t_w X^w \equiv f_{i+1}(X) \pmod J. \tag{3.10}$$

By (2.5), and the note preceding Lemma 2.3,

$$(a, x^p; r) = (a, x; pr) \prod_{w=r}^{pr-1} (a, x; w)^{pn_w}, \tag{3.11}$$

where the n_w are positive integers. Hence $\sum t_w X^w$ is $f_i(X)$ with X^r replaced by

$$X^{pr} + pX^r g_r(X), \qquad r \geqslant 1; \tag{3.12}$$

here

$$g_r(X) = \sum_{w=r}^{pr-1} n_w X^{w-r} \in Z[X].$$

Making this replacement in each term of (3.7), we obtain

$$\sum t_w X^w = \sum_{j=1}^{e} p^{e-j}[X^{p^{i+j}-p^i} + pX^{p^{i+j-1}-p^{i-1}}h_j(X)]a'_{ij}(X), \tag{3.13}$$

where $h_j(X) = g_{p^{i+j-1}-p^{i-1}}(X)$, and $a'_{ij}(X)$ has the same constant coefficient as $a_{ij}(X)$. We now use the fact that $p^e \in J$ and rearrange terms:

$$\sum t_w X^w \equiv \sum_{j=1}^{e} p^{e-j}X^{p^{i+j}-p^i}a_{i+1,j}(X) = f_{i+1}(X) \quad (\text{mod } J), \tag{3.14}$$

where

$$(3.15)\ p^{e-j}X^{p^{i+j}-p^i}a_{i+1,j}(X)$$
$$= p^{e-j}[X^{p^{i+j}-p^i}a'_{ij}(X) + X^{p^{i+j}-p^{i-1}}h_{j+1}(X)a'_{i,j+1}(X)] \ (1 \leqslant j \leqslant e-1),$$

and

$$a_{i+1,e}(X) = a'_{ie}(X) = a_{ie}(X) = 1.$$

Since $a_{ij}(X)$ has constant coefficient prime to p for all $1 \leqslant j \leqslant e$, and since

$$p^{i+j} - p^{i-1} > p^{i+j} - p^i, \tag{3.16}$$

we conclude that $a_{i+1,j}(X)$ has constant coefficient prime to p for all

$$1 \leqslant j \leqslant e.$$

We now prove an elimination theorem, which yields the important formula (3.29).

THEOREM 3.2. *Let I be an ideal of $Z[X]$ with the property that if $a[X] \in Z[X]$ is invertible* (mod I), *then so is $a(X) + Xb(X)$ for all $b(X) \in Z[X]$. Suppose that I contains the polynomials*

$$(3.17)\quad \begin{aligned} f_1(X) &= A_1 a_{11} X^{m_{11}} + A_2 a_{12} X^{m_{12}} + \ldots + A_e a_{1e} X^{m_{1e}},\\ f_2(X) &= A_1 a_{21} X^{m_{21}} + A_2 a_{22} X^{m_{22}} + \ldots + A_e a_{2e} X^{m_{2e}},\\ &\ldots\\ f_e(X) &= A_1 a_{e1} X^{m_{e1}} + A_2 a_{e2} X^{m_{e2}} + \ldots + A_e a_{ee} X^{m_{ee}}, \end{aligned}$$

where the A_j and a_{ij} are in $Z[X]$, the a_{ij} are invertible (mod I), *and the m_{ij} are non-negative integers such that*

$$m_{i1} < m_{i2} < \ldots < m_{ie}, \qquad 1 \leqslant i \leqslant e, \tag{3.18}$$

$$m_{11} < m_{21} < \ldots < m_{e1}, \tag{3.19}$$

$$(m_{ij} + m_{i+1,j+t}) - (m_{i,j+t} + m_{i+1,j}) > 0, \qquad 1 \leqslant t \leqslant e-j,\ 1 \leqslant i, j, i+1 \leqslant e. \tag{3.20}$$

Then there exist $c_k(X) \in Z[X]$ such that

$$A_k X^{m_k} \equiv X^{m_k+1}c_k(X) \quad (\text{mod } I), \qquad 1 \leqslant k \leqslant e, \tag{3.21}$$

where

(3.22) $$m_k = \sum_{r+s=k+1} m_{rs} - \sum_{r+s=k} m_{rs}$$

(*with the understanding that* $m_1 = m_{11}$).

