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Boundedness from Below of Composition Operators on α-Bloch Spaces

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Abstract. We give a necessary and sufficient condition for a composition operator on an α-Bloch space with $\alpha \geq 1$ to be bounded below. This extends a known result for the Bloch space due to P. Ghatage, J. Yan, D. Zheng, and H. Chen.

1 Introduction

Let D be the unit disk in the complex plane $\mathbb{C}$, and $H(D)$ the space of all holomorphic functions on D. For $\alpha > 0$, a function $f \in H(D)$ is called an α-*Bloch* function if

$$\|f\|_\alpha = \sup\{(1 - |z|^2)^\alpha |f'(z)| : z \in D\} < \infty.$$

For fixed α, the family of all α-Bloch functions with the norm $\|f\|_{\mathcal{B}^\alpha} = |f(0)| + \|f\|_\alpha$ forms a complex Banach space, which is called α-*Bloch space* and denoted by $\mathcal{B}^\alpha$. When $\alpha = 1$ we obtain the *Bloch functions* and corresponding *Bloch space*, which is denoted by $\mathcal{B}$. For the general theory of Bloch functions and α-Bloch functions, see [2, 6].

The *pseudo distance* on the unit disk is defined by

$$\rho(z, w) = \left| \frac{z - w}{1 - \overline{w}z} \right| \quad \text{for } z, w \in D.$$

A subset E of D is called a *pseudo r-net*, $0 < r < 1$, if for every $w \in D$, there exists a $z \in E$ such that $\rho(z, w) \leq r$. If we define $\rho(z, E) = \inf\{\rho(z, w) : w \in E\}$ for a set $E \subset D$, then a relatively closed subset E of D is an r-net if and only if $\rho(z, E) \leq r$.

In this paper, ϕ always denotes a holomorphic self-mapping of the unit disk D. Let

$$\tau_\phi(z) = \frac{(1 - |z|^2)|\phi'(z)|}{1 - |\phi(z)|^2} \quad \text{for } z \in D.$$

The Schwarz–Pick lemma [1] says that

$$\tau_\phi(z) \leq 1 \quad \text{for } z \in D. \tag{1.1}$$

For $\epsilon > 0$, let

$$\Omega_e = \{z \in D : \tau_\phi(z) \geq \epsilon\}, \quad G_\epsilon = \phi(\Omega_\epsilon).$$

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The composition operator C_ϕ on $H(D)$, induced by ϕ, is defined by

$$C_\phi(f) = f \circ \phi \quad \text{for } f \in H(D).$$

It follows from (1.1) that C_ϕ is always a bounded operator on $\mathcal{B}$. For the boundedness from below, the following result is known:

In order that C_ϕ be bounded below on $\mathcal{B}$, it is sufficient and necessary that there exist $\epsilon > 0$ and $0 < r < 1$ such that G_ϵ is a pseudo r-net.

We recall that a bounded linear operator T of a Banach space $\mathcal{S}_1$ into another one $\mathcal{S}_2$ is said to be *bounded below* if $\|T(s)\|_{\mathcal{S}_2} \geq k\|s\|_{\mathcal{S}_1}$ for $s \in \mathcal{S}_1$ with a $k > 0$ independent of s. P. Ghatage, P. Yan and D. Zheng [4] proved the necessity of the condition as well as the sufficiency with the restriction $r < 1/4$. Shortly after, H. Chen [3] showed that the condition is sufficient without any restriction on the value of r. Recently, P. Ghatage, D. Zheng and N. Zorboska proved the sufficiency of the condition for a univalent ϕ [5].

The purpose of this short paper is to generalize the above result to the case of α-Bloch spaces with $\alpha \geq 1$. To this end, instead of τ_ϕ, Ω_ϵ and G_ϵ, we should consider τ_ϕ^α, Ω_ϵ^α and G_ϵ^α, respectively, which are defined by

$$\tau_\phi^\alpha(z) = \frac{(1-|z|^2)^\alpha |\phi'(z)|}{(1-|\phi(z)|^2)^\alpha} \quad \text{for } z \in D,$$

and

$$\Omega_e^\alpha = \{z \in D : \tau_\phi^\alpha(z) \geq \epsilon\}, \quad G_\epsilon^\alpha = \phi(\Omega_\epsilon^\alpha).$$

Our main result is that C_ϕ is bounded below on $\mathcal{B}^\alpha$ with $\alpha \geq 1$ if and only if there exist $\epsilon > 0$ and $r \in (0,1)$ such that G_ϵ^α is a pseudo r-net.

