Minimum Deviation through a Prism, etc.

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Let $\mu \equiv$ refractive index of the prism.

Let α, α', β', β be the successive angles of incidence and refraction of a ray, in a plane perpendicular to the edge of the prism.

Then $$\left.\begin{aligned}\sin\alpha &= \mu\sin\alpha' \\ \sin\beta &= \mu\sin\beta'\end{aligned}\right\} \qquad (1)$$

Consider the case when $\mu > 1$, so that $\alpha > \alpha'$ and $\beta > \beta'$. $\qquad (2)$

First Method. (Fig. 26.)

Let O be the centre of a circle of unit radius and let $\angle$ XOA′ and $\angle$ XOB′ on opposite sides of OX be the angles α' and β' respectively, so that $\angle \text{A}'\text{OB}' = i$, the angle of the prism.

Take $\text{OA}' = \text{OB}' = \mu$.

Draw A′A, B′B parallel to XO to meet the circle in A, B.

Join OA, OB.

We have $\quad \sin\text{OAA}'/\sin\text{AA}'\text{O} = \text{OA}'/\text{AO} = \mu$;

$\therefore \angle \text{AOX} = \alpha$. Similarly $\angle \text{BOX} = \beta$.

Suppose now $\alpha' \neq \beta'$, say $\alpha' > \beta'$, then $\angle \text{AA}'\text{O} > \angle \text{BB}'\text{O}$.

Hence, since A′ and B′ are equally distant from O, we have, by a slight extension of *Euc.* III. 8, $\text{A}'\text{A} > \text{B}'\text{B}$.

From AA′ cut off $\text{AK} = \text{BB}'$ and join AB, B′K, so that ABB′K is a parallelogram.

Now $\quad \angle \text{AA}'\text{O} > \angle \text{BB}'\text{O}$ and $\angle \text{OA}'\text{B}' = \angle \text{OB}'\text{A}'$;

$$\therefore \angle \text{AA}'\text{B}' > \angle \text{BB}'\text{A}' > \angle \text{BB}'\text{K}$$
$$> \angle \text{B}'\text{KA}';$$
$$\therefore \text{B}'\text{K} > \text{B}'\text{A}';$$
$$\therefore \text{BA} > \text{B}'\text{A}'.$$

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On the other hand, if $\alpha' = \beta'$, it is obvious that $AB = A'B'$.

Now, for a given prism, $A'B'$ is a fixed length, since $OA' = OB' = \mu$, and $\angle A'OB' = i$.

Hence when $\alpha' = \beta'$, AB is a minimum ; $\therefore \angle AOB$ (or $\alpha + \beta$) is a minimum, $\therefore$ $\alpha + \beta - i$ is a minimum.

Thus the deviation is least when $\alpha' = \beta'$.

Second Method. (Fig. 27.)

Let ADB be a circle of unit radius and centre O.

Let $CO = \mu$, $\angle OCB = \beta'$ and $\angle OCA = \alpha'$;

$\therefore$ $\angle AOC = \alpha - \alpha'$ and $\angle BOC = \beta - \beta'$.

Let Q be the centre of the circle circumscribing ABC.

Let OQ meet this circle in E and the other in D.

Join CD, CE.

Thus $\angle OCE = \dfrac{\alpha' + \beta'}{2} = \dfrac{i}{2}$ and $\angle COD = \dfrac{\alpha + \beta - \alpha' - \beta'}{2} = \dfrac{\delta}{2}$,

where $\delta \equiv$ deviation of ray.

Let CE meet the arc ADB in F.

As α' approaches equality with β', A, B and E tend towards coincidence at F. Hence, since F, C, O are fixed points, the angle COE or $\delta/2$ is least in the limiting case when E coincides with F, *i.e.*, when $\alpha' = \beta'$.

Third Method. (Fig. 28.)

This is a modification of the second.

Let O, C, A, B, F be the same as before, so that CF bisects $\angle$ BCA.

Join AF, and produce BF to meet CA in G.

Now $\angle CBF$ is obtuse, $\therefore$ $CB < CF < CG$.

But $BF : FG = BC : CG$,

$\therefore$ $BF < FG$

$< FA$, since $\angle FGA$ is obtuse.

