

CALCULUS RULES FOR COMBINATIONS OF ELLIPSOIDS AND APPLICATIONS

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We derive formulas for the Minkowski sum, the convex hull, the intersection, and the inverse sum of a finite family of ellipsoids. We show how these formulas can be used to obtain inner and outer ellipsoidal approximations of a convex polytope.

1. INTRODUCTION

There is a one-to-one correspondence between the set P_n of symmetric positive semidefinite $n \times n$ matrices, and the set of ellipsoids in \mathbb{R}^n which are bounded and centered at the origin. One can consider, for instance, the correspondence

$$A \in P_n \mapsto E(A) := \{z \in \mathbb{R}^n : \langle z, x \rangle \leq [(x, Ax)]^{1/2} \text{ for all } x \in \mathbb{R}^n\},$$

where $\langle \cdot, \cdot \rangle$ stands for the usual inner product in \mathbb{R}^n . The set $E(A)$ is referred to as the ellipsoid associated with $A \in P_n$. Note that this set is bounded because its support function

$$s[x; E(A)] := \sup_{z \in E(A)} \langle z, x \rangle = [(x, Ax)]^{1/2}$$

is finite. The ellipsoid $E(A)$ is centered at $0 \in \mathbb{R}^n$, but it may not contain 0 in its interior. Since there are matrices in P_n which are singular, the possibility of having flat ellipsoids is not excluded.

Combinations of two or more ellipsoids appear quite often in applications. For instance, Hiriart-Urruty and Seeger [1, Corollary 6.2] obtained the bounds

$$\bigcap_{i \in I(\bar{x})} E(\nabla^2 f_i(\bar{x})) \subset \partial^2 f(\bar{x}) \subset \text{co} \left[\bigcup_{i \in I(\bar{x})} E(\nabla^2 f_i(\bar{x})) \right]$$

for the second-order subdifferential $\partial^2 f(\bar{x})$ of a convex function f of the type

$$x \in \mathbb{R}^n \mapsto f(x) = \text{Max}_{1 \leq i \leq p} \{f_i(x)\}.$$

(Each $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$ is assumed to be convex and twice differentiable at $\bar{x} \in \mathbb{R}^n$, and $I(\bar{x}) = \{i : f(\bar{x}) = f_i(\bar{x})\}$ denotes the set of indices which are active at \bar{x} .) As a

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second example we mention that Kurzhanski and Valyi [2] were concerned with a state estimation problem for dynamic systems under uncertainty, and they needed to estimate the intersection and the Minkowski sum of a finite number of ellipsoids. Unfortunately, the estimates announced in [2] were given without proof.

The purpose of this note is to derive, in a unified way, formulas for the Minkowski sum, the convex hull, the intersection, and the inverse sum of a finite family of ellipsoids.

For convenience we recall first some basic definitions and results. Let $\{A_1, \dots, A_p\}$ be a finite collection of matrices in P_n . The series sum of the A_i 's is the ordinary sum $A_1 + \dots + A_p \in P_n$. The parallel sum of the A_i 's is the matrix $A_1 \square \dots \square A_p \in P_n$ defined by

$$(1.1) \quad \langle x, [A_1 \square \dots \square A_p] x \rangle = \inf_{x_1 + \dots + x_p = x} \{ \langle x_1, A_1 x_1 \rangle + \dots + \langle x_p, A_p x_p \rangle \} \quad \text{for all } x \in \mathbb{R}^n.$$

If the A_i 's are nonsingular, then it is possible to write the following pair of dual formulas:

$$A_1 \square \dots \square A_p = (A_1^{-1} + \dots + A_p^{-1})^{-1},$$

$$A_1 + \dots + A_p = (A_1^{-1} \square \dots \square A_p^{-1})^{-1}.$$

For a detailed discussion on the parallel addition of matrices, see for instance Mazure [3, 4] and references therein. The following result is taken from Seeger [6, Theorem 8.1].