2 Preliminaries

Let $\operatorname{Aut}(D)$ denote the group of all Möbius mappings of D. If $\phi \in \operatorname{Aut}(D)$, it is easy to verify that

$$\frac{|\phi'(z)|}{1-|\phi(z)|^2} = \frac{1}{1-|z|^2}, \quad \text{for } z \in D, \tag{2.1}$$

and, consequently,

$$(1-|z|^2)|(f \circ \phi)'(z)| = (1-|\phi(z)|^2)|f'(\phi(z))| \tag{2.2}$$

holds for $f \in H(D)$ and $z \in D$. For $w \in D$, by ϕ_w we denote the mapping in $\operatorname{Aut}(D)$ that exchanges 0 and w. The following identity is easy to verify:

$$1-|\phi_w(z)|^2 = \frac{(1-|z|^2)(1-|w|^2)}{|1-\overline{w}z|^2} \quad \text{for } z \in D. \tag{2.3}$$

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Let $a = \phi(0)$. By (2.1) and a direct calculation, if $\phi \in \mathrm{Aut}(D)$, we have

$$\frac{1-|a|}{1+|a|} \le \frac{1-|\phi(z)|^2}{1-|z|^2} = |\phi'(z)| \le \frac{1+|a|}{1-|a|} \quad \text{for } z \in D. \tag{2.4}$$

In general, if ϕ is a holomorphic self-mapping of D, letting σ be the holomorphic self-mapping of D such that $\sigma(0) = 0$ and $\phi = \phi_a \circ \sigma$, we have by the Schwarz lemma and (2.4),

$$\frac{1-|z|^2}{1-|\phi(z)|^2} = \frac{1-|z|^2}{1-|\sigma(z)|^2}\frac{1-|\sigma(z)|^2}{1-|\phi_a(\sigma(z))|^2} \le \frac{1+|a|}{1-|a|} \quad \text{for } z \in D. \tag{2.5}$$

It is easy to prove that C_ϕ is a bounded operator of $\mathcal{B}^\alpha$ into $\mathcal{B}^\beta$ if and only if $\|C_\phi(f)\|_\beta \le M\|f\|_\alpha$ for $f \in \mathcal{B}^\alpha$ with $M \ge 0$ independent of f and that a bounded composition operator C_ϕ of $\mathcal{B}^\alpha$ into $\mathcal{B}^\beta$ is bounded below if and only if $\|C_\phi(f)\|_\beta \ge m\|f\|_\alpha$ for $f \in \mathcal{B}^\alpha$ with $m > 0$ independent of f.

For $\alpha > 0$, $w \in D$, we define

$$f_w(z) = \frac{1}{\alpha\overline{w}}\frac{(1-|w|^2)}{(1-\overline{w}z)^\alpha} \quad \text{for } z \in D.$$

Then for $z \in D$,

$$\begin{aligned}(1-|z|^2)^\alpha|f_w'(z)| &= \frac{(1-|z|^2)^\alpha(1-|w|^2)}{|1-\overline{w}z|^{\alpha+1}} \\ &\le \frac{(1-|z|^2)^\alpha}{(1-|z|)^\alpha}\frac{1-|w|^2}{1-|w|} \le 2^{\alpha+1}.\end{aligned}$$

On the other hand, $(1-|w|^2)^\alpha|f'(w)| = 1$. Thus,

$$1 \le \|f_w\|_\alpha \le 2^{\alpha+1}. \tag{2.6}$$

It is easy to see that f_w converges to 0, locally uniformly in D, as $w \to \partial D$.