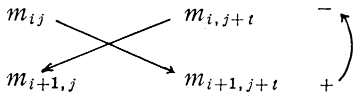
Comments. For our purposes, we shall be using Theorem 3.2 with $I = J$ as described in (3.6) and the $f_i(X)$ being the $f_i(X)$ of (3.7); i.e.,

$$A_j = p^{e-j}, \qquad a_{ij} = a_{ij}(X), \qquad m_{ij} = p^{i+j-1} - p^{i-1}.$$

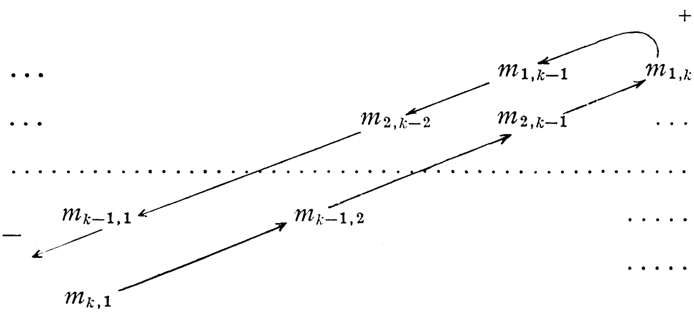
For this value of m_{ij}, one readily verifies that

(3.23) $$m_k = k(p^k - p^{k-1}).$$

Condition (3.20) can be considered as a requirement that all 2 × 2 "determinants"



be positive. The right-hand side of formula (3.22) can be thought of as the "sum" of the "kth diagonal" minus the "sum" of the "(k − 1)st diagonal" of the matrix of the m_{ij}:



Proof of Theorem 3.2. We use induction on e. If $e = 1$, then (3.17) is just

$$f_1(X) = A_1 a_{11} X^{m_{11}} \in I.$$

But a_{11} invertible (mod I) implies that

$$A_1 X^{m_{11}} \equiv 0 \pmod{I};$$

so simply let $c_1(X) = 0$ and (3.21) will be satisfied.

Now suppose the theorem is true for $e - 1$ and consider (3.17). We rewrite $f_1(X) \in I$ as

(3.24) $$A_1 a_{11} X^{m_{11}} \equiv -\sum_{i=2}^{e} A_i a_{1i} X^{m_{1i}} \pmod{I}.$$

Using (3.18) with $i = 1$ and the invertibility of a_{11}, we obtain (3.21) with $k = 1$ and

$$c_1(X) = -\sum_{i=2}^{e} A_i a_{11}^{-1} a_{1i} X^{m_{1i}-m_{11}-1}.$$

We now essentially strike out the first row and the first column from the right-hand side of (3.17). Since the $a_{ij}(X)$ are invertible (mod I) and the product of two invertible polynomials is invertible, we may assume that $a_{i1} \equiv 1$ (mod I), $1 \leqslant i \leqslant e$. (This amounts to multiplying each $f_i(X)$ by $a_{i1}^{-1}(X)$.) Let

$$(3.25)\ f_i^{(2)}(X) = X^{m_{i1}-m_{i-1,1}} f_{i-1}(X) - f_i(X) = \sum_{j=2}^{e} A_j b_{ij} X^{m_{ij}^{(2)}}, \qquad 2 \leqslant i \leqslant e,$$

where

$$b_{ij} X^{m_{ij}^{(2)}} = a_{i-1,j} X^{m_{i1}-m_{i-1,1}+m_{i-1,j}} - a_{ij} X^{m_{ij}},$$

$$(3.26) \qquad m_{ij}^{(2)} = m_{i1} - m_{i-1,1} + m_{i-1,j},$$

and hence

$$(3.27) \qquad b_{ij} = a_{i-1,j} - a_{ij} X^{m_{i-1,1}+m_{ij}-m_{i1}-m_{i-1,j}}.$$

Observe that (3.20) guarantees that the exponent of X in (3.27) is positive.