LEMMA 1.1. *Let $\{A_1, \dots, A_p\}$ be a finite collection of matrices in P_n . Then*

$$E(A_1 + \dots + A_p) = \bigcup_{\lambda \in \Lambda} \{ \sqrt{\lambda_1} E(A_1) + \dots + \sqrt{\lambda_p} E(A_p) \},$$

$$E(A_1 \square \dots \square A_p) = \bigcup_{\lambda \in \Lambda} \{ \sqrt{\lambda_1} E(A_1) \cap \dots \cap \sqrt{\lambda_p} E(A_p) \},$$

where Λ stands for the elementary simplex in \mathbb{R}^p :

$$\Lambda := \{ \lambda = (\lambda_1, \dots, \lambda_p) \in \mathbb{R}^p : \lambda_1 + \dots + \lambda_p = 1 \text{ and } \lambda_i \geq 0 \text{ for all } i = 1, \dots, p \}.$$

The formulas given in Lemma 1.1 are crucial for the understanding of the theorems stated in the next section. For the sake of completeness in our exposition we need to record also the following result:

LEMMA 1.2. *Let $\{A_1, \dots, A_p\}$ be a finite collection of matrices in P_n . Then the following equality is true*

$$(1.2) \quad E(A_1 \square \dots \square A_p) = \bigcap_{\substack{B_1, \dots, B_p \\ B_1 + \dots + B_p = I_n}} E(B_1 A_1 B_1^T + \dots + B_p A_p B_p^T).$$

Here I_n denotes the $n \times n$ identity matrix, and B_1^T, \dots, B_p^T are the transposes of the $n \times n$ matrices B_1, \dots, B_p , respectively.

PROOF: Let $x \in \mathbb{R}^n$ be arbitrary. If $x_1 + \dots + x_p = x$, then

$$\langle x, [A_1 \square \dots \square A_p]x \rangle \leq \langle x_1, A_1 x_1 \rangle + \dots + \langle x_p, A_p x_p \rangle .$$

If one chooses $x_i = B_i^T x$, with $B_1 + \dots + B_p = I_n$, then one gets

$$\langle x, [A_1 \square \dots \square A_p]x \rangle \leq \langle x, [B_1 A_1 B_1^T + \dots + B_p A_p B_p^T]x \rangle,$$

and thus

$$(1.3) \quad E(A_1 \square \dots \square A_p) \subset E(B_1 A_1 B_1^T + \dots + B_p A_p B_p^T) .$$

This proves of course the inclusion $E(A_1 \square \dots \square A_p) \subset K$, where K is the set on the right hand side of (1.2). If the A_i 's are nonsingular, then one can show that the infimum in (1.1) is attained at

$$x_i = A_i^{-1}(A_1 \square \dots \square A_p)x \quad \text{for all } i = 1, \dots, p .$$

This implies that the inclusion (1.3) becomes an equality when

$$B_i^T = A_i^{-1}(A_1 \square \dots \square A_p) \quad \text{for all } i = 1, \dots, p .$$

Summarising, formula (1.2) holds if the A_i 's are nonsingular. To cover the singular case one can use a continuity argument: one applies formula (1.2) to the collection of nonsingular matrices $\{A_1 + \epsilon I_n, \dots, A_p + \epsilon I_n\}$, where $\epsilon > 0$ is understood as a perturbation parameter. One gets in this way

$$\begin{aligned} E((A_1 + \epsilon I_n) \square \dots \square (A_p + \epsilon I_n)) \\ = \bigcap_{\substack{B_1, \dots, B_p \\ B_1 + \dots + B_p = I_n}} E(B_1(A_1 + \epsilon I_n)B_1^T + \dots + B_p(A_p + \epsilon I_n)B_p^T) . \end{aligned}$$