Theorem 2.1 *Let $\beta \ge 1$ and $\alpha \le \beta$. Then C_ϕ is a bounded operator of $\mathcal{B}^\alpha$ into $\mathcal{B}^\beta$, while it is not bounded below if $\alpha < \beta$.*

Proof First we prove the boundedness of C_ϕ. Let $f \in \mathcal{B}^\alpha$. We have, for $z \in D$,

$$\begin{aligned}(1-|z|^2)^\beta|(f\circ\phi)'(z)| &= (1-|z|^2)^\beta|f'(\phi(z))||\phi'(z)| \\ &= \frac{(1-|z|^2)^{\alpha-1}}{(1-|\phi(z)|^2)^{\alpha-1}}(1-|z|^2)^{\beta-\alpha}(1-|\phi(z)|^2)^\alpha \\ &\quad \times |f'(\phi(z))|\tau_\phi(z).\end{aligned}$$

Thus, by (1.1) and (2.5),

$$\|C_\phi(f)\|_\beta = \sup\{(1-|z|^2)^\beta|(f\circ\phi)'(z)| : z \in D\} \le \frac{(1+|a|)^{\alpha-1}}{(1-|a|)^{\alpha-1}} \cdot \|f\|_\alpha,$$

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and C_ϕ is a bounded operator of $\mathcal{B}^\alpha$ into B^β.

Let $w_n \to \partial D$ and $f_n = f_{w_n}$ be the function defined above for $w = w_n$. Assume $\beta > \alpha \geq 1$ first. Then for $z \in D$ and $n = 1, 2, \ldots,$

$$\begin{aligned}(1-|z|^2)^\beta |(f_n \circ \phi)'(z)| &= (1-|z|^2)^\beta |f_n'(\phi(z))||\phi'(z)| \\ &= (1-|z|^2)^{\beta-1}(1-|\phi(z)|^2)|f_n'(\phi(z))|\tau_\phi(z).\end{aligned}$$

Thus, we have by (1.1),

$$(1-|z|^2)^\beta |(f_n \circ \phi)'(z)| \leq |f_n'(\phi(z))|, \tag{2.7}$$

and by (1.1), (2.5) and (2.6),

$$\begin{aligned}(1-|z|^2)^\beta |(f_n \circ \phi)'(z)| &= \frac{(1-|z|^2)^{\alpha-1}}{(1-|\phi(z)|^2)^{\alpha-1}}(1-|z|^2)^{\beta-\alpha}(1-|\phi(z)|^2)^\alpha \\ &\quad \times |f_n'(\phi(z))|\tau_\phi(z) \\ &\leq 2^{\alpha+1}(1-|z|^2)^{\beta-\alpha} \cdot \frac{(1+|a|)^{\alpha-1}}{(1-|a|)^{\alpha-1}}.\end{aligned} \tag{2.8}$$

For $\epsilon > 0$ by (2.8), there exists an $r < 1$ such that

$$(1-|z|^2)^\beta |(f_n \circ \phi)'(z)| < \epsilon \tag{2.9}$$

for $n = 1, 2, \ldots$ and $|z| > r$. Since $f_n' \to 0$ locally uniformly in D by (2.7), there exists an N such that (2.9) holds also for $n > N$ and $|z| \leq r$. This shows that $\|C_\phi(f_n)\|_\beta \to 0$ as $n \to \infty$. However, $\|f_n\|_\alpha \geq 1$ by (2.6). Thus, C_ϕ is not bounded below. The proof for the case $\beta \geq 1 > \alpha$ is similar to the above. This time, (2.8) is replaced by

$$(1-|z|^2)^\beta |(f_n \circ \phi)'(z)| \leq 2^{\alpha+1}(1-|\phi(z)|^2)^{\beta-\alpha} \cdot \frac{(1+|a|)^{\beta-1}}{(1-|a|)^{\beta-1}}.$$

The proof is complete. ∎

3 Sampling Sets and Pseudo r-Nets

Recently, Ghatage, Zheng and Zorboska [5] introduced the notion of sampling sets for the Bloch space. It can be generalized to α-Bloch spaces automatically. For $\alpha > 0$, a subset H of D is called a *sampling set* for $\mathcal{B}^\alpha$ if there exists $k > 0$ such that

$$\|f\|_\alpha \leq k \sup\{(1-|z|^2)^\alpha |f'(z)| : z \in H\}$$

holds for $f \in \mathcal{B}^\alpha$. They proved the following theorem in the special case $\alpha = 1$, and it can be proved generally in a similar way. From now on, we suppose that $\alpha \geq 1$.