Hence $\angle BOF < \angle FOA$; $\therefore$ $\angle COB + \angle COA >$ twice $\angle COF$.

But twice $\angle COF$ is the value of δ when $\alpha' = \beta'$.

Hence in all other cases δ is greater.

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Corollary. We also see from Fig. 27 or Fig. 28 that $\alpha - \alpha'$, the deviation due to a *single* refraction increases as α' increases uniformly, and at an increasing rate. For if we take OCB, OCF and OCA as three successive values of α' increasing by equal increments BCF, FCA, the corresponding deviations increase by the amounts BOF, FOA, of which the latter is the greater.

It may be noted that the Second and Third methods could be modified by taking $OC = 1$ and the radius $OA = \mu$ and interchanging α, α' and β, β'. And from the modified figure we could deduce, as in the previous corollary, that as α increases uniformly, the deviation $\alpha - \alpha'$ increases at an increasing rate.

In Heath's *Treatise* there are proofs of these results by the use of infinitesimals, ascribed to the late Professor Tait.

The following propositions in Geometry, amongst others that could be stated, are corollaries to what precedes :—

I. If from two fixed points without a fixed circle, and equi-distant from its centre, two parallel lines be drawn cutting the circle, the equal arcs intercepted by them on the circumference are *least* when the parallel lines are equi-distant from the centre.

II. If from a fixed point without a fixed circle, a pair of lines including an angle of fixed size are drawn to cut the circle, the two arcs intercepted between them on the circumference are both *least* when the lines are equi-distant from the centre.

III. If from the centre of a fixed circle two radii are drawn, including an angle fixed in size, then the angle subtended at a fixed external point by the arc between the extremities of the radii is *greatest* when the radii are equally distant from the fixed point.

In seeking for a concise *trigonometrical* proof of the minimum deviation theorem for the prism, I arrived at a formula, which I found to be none other than that given in Parkinson's Optics. It is one that seems to leave nothing to be desired in the way of conciseness and neatness, but its popularity has perhaps suffered from the rather unsymmetrical and difficult way in which Parkinson deduces it.

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I recall it here partly in order to indicate a simpler proof of the formula, and partly to associate with it a companion formula which enables us to deduce the theorem as to a single refraction given as a corollary above.

If α, α', β', β have the same significations as before, and we wish to deduce results connecting $i = \alpha' + \beta'$ and $\delta = \alpha + \beta - \alpha' - \beta'$ from the law of refraction, which gives $\sin\alpha = \mu\sin\alpha'$, $\sin\beta = \mu\sin\beta'$; let us consider the formulae

$$\begin{aligned}\cos(\alpha-\beta) - \cos(\alpha+\beta) &= 2\sin\alpha\sin\beta \\ &= 2\mu^2\sin\alpha'\sin\beta' \\ &= \mu^2\{\cos(\alpha'-\beta') - \cos(\alpha'+\beta')\} ; \\ \cos(\alpha-\beta)\cos(\alpha+\beta) - 1 &= -\sin^2\alpha - \sin^2\beta \\ &= -\mu^2(\sin^2\alpha' + \sin^2\beta') \\ &= \mu^2\{\cos(\alpha'-\beta')\cos(\alpha'+\beta') - 1\}.\end{aligned}$$

Putting

$$x \equiv \cos(\alpha-\beta),\ y \equiv \cos(\alpha+\beta),\ z \equiv \cos(\alpha'-\beta'),\ c \equiv \cos(\alpha'+\beta') = \cos i,$$

we may write the above results thus :—

$$x - y = \mu^2(z - c),$$
$$xy - 1 = \mu^2(zc - 1).$$

Eliminating z, we get $(c + x)(c - y) = (\mu^2 - 1)(1 - c^2) = (\mu^2 - 1)\sin^2 i$, which is Parkinson's formula.

Eliminating c, we get $(z - x)(z + y) = (\mu^2 - 1)(1 - z^2)$.

From the latter formula, if we suppose z constant, we find that x decreases with y, $\therefore$ $\alpha - \beta$ increases with $\alpha + \beta$. Thus, for a given value of $\alpha' - \beta'$, if β increases so does β' ; $\therefore$ α' increases ; $\therefore$ α increases ; $\therefore$ $\alpha + \beta$ increases ; $\therefore$ $\alpha - \beta$ increases.