Taking into account that

$$E(B_1(A_1 + \epsilon I_n)B_1^T + \dots + B_p(A_p + \epsilon I_n)B_p^T) \supset E(B_1 A_1 B_1^T + \dots + B_p A_p B_p^T),$$

one obtains

$$E((A_1 + \epsilon I_n) \square \dots \square (A_p + \epsilon I_n)) \supset K .$$

It suffices now to let $\epsilon > 0$ tend to zero in order to get the reverse inclusion $E(A_1 \square \dots \square A_p) \supset K$, and the proof is complete. □

2. CALCULUS RULES FOR COMBINATIONS OF BOUNDED ELLIPSOIDS

Let $\{A_1, \dots, A_p\}$ be a finite collection of matrices in P_n . Our first rule concerns the convex hull of the ellipsoids $E(A_1), \dots, E(A_p)$.

THEOREM 2.1. (Calculus rule for the convex hull). *The following equality is true:*

$$(2.1) \quad \text{co} \left[\bigcup_{i=1}^p E(A_i) \right] = \bigcup_{\alpha \in \Lambda} E(\alpha_1 A_1 + \dots + \alpha_p A_p).$$

PROOF: Let K be the set on the right hand side of (2.1). According to Lemma 1.1 one can write

$$K = \bigcup_{\alpha \in \Lambda} \bigcup_{\lambda \in \Lambda} \{ \sqrt{\lambda_1} E(\alpha_1 A_1) + \dots + \sqrt{\lambda_p} E(\alpha_p A_p) \}.$$

Taking into account that

$$E(\alpha_i A_i) = \sqrt{\alpha_i} E(A_i) \quad \text{for all } i = 1, \dots, p,$$

one gets

$$K = \bigcup_{\substack{\alpha \in \Lambda \\ \lambda \in \Lambda}} \{ \sqrt{\lambda_1} \sqrt{\alpha_1} E(A_1) + \dots + \sqrt{\lambda_p} \sqrt{\alpha_p} E(A_p) \}.$$

The coefficients α'_i s and λ'_i s play the same role in the above union, so that one can impose on them the additional constraint

$$\alpha_i = \lambda_i \quad \text{for all } i = 1, \dots, p.$$

One gets in this way

$$(2.2) \quad K = \bigcup_{\alpha \in \Lambda} \{ \alpha_1 E(A_1) + \dots + \alpha_p E(A_p) \}.$$

But this is precisely the convex hull of the sets $E(A_1), \dots, E(A_p)$ (see Rockafellar [5, p.80]). □

REMARK 2.1. The convex hull of the ellipsoids $E(A_1), \dots, E(A_p)$ admits the alternative formula

$$(2.3) \quad \text{co} \left[\bigcup_{i=1}^p E(A_i) \right] = \overline{\bigcup_{\alpha \in \Lambda^+} E(\alpha_1 A_1 + \dots + \alpha_p A_p)},$$

where \overline{C} stands for the closure of the set $C \subset \mathbb{R}^n$, and

$$\Lambda^+ := \{ \lambda = (\lambda_1, \dots, \lambda_p) \in \mathbb{R}^p : \lambda_1 + \dots + \lambda_p = 1, \quad \lambda_i > 0 \quad \text{for all } i = 1, \dots, p \}.$$

Formula (2.3) follows from Theorem 2.1 and the fact that the mapping $\lambda \in \Lambda \mapsto E(\lambda_1 A_1 + \dots + \lambda_p A_p)$ is continuous with respect to the Hausdorff distance.