We consider now the following polynomials in I.

$$(3.28) \qquad \begin{aligned} f_2^{(2)}(X) &= A_2 b_{22} X^{m_{22}^{(2)}} + A_3 b_{23} X^{m_{23}^{(2)}} + \ldots + A_e b_{2e} X^{m_{2e}^{(2)}}, \\ f_3^{(2)}(X) &= A_2 b_{32} X^{m_{32}^{(2)}} + A_3 b_{33} X^{m_{33}^{(2)}} + \ldots + A_e b_{3e} X^{m_{3e}^{(2)}}, \\ &\ldots \\ f_e^{(2)}(X) &= A_2 b_{e2} X^{m_{e2}^{(2)}} + A_3 b_{e3} X^{m_{e3}^{(2)}} + \ldots + A_e b_{ee} X^{m_{ee}^{(2)}}. \end{aligned}$$

It is easy to verify that the b_{ij} and the $m_{ij}^{(2)}$, $2 \leqslant i \leqslant e$, satisfy all of the hypotheses of the theorem and hence, by induction, there exist $c_k(X) \in Z[X]$ such that

$$A_k X^{m_k^{(2)}} \equiv X^{m_k^{(2)}+1} c_k(X) \pmod I$$

for all $2 \leqslant k \leqslant e$, where

$$m_k^{(2)} = \sum_{r+s=k+2} m_{rs}^{(2)} - \sum_{r+s=k+1} m_{rs}^{(2)} \qquad (2 \leqslant r, s \leqslant e).$$

By (3.26), the right-hand side of the above reduces to m_k, and the proof is complete.

THEOREM 3.3. *Let A be an abelian normal subgroup of a group G of exponent p^e. Then*

$$(3.29) \qquad (a, x; k(p^k - p^{k-1}))^{p^{e-k}} \in G_m \quad \textit{for all } a \in A, x \in G,$$

where

$$m = k(p^k - p^{k-1}) + 2.$$

Proof. As indicated in the comment above equation (3.23), we wish to apply Theorem 3.2 with $I = J$ as described in (3.6) and the $f_i(X)$ of (3.17) being the $f_i(X)$ of (3.7), i.e.,

$$A_j = p^{e-j}, \qquad a_{ij} = a_{ij}(X), \qquad m_{ij} = p^{i+j-1} - p^{i-1}.$$

The only hypotheses of Theorem 3.2 that are not trivial to verify are that the $a_{ij}(X)$ are invertible, and that the invertibility of $a(X)$ implies that of $a(X) + Xb(X)$. Both of these facts will follow if we can show that if $a(X)$ has constant coefficient prime to p, then $a(X)$ is invertible (mod J).

By (3.7) with $i = 1$ (recall that $a_{ie}(X) = 1$),

$$\text{(3.30)} \qquad X^{p^e-1} \equiv ph(X) \pmod{J}, \qquad h(X) \in Z[X].$$

Raising each side to the eth power and observing (see (3.4) and (3.6)) that $p^e \in J$, we obtain

$$\text{(3.31)} \qquad X^s \in J,$$

for $s = e(p^e - 1)$. (In general, this is not the minimal s for which (3.31) is true.) Hence if

$$a(X) = a_0 + a_1 X + \ldots + a_j X^j \in Z[X], \qquad (a_0, p) = 1,$$

then one can find $a'(X)$ such that

$$a(X)a'(X) \equiv 1 \pmod{J}$$

by solving a finite collection of congruences of the form

$$a_0 z \equiv w \pmod{p^e} \qquad (w \in Z)$$

(which is possible since $(a_0, p^e) = 1$); see **(1)** for a similar argument.

4. Let G be a group of exponent p^e, let $K(y, x)$ denote the subgroup of G generated by y and x, and let $K_{m,n}(y, x)$ denote the subgroup of $K(y, x)$ generated by all (complex) commutators built up from at least m y's and at least n x's (and nothing else) **(2**, p. 138**)**. It is not difficult to show that $K_{m,n}(y, x)$ is normal in $K(y, x)$.

LEMMA 4.1. *Let G be a group of exponent p^e and let d be a positive integer divisible by $p - 1$. Let t be an arbitrary positive integer. If*

$$\text{(4.1)} \qquad (y, x; d)^t \in K_{1,d+1}(y, x)K_{2,1}(y, x) \quad \textit{for all } y, x \in G,$$

then

$$\text{(4.2)} \qquad (y, x; d)^t \in K_{1,d+1}(y, x)K_{p,p-1}(y, x)K_{2,d}(y, x) \quad \textit{for all } y, x \in G.$$

Proof. Assume inductively that

https://doi.org/10.4153/CJM-1966-056-3 Published online by Cambridge University Press

$$(y, x; d)^t \in K_{1,d+1}(y, x)K_{n,1}(y, x)K_{2,d}(y, x) \qquad (2 \leqslant n \leqslant p - 1).$$