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Theorem 3.1 *C_ϕ is bounded below on $\mathcal{B}^\alpha$ if and only if there exists $\epsilon > 0$ such that G^α_ϵ is a sampling set for $\mathcal{B}^\alpha$.*

Proof Assume that C_ϕ is bounded below on $\mathcal{B}^\alpha$, *i.e.*, $\|C_\phi(f)\|_\alpha \geq m\|f\|_\alpha$ for $f \in \mathcal{B}^\alpha$ with $m > 0$ independent of f. Then for $f \in \mathcal{B}^\alpha$ with $\|f\|_\alpha > 0$, there is a $z_f \in D$ such that

$$\tau^\alpha_\phi(z_f)(1-|\phi(z_f)|^2)^\alpha|f'(\phi(z_f))| = (1-|z_f|^2)^\alpha|(C_\phi(f))'(z_f)| \geq (m/2)\|f\|_\alpha.$$

Thus,

$$\tau^\alpha_\phi(z_f) \geq m/2 \quad \text{and} \quad (1-|\phi(z_f)|^2)^\alpha|f'(\phi(z_f))| \geq \frac{m(1-|a|)^{\alpha-1}}{2(1+|a|)^{\alpha-1}}\|f\|_\alpha, \tag{3.1}$$

since $(1-|\phi(z_f)|^2)^\alpha|f'(\phi(z_f))| \leq \|f\|_\alpha$ and, by (1.1) and (2.5),

$$\tau^\alpha_\phi(z_f) = \frac{(1-|z_f|^2)^{\alpha-1}\tau_\phi(z_f)}{(1-|\phi(z_f)|^2)^{\alpha-1}} \leq \Big(\frac{1+|a|}{1-|a|}\Big)^{\alpha-1}.$$

If we take $\epsilon = m/2$, by (3.1) G^α_ϵ contains all $\phi(z_f)$ and is a sampling set for $\mathcal{B}^\alpha$.

Now assume that G^α_ϵ with some $\epsilon > 0$ is a sampling set for $\mathcal{B}^\alpha$. Then, for $f \in \mathcal{B}^\alpha$, we have $z_f \in D$ such that $\phi(z_f) \in G^\alpha_\epsilon$, $\tau_\phi(z_f) \geq \epsilon$ and

$$\|f\|_\alpha \leq k\sup\{(1-|w|^2)^\alpha|f'(w)| : w \in G^\alpha_\epsilon\} \leq 2k(1-|\phi(z_f)|^2)^\alpha|f'(\phi(z_f))|,$$

where $k > 0$ is independent of f. Thus,

$$\begin{aligned}\|C_\phi(f)\|_\alpha &\geq (1-|z_f|^2)^\alpha|f'(\phi(z_f))||\phi'(z_f)| \\ &= \tau^\alpha_\phi(z_f)(1-|\phi(z_f)|^2)^\alpha|f'(\phi(z_f))| \geq \frac{\epsilon\|f\|_\alpha}{2k}.\end{aligned}$$

This shows that C_ϕ is bounded below on $\mathcal{B}^\alpha$. The theorem is proved. ∎

For $w \in \mathbb{C}$ and $r > 0$, we denote by $D(w,r)$ the disk with radius r and centered at w, while for $w \in D$ and $0 < r < 1$, we denote by $\Delta(w,r)$ the pseudo disk $\Delta(w,r) = \{z \in D : \rho(z,w) < r\}$. Let $\overline{\Delta}(w,r)$ and $\overline{D}(w,r)$ denote their closures respectively.

Theorem 3.2 *A sampling set for $\mathcal{B}^\alpha$ is a pseudo r-net and, conversely, if E is a pseudo r-net, then for any $\delta > 0$, the set $E_\delta = \bigcup_{z\in E}\overline{\Delta}(z,\delta)$ is a sampling set for $\mathcal{B}^\alpha$.*

Proof First we assume that H is a sampling set for $\mathcal{B}^\alpha$, *i.e.*, there exists a $k > 0$ such that

$$\|f\|_\alpha \leq k\sup\{(1-|z|^2)^\alpha|f'(z)| : z \in H\} \quad \text{for } f \in \mathcal{B}^\alpha. \tag{3.2}$$

Let $w \in D$ and f_w be the function defined above. Then by (3.2), there exists a $z \in H$ such that $\|f_w\|_\alpha \leq 2k(1-|z|^2)^\alpha|f'_w(z)$, and by (2.6) and (2.3),

$$\begin{aligned}1 \leq 2k(1-|z|^2)^\alpha|f'_w(z) &= \frac{2k(1-|z|^2)^\alpha(1-|w|^2)}{|1-\overline{w}z|^{\alpha+1}} \\ &= \frac{2k(1-|z|^2)^{\alpha-1}}{|1-\overline{w}z|^{\alpha-1}}\frac{(1-|z|^2)(1-|w|^2)}{|1-\overline{w}z|^2} \leq 2^\alpha k(1-|\phi_w(z)|^2).\end{aligned}$$

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Thus, $\rho(z,w) = |\phi_w(z)| \le r = \sqrt{1 - 1/(2^\alpha k)}$, and H is a pseudo r-net.