The next rule refers to the operation called inverse addition (see Rockafellar [5, p.21]). Recall that the inverse sum of the ellipsoids $E(A_1), \dots, E(A_p)$ is the set

$$(2.4) \quad E(A_1) \otimes \dots \otimes E(A_p) := \bigcup_{\alpha \in \Lambda} \{ \alpha_1 E(A_1) \cap \dots \cap \alpha_p E(A_p) \} .$$

THEOREM 2.2. (Calculus rule for the inverse sum). *The following formula holds:*

$$(2.5) \quad E(A_1) \otimes \dots \otimes E(A_p) = \bigcup_{\alpha \in \Lambda} E(\alpha_1 A_1 \square \dots \square \alpha_p A_p) .$$

PROOF: The proof of equality (2.5) follows the same steps as in the previous theorem. This time one has to apply, of course, the second formula appearing in Lemma 1.1. □

REMARK 2.2. If $\alpha_i = 0$ for some $i \in \{1, \dots, p\}$, then the parallel sum $\alpha_1 A_1 \square \dots \square \alpha_p A_p$ reduces to the zero matrix, and consequently

$$E(\alpha_1 A_1 \square \dots \square \alpha_p A_p) = \{0\} .$$

This means that the union in (2.5) remains unchanged if one replaces Λ by the smaller set Λ^+ . Thus, one can also write

$$(2.6) \quad E(A_1) \otimes \dots \otimes E(A_p) = \bigcup_{\alpha \in \Lambda^+} E(\alpha_1 A_1 \square \dots \square \alpha_p A_p) .$$

We continue now with a rule for estimating the Minkowski sum of the ellipsoids $E(A_1), \dots, E(A_p)$.

THEOREM 2.3. (Calculus rule for the Minkowski sum). *One has the formula*

$$(2.7) \quad E(A_1) + \dots + E(A_p) = \bigcap_{\alpha \in \Lambda^+} E\left(\frac{1}{\alpha_1} A_1 + \dots + \frac{1}{\alpha_p} A_p\right) .$$

PROOF: Suppose first that the matrices A_1, \dots, A_p are non-singular. If one applies Theorem 2.2 to the collection $\{A_1^{-1}, \dots, A_p^{-1}\}$, one gets

$$E(A_1^{-1}) \otimes \dots \otimes E(A_p^{-1}) = \bigcup_{\alpha \in \Lambda} E(\alpha_1 A_1^{-1} \square \dots \square \alpha_p A_p^{-1}) ,$$

or equivalently,

$$E(A_1^{-1}) \otimes \dots \otimes E(A_p^{-1}) = \bigcup_{\alpha \in \Lambda^+} E(\alpha_1 A_1^{-1} \square \dots \square \alpha_p A_p^{-1}) .$$

Now we compute the polar of each set appearing in the above equality. Recall that the polar C° of a set $C \subset \mathbb{R}^n$ is given by

$$C^\circ := \{z \in \mathbb{R}^n : \langle z, x \rangle \leq 1 \text{ for all } x \in C\}.$$

Applying standard rules for computing polar sets, one gets for the left hand side

$$[E(A_1^{-1}) \otimes \cdots \otimes E(A_p^{-1})]^\circ = [E(A_1^{-1})]^\circ + \cdots + [E(A_p^{-1})]^\circ = E(A_1) + \cdots + E(A_p),$$

and for the right hand side

$$\begin{aligned} \left[\bigcup_{\alpha \in \Lambda^+} E(\alpha_1 A_1^{-1} \square \cdots \square \alpha_p A_p^{-1}) \right]^\circ &= \bigcap_{\alpha \in \Lambda^+} [E(\alpha_1 A_1^{-1} \square \cdots \square \alpha_p A_p^{-1})]^\circ \\ &= \bigcap_{\alpha \in \Lambda^+} E([\alpha_1 A_1^{-1} \square \cdots \square \alpha_p A_p^{-1}]^{-1}) \\ &= \bigcap_{\alpha \in \Lambda^+} E\left(\frac{1}{\alpha_1} A_1 + \cdots + \frac{1}{\alpha_p} A_p\right). \end{aligned}$$