Then

$$(4.3) \qquad (y, x; d)^t \equiv \prod k_i \pmod{K_{1,d+1}(y, x)K_{n+1,1}(y, x)K_{2,d}(y, x)},$$

where the k_i are complex commutators in x and y containing exactly n y's. Let m be a primitive root (mod p); i.e.,

$$(4.4) \qquad m^{p-1} \equiv 1 \pmod{p}, \qquad m^j \not\equiv 1 \pmod{p} \quad \text{if } j < p - 1.$$

Replace y by y^m in (4.3). Observe that repeated use of (2.1) and (2.2) yields

$$K_{r,s}(y^m, x) \leqslant K_{r,s}(y, x).$$

Further use of (2.1) and (2.2) yields

$$(4.5) \quad [(y, x; d)^t]^m \equiv (\prod k_i)^{m^n} \pmod{K_{1,d+1}(y, x)K_{n+1,1}(y, x)K_{2,d}(y, x)}.$$

Next raise both sides of (4.3) to the m^nth power to get

$$(4.6) \quad [(y, x; d)^t]^{m^n} \equiv (\prod k_i)^{m^n} \pmod{K_{1,d+1}(y, x)K_{n+1,1}(y, x)K_{2,d}(y, x)}.$$

A comparison of (4.5) and (4.6) yields

$$[(y, x; d)^t]^{m^n - m} \equiv 1 \pmod{K_{1,d+1}(y, x)K_{n+1,1}(y, x)K_{2,d}(y, x)}.$$

But, by (4.4), $m^n - m$ is prime to p and hence to p^e. This means that

$$(y, x; d)^t \equiv 1 \pmod{K_{1,d+1}(y, x)K_{n+1,1}(y, x)K_{2,d}(y, x)}.$$

By induction

$$(4.7) \qquad (y, x; d)^t \equiv 1 \pmod{K_{1,d+1}(y, x)K_{p,1}(y, x)K_{2,d}(y, x)}.$$

Next we proceed from (4.7) to the inductive assumption

$$(y, x; d)^t \equiv 1 \pmod{K_{1,d+1}(y, x)K_{p,n}(y, x)K_{2,d}(y, x)} \qquad (1 \leqslant n \leqslant p - 2).$$

Then

$$(4.8) \qquad (y, x; d)^t \equiv \prod k_i \pmod{K_{1,d+1}(y, x)K_{p,n+1}(y, x)K_{2,d}(y, x)},$$

where the k_i are complex commutators in x and y containing exactly n x's. Choose m as in (4.4) and replace x by x^m in (4.8). Repeat the previous argument to obtain

$$[(y, x; d)^t]^{m^d - m^n} \equiv 1 \pmod{K_{1,d+1}(y, x)K_{p,n+1}(y, x)K_{2,d}(y, x)}$$

and hence, using the divisibility of d by $p - 1$,

$$(y, x; d)^t \equiv 1 \pmod{K_{1,d+1}(y, x)K_{p,n+1}(y, x)K_{2,d}(y, x)}.$$

Induction now yields (4.2).

5. Given the group G of exponent p^e, we are to prove (1.1). We begin by taking arbitrary elements $y, x \in G$ and forming the subgroup $K = K(y, x)$

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generated by them. We define $K_{m,n}$ as in §4 and verify that $K_{1,0}/K_{2,1}$ is an abelian normal subgroup of $K/K_{2,1}$. Hence, by (3.29),

$$(y, x; k(p^k - p^{k-1}))^{p^{e-k}} \in K_{k(p^k-p^{k-1})+2}K_{2,1}.$$

Since K_r is generated by complex commutators in x and y of weight r and greater,

$$K_{n+2}K_{2,1} = K_{1,n+1}K_{2,1} = K_{1,n+1}K_{2,1}K_{2,n}$$

for all positive integers n. Hence

$$(y, x; k(p^k - p^{k-1}))^{p^{e-k}} \in K_{1,k(p^k-p^{k-1})+1}K_{2,1}.$$

Since $k(p^k - p^{k-1})$ is divisible by $p - 1$, we can apply Lemma 4.1 to obtain

$$(y, x; k(p^k - p^{k-1}))^{p^{e-k}} \in K_{1,k(p^k-p^{k-1})+1}K_{p,p-1}K_{2,k(p^k-p^{k-1})}.$$

In particular, if $y = y_m \in G_m$, then

$$(y_m, x; k(p^k - p^{k-1}))^{p^{e-k}} \in K_q \leqslant G_q,$$

where q is given in (1.4).

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