Now, we assume that E is a pseudo r-net. We want to prove that E_δ, for any δ, is a sampling set for $\mathcal{B}^\alpha$. Suppose on the contrary that there are a $\delta > 0$ and a sequence $f_n \in \mathcal{B}$ such that

$$\|f_n\|_\alpha = 1 \quad \text{for } n = 1, 2, \dots, \tag{3.3}$$

and

$$\sup\{(1 - |z|^2)^\alpha |f_n'(z)| : z \in E_\delta\} = \epsilon_n \to 0 \quad \text{as } n \to \infty. \tag{3.4}$$

For $n = 1, 2, \dots$, let $z_n \in D$ be such that

$$(1 - |z_n|^2)^\alpha |f_n'(z_n)| \ge 1/2. \tag{3.5}$$

Since E is a pseudo r-net, we have a sequence $z_n' \in E$ such that $\rho(z_n', z_n) \le r$ for $n = 1, 2, \dots$. Let $w_n = \phi_{z_n'}(z_n)$ and $g_n = f_n \circ \phi_{z_n'}$ for $n = 1, 2, \dots$.

Let $n \ge 1$ be fixed. We have

$$|w_n| = \rho(z_n', z_n) \le r, \tag{3.6}$$

and by (2.2), for $w \in D$,

$$\begin{aligned}(1 - |w|^2)^\alpha |g_n'(w)| &= (1 - |w|^2)^{\alpha-1}(1 - |w|^2)|g_n'(w)| \\ &= (1 - |w|^2)^{\alpha-1}(1 - |\phi_{z_n'}(w)|^2)|f_n'(\phi_{z_n'}(w))| \\ &= \frac{(1 - |w|^2)^{\alpha-1}}{(1 - |\phi_{z_n'}(w)|^2)^{\alpha-1}} \cdot (1 - |\phi_{z_n'}(w)|^2)^\alpha |f_n'(\phi_{z_n'}(w))|.\end{aligned} \tag{3.7}$$

It follows from (3.7), (2.4) and (3.3) that

$$(1 - |w|^2)^\alpha |g_n'(w)| \le \frac{(1 + |z_n'|)^{\alpha-1}}{(1 - |z_n'|)^{\alpha-1}} \le \frac{4^{\alpha-1}}{(1 - |z_n'|^2)^{\alpha-1}} \quad \text{for } w \in D. \tag{3.8}$$

In particular, letting $w = w_n$ in (3.7) and by (2.1), (3.5) and (3.6), we have

(3.9)

$$\begin{aligned}(1 - |w_n|^2)^\alpha |g_n'(w_n)| &= \frac{(1 - |w_n|^2)^{\alpha-1}}{(1 - |\phi_{z_n'}(w_n)|^2)^{\alpha-1}} \cdot (1 - |\phi_{z_n'}(w_n)|^2)^\alpha |f_n'(\phi_{z_n'}(w_n))| \\ &= \frac{(1 - |w_n|^2)^{\alpha-1}}{(1 - |z_n|^2)^{\alpha-1}} \cdot (1 - |z_n|^2)^\alpha |f_n'(z_n)| \\ &\ge \frac{1}{2|\phi_{z_n'}'(w_n)|^{\alpha-1}} \\ &= \frac{|1 - \overline{z}_n' w_n|^2}{2(1 - |z_n'|^2)} \ge \frac{(1 - r)^2}{2(1 - |z_n'|^2)}.\end{aligned}$$