Summarising, we have proven that equality (2.7) holds when the matrices A_1, \dots, A_p are nonsingular. Consider now the singular case. Let K be the set on the right hand side of (2.7). According to Lemma 1.1 one can write

$$\begin{aligned} K &= \bigcap_{\alpha \in \Lambda^+} \bigcup_{\lambda \in \Lambda} \left\{ \sqrt{\lambda_1} E\left(\frac{1}{\alpha_1} A_1\right) + \cdots + \sqrt{\lambda_p} E\left(\frac{1}{\alpha_p} A_p\right) \right\} \\ &= \bigcap_{\alpha \in \Lambda^+} \bigcup_{\lambda \in \Lambda} \left\{ \frac{\sqrt{\lambda_1}}{\sqrt{\alpha_1}} E(A_1) + \cdots + \frac{\sqrt{\lambda_p}}{\sqrt{\alpha_p}} E(A_p) \right\}. \end{aligned}$$

Since

$$\bigcup_{\lambda \in \Lambda} \left\{ \frac{\sqrt{\lambda_1}}{\sqrt{\alpha_1}} E(A_1) + \cdots + \frac{\sqrt{\lambda_p}}{\sqrt{\alpha_p}} E(A_p) \right\} \supset E(A_1) + \cdots + E(A_p)$$

for all $\alpha \in \Lambda^+$, one has the trivial inclusion $K \supset E(A_1) + \cdots + E(A_p)$. To prove the reverse inclusion, we apply formula (2.7) to the nonsingular matrices $\{A_1 + \varepsilon I_n, \dots, A_p + \varepsilon I_n\}$. In this way one gets

$$E(A_1 + \varepsilon I_n) + \cdots + E(A_p + \varepsilon I_n) = \bigcap_{\alpha \in \Lambda^+} E\left(\frac{1}{\alpha_1}(A_1 + \varepsilon I_n) + \cdots + \frac{1}{\alpha_p}(A_p + \varepsilon I_n)\right).$$

But, for all $\alpha \in \Lambda^+$, one has

$$E\left(\frac{1}{\alpha_1}(A_1 + \varepsilon I_n) + \cdots + \frac{1}{\alpha_p}(A_p + \varepsilon I_n)\right) \supset E\left(\frac{1}{\alpha_1} A_1 + \cdots + \frac{1}{\alpha_p} A_p\right).$$

Hence $E(A_1 + \varepsilon I_n) + \cdots + E(A_p + \varepsilon I_n) \supset K$.

To get the desired inclusion, it suffices now to let the parameter $\varepsilon > 0$ tend to zero. \square

Formula (2.7) has been announced in a slightly different form by Kurzhanski and Vályi [2, Lemma 4.1]. These authors gave without proof the following result:

$$(2.8) \quad E(a_1; A_1) + \dots + E(a_p; A_p) = \bigcap_{\pi_i > 0} E(a_1 + \dots + a_p; A(\pi)),$$

where
$$A(\pi) := (\pi_1 + \dots + \pi_p) \left(\frac{1}{\pi_1} A_1 + \dots + \frac{1}{\pi_p} A_p \right).$$

The notation $E(a; A)$ stands for the ellipsoid associated with $A \in P_n$ which is bounded and centered at $a \in \mathbb{R}^n$, that is,

$$E(a; A) := \{z \in \mathbb{R}^n : \langle z, x \rangle \leq \langle a, x \rangle + [(\langle x, Ax \rangle)]^{1/2} \text{ for all } x \in \mathbb{R}^n\} = a + E(A).$$

A simple exercise shows that formulas (2.7) and (2.8) are equivalent.

We state finally a rule for estimating the intersection of the ellipsoids $E(A_1), \dots, E(A_p)$.