If $|w| \le \delta$, then $\rho(\phi_{z_n'}(w), z_n') = \rho(\phi_{z_n'}(w), \phi_{z_n'}(0)) = \rho(w, 0) = |w| \le \delta$ and consequently, $\phi_{z_n'}(w) \in E_\delta$. Thus, by (3.4), $(1 - |\phi_{z_n'}(w)|^2)^\alpha |f_n'(\phi_{z_n'}(w))| \le \epsilon_n$, and by (3.7) and (2.4),

$$(3.10) \qquad (1 - |w|^2)^\alpha |g_n'(w)| \le \frac{\epsilon_n (1 + |z_n'|)^{\alpha-1}}{(1 - |z_n'|)^{\alpha-1}} \le \frac{4^{\alpha-1}\epsilon_n}{(1 - |z_n'|^2)^{\alpha-1}} \quad \text{for } |w| \le \delta.$$

For $n = 1, 2, \dots$, let $h_n(w) = (1 - |z_n'|^2)^{\alpha-1} g_n'(w)$ for $w \in D$. By (3.8), h_n is bounded locally uniformly in D. Using Montel's theorem and choosing a subsequence if necessary, we may assume that h_n converges to a holomorphic function h, locally uniformly in D, and $w_n \to w_0$ with $|w_0| \le r$ because of (3.6). Letting $n \to \infty$ in (3.9) and (3.10), we obtain $|h(w_0)| \ge (1 - r)^2 / (2(1 - |w_0|^2)^\alpha)$ and $h(w) = 0$ for $|w| \le \delta$. We arrive at a contradiction and it is proved that E_δ is a sampling set for any δ. The theorem is proved. ∎

4 The Main Result and Its Proof

Lemma 4.1 *Let h be a holomorphic self-mapping of D such that $h(0) = 0$. If $|h'(0)| \ge \epsilon > 0$, then there exist $\delta_1, \delta_2 > 0$, depending only on ϵ, such that*

(i) $|h'(z)| \ge \epsilon/2$ *for* $z \in D(0, \delta_1)$,
(ii) $\overline{D}(0, \delta_2) \subset h(D(0, \delta_1))$.

Proof First we want to prove that there exists a $\delta_1 > 0$ with property (i). Suppose on the contrary that there exists a sequence of functions h_n which satisfies the assumption for h in the lemma, and a sequence $z_n \to 0$, such that $h_n'(z_n) < \epsilon/2$ for $n = 1, 2, \dots$. Using Montel's theorem, we may assume that h_n converges to h_0 locally uniformly in D. Then $h_0'(0) = \lim h_n'(0) \ge \epsilon$. On the other hand, since $z_n \to 0$ and $h_n'(z_n) < \epsilon/2$ for $n = 1, 2, \dots$, we have $h_0'(0) = \lim h_n'(z_n) \le \epsilon/2$, a contradiction. The existence of δ_1 satisfying (i) is proved.

Now we fix $\delta_1 > 0$ that satisfies (i). To prove the existence of δ_2, suppose that there exists a sequence of functions h_n which satisfies the assumption for h in the lemma and a sequence $w_n \to 0$ such that h_n does not assume w_n in $D(0, \delta_1)$ for $n = 1, 2, \dots$. Using Montel's theorem again, we may assume that h_n converges to h_0 locally uniformly in D. Then $h_0(0) = 0$, $|h_0'(0)| \ge \epsilon$, and h_0 is not a constant. Thus, by using Rouché's theorem, a usual argument shows that there exist a $\delta' > 0$ and a positive integer N such that h_n assumes every $w \in D(0, \delta')$ in $D(0, \delta_1)$ if $n > N$. We arrive at a contradiction again and the lemma is proved. ∎

Lemma 4.2 *For $\epsilon > 0$, there exist $\delta, \epsilon' > 0$, which depend on $|\phi(0)|$, ϵ and α only, such that*

$$(G_\epsilon^\alpha)_\delta = \bigcup_{w' \in G_\epsilon^\alpha} \overline{\Delta}(w', \delta) \subset G_{\epsilon'}^\alpha.$$

Proof Let $w' = \phi(z') \in G_\epsilon^\alpha$ and $\tau_\phi^\alpha(z') \ge \epsilon$, and let $h = \phi_{w'} \circ \phi \circ \phi_{z'}$. It follows from

$$\epsilon \le \tau_\phi^\alpha(z') = \frac{(1 - |z'|^2)^{\alpha-1} \tau_\phi(z')}{(1 - |w'|^2)^{\alpha-1}}$$

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and $\tau_\phi(z') \le 1$ that

$$\frac{(1-|z'|^2)^{\alpha-1}}{(1-|w'|^2)^{\alpha-1}} \ge \epsilon. \tag{4.1}$$

We have by (2.5),

$$|h'(0)| = \frac{(1-|z'|^2)|\phi'(z')|}{1-|w'|^2} = \frac{(1-|w'|^2)^{\alpha-1}\tau_\phi^\alpha(z')}{(1-|z'|^2)^{\alpha-1}} \ge \frac{(1-|a|)^{\alpha-1}\epsilon}{(1+|a|)^{\alpha-1}} = \epsilon_1.$$

By the above lemma, there exist $\delta_1, \delta_2 > 0$ satisfying (i) and (ii) with ϵ replaced by ϵ_1.