THEOREM 2.4. (Calculus rule for the intersection). *The following formula holds:*

$$(2.9) \quad E(A_1) \cap \dots \cap E(A_p) = \bigcap_{\alpha \in \Lambda^+} E\left(\frac{1}{\alpha_1} A_1 \square \dots \square \frac{1}{\alpha_p} A_p\right).$$

PROOF: This theorem can be proven in the same way as the previous one. We consider first the case in which the matrices A_1, \dots, A_p are nonsingular. Theorem 2.1 applied to the collection $\{A_1^{-1}, \dots, A_p^{-1}\}$ yields

$$\text{co} \left[\bigcup_{i=1}^p E(A_i^{-1}) \right] = \bigcup_{\alpha \in \Lambda} E(\alpha_1 A_1^{-1} + \dots + \alpha_p A_p^{-1}),$$

or equivalently,

$$\text{co} \left[\bigcup_{i=1}^p E(A_i^{-1}) \right] = \overline{\bigcup_{\alpha \in \Lambda^+} E(\alpha_1 A_1^{-1} + \dots + \alpha_p A_p^{-1})}.$$

Now we use the technique of computing the corresponding polar sets. One has on the

right hand side

$$\left\{ \text{co} \left[\bigcup_{i=1}^p E(A_i^{-1}) \right] \right\}^\circ = \bigcap_{i=1}^p \{E(A_i^{-1})\}^\circ = \bigcap_{i=1}^p E(A_i),$$

and on the left hand side

$$\begin{aligned} \left\{ \overline{\bigcup_{\alpha \in \Lambda^+} E(\alpha_1 A_1^{-1} + \dots + \alpha_p A_p^{-1})} \right\}^\circ &= \bigcap_{\alpha \in \Lambda^+} \{E(\alpha_1 A_1^{-1} + \dots + \alpha_p A_p^{-1})\}^\circ \\ &= \bigcap_{\alpha \in \Lambda^+} E([\alpha_1 A_1^{-1} + \dots + \alpha_p A_p^{-1}]^{-1}) \\ &= \bigcap_{\alpha \in \Lambda^+} E\left(\frac{1}{\alpha_1} A_1 \square \dots \square \frac{1}{\alpha_p} A_p\right). \end{aligned}$$

Finally, we consider the singular case. Let K be the set on the right hand side of (2.9). Lemma 1.1 yields

$$\begin{aligned} K &= \bigcap_{\alpha \in \Lambda^+} \bigcup_{\lambda \in \Lambda} \left\{ \sqrt{\lambda_1} E\left(\frac{1}{\alpha_1} A_1\right) \cap \dots \cap \sqrt{\lambda_p} E\left(\frac{1}{\alpha_p} A_p\right) \right\} \\ &= \bigcap_{\alpha \in \Lambda^+} \bigcup_{\lambda \in \Lambda} \left\{ \frac{\sqrt{\lambda_1}}{\sqrt{\alpha_1}} E(A_1) \cap \dots \cap \frac{\sqrt{\lambda_p}}{\sqrt{\alpha_p}} E(A_p) \right\}. \end{aligned}$$

One then has the trivial inclusion $K \supset E(A_1) \cap \dots \cap E(A_p)$. To prove the reverse inclusion, we apply formula (2.9) to the collection $\{A_1 + \varepsilon I_n, \dots, A_p + \varepsilon I_n\}$. In this way one gets

$$E(A_1 + \varepsilon I_n) \cap \dots \cap E(A_p + \varepsilon I_n) = \bigcap_{\alpha \in \Lambda^+} E\left(\frac{1}{\alpha_1} (A_1 + \varepsilon I_n) \square \dots \square \frac{1}{\alpha_p} (A_p + \varepsilon I_n)\right).$$

But, for all $\alpha \in \Lambda^+$, one has

$$E\left(\frac{1}{\alpha_1} (A_1 + \varepsilon I_n) \square \dots \square \frac{1}{\alpha_p} (A_p + \varepsilon I_n)\right) \supset E\left(\frac{1}{\alpha_1} A_1 \square \dots \square \frac{1}{\alpha_p} A_p\right).$$

Hence $E(A_1 + \varepsilon I_n) \cap \dots \cap E(A_p + \varepsilon I_n) \supset K$