For $w \in \overline{\Delta}(w', \delta_2)$, let $\omega = \phi_{w'}(w) \in \overline{D}(0, \delta_2)$. Then by (i) and (ii), there exists a $\zeta \in D(0, \delta_1)$ such that $h(\zeta) = \omega$ and $h'(\zeta) \ge \epsilon_1/2$. Let $z = \phi_{z'}(\zeta)$. Then $\phi(z) = w$ and by (2.1) and (4.1),

$$\begin{aligned}
\tau_\phi^\alpha(z) &= \frac{(1-|z|^2)^\alpha|\phi'(z)|}{(1-|w|^2)^\alpha} \\
&= \frac{(1-|z|^2)^\alpha}{(1-|\zeta|^2)^\alpha}\frac{(1-|\zeta|^2)^\alpha}{(1-|\omega|^2)^\alpha}\frac{(1-|\omega|^2)^\alpha}{(1-|w|^2)^\alpha}\cdot|\phi'_{w'}(\omega)h'(\zeta)\phi_{z'}(z)| \\
&\ge \frac{\epsilon_1}{2}\frac{|\phi'_{z'}(\zeta)|^{\alpha-1}}{|\phi'_{w'}(\omega)|^{\alpha-1}}\frac{(1-|\zeta|^2)^\alpha}{(1-|\omega|^2)^\alpha} \ge \frac{\epsilon_1}{2}\frac{(1-\delta_2)^{2(\alpha-1)}(1-\delta_1^2)^\alpha}{(1+\delta_1)^{2(\alpha-1)}}\frac{(1-|z'|^2)^{\alpha-1}}{(1-|w'|^2)^{\alpha-1}} \\
&\ge \frac{\epsilon_1\epsilon}{2}\frac{(1-\delta_2)^{2(\alpha-1)}(1-\delta_1^2)^\alpha}{(1+\delta_1)^{2(\alpha-1)}} = \epsilon'.
\end{aligned}$$

This shows that $\overline{\Delta}(w', \delta_2) \subset G^\alpha_{\epsilon'}$ for $w' \in G^\alpha_\epsilon$. The lemma is proved. ∎

Now our main result follows directly from Lemma 4.2 and Theorems 3.1 and 3.2.

Theorem 4.3 *C_ϕ is bounded below on $\mathcal{B}^\alpha$ if and only if there exist an $\epsilon > 0$ and an r with $0 < r < 1$ such that G^α_ϵ is a pseudo r-net.*

Proof If C_ϕ is bounded below, by using Theorem 3.1, there exists an $\epsilon > 0$ such that G^α_ϵ is a sampling set for $\mathcal{B}^\alpha$ and, consequently, is a pseudo r-net with $0 < r < 1$ by Theorem 3.2. Conversely, assume that there exist an $\epsilon > 0$ and an r with $0 < r < 1$ such that G^α_ϵ is a pseudo r-net . Then by Lemma 4.2, there exist $\delta, \epsilon' > 0$ such that $(G^\alpha_\epsilon)_\delta \subset G^\alpha_{\epsilon'}$. Using Theorem 3.2, we see that $(G^\alpha_\epsilon)_\delta$ is a sampling set for $\mathcal{B}^a$ and, consequently, so is $G^\alpha_{\epsilon'}$, since $(G^\alpha_\epsilon)_\delta \subset G^\alpha_{\epsilon'}$. The theorem is proved. ∎

We proved our main result by using Theorems 3.1 and 3.2, in which the notion of sample set is involved. In fact, it can be proved in a more direct way, without making use of the notion of sample set, as in [3] for $\alpha = 1$.