It suffices now to let $\varepsilon > 0$ tend to zero. □

Kurzhanski and Vályi [2, Lemma 4.2] gave a different estimate for the intersection of a finite family of ellipsoids. Their formula reads as follows:

(2.10)

$$\bigcap_{i=1}^p E(a_i; A_i) = \bigcap_{\substack{B_1, \dots, B_p \\ B_1 + \dots + B_p = I_n}} \{E(B_1 a_1; B_1 A_1 B_1^T) + \dots + E(B_p a_p; B_p A_p B_p^T)\}.$$

If the ellipsoids are centered at the origin, then (2.10) takes the form

$$(2.11) \quad \bigcap_{i=1}^p E(A_i) = \bigcap_{\substack{B_1, \dots, B_p \\ B_1 + \dots + B_p = I_n}} \{E(B_1 A_1 B_1^T) + \dots + E(B_p A_p B_p^T)\} .$$

Let us prove briefly that the complicated expression appearing on the right hand side of (2.11) reduces to the simpler estimate given in Theorem 2.4. If one applies Theorem 2.3 to the collection $\{B_1 A_1 B_1^T, \dots, B_p A_p B_p^T\}$, one obtains

$$\bigcap_{i=1}^p E(A_i) = \bigcap_{\substack{B_1, \dots, B_p \\ B_1 + \dots + B_p = I_n}} \bigcap_{\alpha \in \Lambda^+} E\left(\frac{1}{\alpha_1} B_1 A_1 B_1^T + \dots + \frac{1}{\alpha_p} B_p A_p B_p^T\right) ,$$

or equivalently

$$\bigcap_{i=1}^p E(A_i) = \bigcap_{\alpha \in \Lambda^+} \bigcap_{\substack{B_1, \dots, B_p \\ B_1 + \dots + B_p = I_n}} E\left(B_1 \left(\frac{1}{\alpha_1} A_1\right) B_1^T + \dots + B_p \left(\frac{1}{\alpha_p} A_p\right) B_p^T\right) .$$

It suffices now to apply Lemma 1.2 to obtain the formula given in Theorem 2.4.

3. APPLICATION: ELLIPSOIDAL BOUNDS FOR A CONVEX POLYTOPE

Theorems 2.1. and 2.3 can be used to construct inner and outer ellipsoidal approximations of a convex polytope which admits the representation

$$L(R) := \{Lx : x \in R\} .$$

Here $L : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a linear mapping and R stands for the bounded rectangular box

$$R = [a_1, b_1] \times \dots \times [a_n, b_n] , \quad \text{with } a_i \leq b_i \quad \text{for all } i = 1, \dots, n .$$

For notational convenience, L is identified with the corresponding matrix relative to the canonical basis $\{e_1, \dots, e_n\}$ of \mathbb{R}^n , and vectors of \mathbb{R}^n are written in the column form. The rectangular box R admits the expression $R = c_R + D_R$, where

$$c_R := \left(\frac{a_1 + b_1}{2}, \dots, \frac{a_n + b_n}{2}\right)^T$$

corresponds to its centre, and

$$D_R := [-d_1, d_1] \times \dots \times [-d_n, d_n] , \quad \text{with } d_i := (b_i - a_i)/2 \quad \text{for all } i = 1, \dots, n ,$$

is a translation of R centered at the origin. The next proposition is a consequence of the fact that D_R can be written as a Minkowski sum of ellipsoids. To see this, observe

that the closed segment joining an arbitrary vector $v \in \mathbb{R}^n$ and its opposite $-v$ is a flat ellipsoid. Indeed, it is the ellipsoid associated with the rank one matrix vv^T , that is,

$$\text{co}\{v, -v\} = E(vv^T) \quad \text{for all } v \in \mathbb{R}^n.$$

Thus $D_R = \text{co}\{d_1 e_1, -d_1 e_1\} + \dots + \text{co}\{d_n e_n, -d_n e_n\}$ can be written in the form

$$(3.1) \quad D_R = E(d_1^2 e_1 e_1^T) + \dots + E(d_n^2 e_n e_n^T).$$

To underline the number of ellipsoids involved in the representation of D_R , we use the notation

$$\Lambda_n^+ := \{\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n : \lambda_1 + \dots + \lambda_n = 1 \text{ and } \lambda_i > 0 \text{ for all } i = 1, \dots, n\}.$$

PROPOSITION 3.1. (Outer ellipsoidal approximation). *Let $\ell_i = Le_i$ denote the i^{th} column of L . Then, for each $\lambda \in \Lambda_n^+$, one has the upper bound*

$$(3.2) \quad L(R) \subset Lc_R + E\left(\frac{d_1^2}{\lambda_1} \ell_1 \ell_1^T + \dots + \frac{d_n^2}{\lambda_n} \ell_n \ell_n^T\right).$$

If L is invertible, one has more precisely

$$(3.3) \quad L(R) = Lc_R + \bigcap_{\lambda \in \Lambda_n^+} E\left(\frac{d_1^2}{\lambda_1} \ell_1 \ell_1^T + \dots + \frac{d_n^2}{\lambda_n} \ell_n \ell_n^T\right).$$

PROOF: Theorem 2.3 applied to the Minkowski sum (3.1) yields

$$D_R = \bigcap_{\lambda \in \Lambda_n^+} E\left(\frac{d_1^2}{\lambda_1} e_1 e_1^T + \dots + \frac{d_n^2}{\lambda_n} e_n e_n^T\right).$$

From this and the equality $L(R) = Lc_R + L(D_R)$, one deduces that

$$(3.4) \quad L(R) \subset Lc_R + \bigcap_{\lambda \in \Lambda_n^+} L\left[E\left(\frac{d_1^2}{\lambda_1} e_1 e_1^T + \dots + \frac{d_n^2}{\lambda_n} e_n e_n^T\right)\right].$$

To prove the inclusion (3.2), it suffices now to apply the general formula

$$L(E(A)) = E(LAL^T) \quad \text{for all } A \in P_n.$$

If L is invertible, then the inclusion (3.4) becomes an equality. □

To obtain an inner ellipsoidal approximation of $L(R)$, we write this time

$$(3.5) \quad D_R = \text{co}[\cup\{E(vv^T) : v \text{ is a vertex of } D_R\}]$$

as a convex hull of ellipsoids. The 2^n vertices of D_R are vectors of the form $(\pm d_1, \dots, \pm d_n)^T$. Since $E(vv^T) = E((-v)(-v)^T)$, it suffices to take only half of them in the representation (3.5) (for instance, all the vertices with the sign + preceding the first component d_1). Theorem 2.1 yields in this case:

PROPOSITION 3.2. (Inner ellipsoidal approximation). *Let $\{v_1, \dots, v_p\}$ be a collection of $p = 2^{n-1}$ vertices of D_R . Suppose this collection does not contain two opposite vertices. Denote the elementary simplex in \mathbb{R}^p by Λ_p , and set $u_i := Lv_i$ for all $i = 1, \dots, p$.*

$$\text{Then} \quad L(R) = Lc_R + \bigcup_{\lambda \in \Lambda_p} E(\lambda_1 u_1 u_1^T + \dots + \lambda_p u_p u_p^T).$$

In particular, for each $\lambda \in \Lambda_p$, one has the inner bound

$$Lc_R + E(\lambda_1 u_1 u_1^T + \dots + \lambda_p u_p u_p^T) \subset L(R).$$

PROOF: Start by applying Theorem 2.1 to the convex hull

$$D_R = \text{co} \left[\bigcup_{i=1}^p E(v_i v_i^T) \right].$$

The remaining part of the proof is similar to that of Proposition 3.1. The sole difference is that this time L has to be permuted with a union symbol (instead of an intersection). This is always possible, even if L is not invertible. \square

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