Remark If $\alpha > 1$, by (1.1) and (2.5), $\tau_\phi^\alpha(z) \ge \epsilon$ implies

$$\frac{1-|z|^2}{1-|\phi(z)|^2} \ge \epsilon^{1/(\alpha-1)} \quad \text{and} \quad \tau_\phi(z) \ge \frac{(1-|a|)^{\alpha-1}\epsilon}{(1+|a|)^{\alpha-1}}.$$

Conversely, $\tau_\phi^\alpha(z) \geq \epsilon^{\alpha+1}$ if

$$\frac{1-|z|^2}{1-|\phi(z)|^2} \geq \epsilon \quad \text{and} \quad \tau_\phi(z) \geq \epsilon.$$

So, if we define

$$\Omega'_\epsilon = \left\{ z : \tau_\phi(z) \geq \epsilon,\ \frac{1-|z|^2}{1-|\phi(z)|^2} \geq \epsilon \right\}, \quad G'_\epsilon = \phi(\Omega'_\epsilon),$$

then G_ϵ^α can be replaced by G'_ϵ in Theorem 4.3 in the case $\alpha > 1$. As a consequence, if C_ϕ is bounded below on $\mathcal{B}^\alpha$ for some $\alpha > 1$, then so is C_ϕ for all $\alpha \geq 1$. In the following section, we will give an example of the function ϕ to show that it is really possible that C_ϕ is bounded below on $\mathcal{B}$, but not on $\mathcal{B}^\alpha$ with $\alpha > 1$.

5 An example

Let ϕ be the conformal mapping of D to the domain

$$\Lambda = \{w : 0 < |w| < 1,\ 0 < \arg w < 2\pi\},$$

which is the unit disk with the positive radius l (including the origin) deleted, such that $\phi(1) = 0$ and $\phi(\pm i) = 1$. Define $U = \bigcup_{w' \in l} \Delta(w', 1/2)$. We claim that $E = D \setminus U \subset G_{1/2}$. In fact, if $w = \phi(z) \in E$, then $\Delta(w, 1/2) \subset \Lambda$, $\lambda = \phi_z \circ \phi^{-1} \circ \phi_w$ is holomorphic on the disk $D(0, 1/2)$ and $\lambda(0) = 0$. Thus, by the Schwarz lemma,

$$2 \geq |\lambda'(0)| = \frac{1-|w|^2}{(1-|z|^2)|\phi'(z)|} = \tau_\phi(z)^{-1}.$$

To show that E is a pseudo r-net with $r < 1$, we fix a pseudo disk $\Delta(w', 1/2)$ with $w' \in l$. Then $\phi_{w'}(i/2)$ is a point in $\partial\Delta(w', 1/2)$ and has a pseudo distance greater than $1/2$ to points in l other than w'. So $\phi_{w'}(i/2) \in E$ and $\rho(w, \phi_{w'}(i/2)) < r$ for $w \in \Delta(w', 1/2)$ with $r = 4/5$. Since w' may be an arbitrary point, it is proved that E is a pseudo $4/5$-net and, consequently, $G_{1/2}$ is also. Thus, C_ϕ is bounded below on $\mathcal{B}$.

It follows from the general theory of conformal mappings that $(\phi^{-1})'(w) \to 0$ as $w \to 1$. In fact, it is easy to verify, since in our special case,

$$\psi^{-1}(\phi^{-1}(\psi(\omega))) = -i\sqrt{1+\omega^2},$$

where $\psi(\omega) = -(\omega - 1)/(\omega + 1)$. Thus,

$$\frac{1-|z|^2}{1-|w|^2} \to 0 \quad \text{as } w \to 1,$$

since

$$\tau_\phi(z)^{-1} = \frac{(1-|w|^2)|(\phi^{-1})'(w)|}{1-|z|^2} \geq 1 \quad \text{for } w \in \Lambda.$$

For $\epsilon > 0$, there exists a $\delta > 0$ such that $(1-|z|^2)/(1-|w|^2) < \epsilon$ for $w \in \Lambda \cap D(1, \delta)$, *i.e.*, $(D \cap D(1, \delta)) \cap G'_\epsilon = \varnothing$. Since $D \cap D(1, \delta)$ contains a pseudo disk with pseudo radius sufficiently close to 1, we see that G'_ϵ cannot be a pseudo r-net for any $r < 1$. By Theorem 4.3 and the remark, we assert that C_ϕ is not bounded below on $\mathcal{B}^\alpha$ with $\alpha > 1$.

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