A necessary and sufficient condition for differentiability

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The familiar Lemma introduced by Goursat in his proof of Cauchy's theorem suggests the following necessary and sufficient condition for differentiability of a complex function $f(z)$.

THEOREM 1. A complex function $f(z)$ is differentiable in the *interior* of a square S, of side a, if

1.1 corresponding to any positive ϵ there is a subdivision of S into a finite number of squares s_r, with sides δ_r, such that, for each value of r, there is a number A_r and

$$| f(z) - f(z') - A_r(z - z') | < \epsilon\delta_r$$

at all points z, z' inside or on the boundary of the square s_r.

Conversely 1.1 holds if $f(z)$ is differentiable inside, and on the boundary of, S.

1.2 The converse follows from Goursat's Lemma, for if $f(z)$ is differentiable in a *closed* square S, then given $\epsilon > 0$, $\kappa > 0$, there is a subdivision of S into a finite number of squares s_r, of sides δ_r, such that in each s_r there is a point z_r and

$$| f(z) - f(z_r) - f'(z_r)(z - z_r) | < \kappa\epsilon \, | z - z_r |$$

for all points z in the closed square s_r. Whence, since $| z - z_r | \leqq \delta_r \sqrt{2}$, taking $\kappa = 1/(2\sqrt{2})$ we have

$$| f(z) - f(z') - f'(z_r)(z - z') |$$
$$= | \{f(z) - f(z_r) - f'(z_r)(z - z_r)\} - \{f(z') - f(z_r) - f'(z_r)(z' - z_r)\} | < \epsilon\delta_r.$$

1.3 If 1.1 holds then $f(z)$ is continuous inside and on the boundary of S; for if z, z' are any two points in the closed square S, such that $| z - z' | < \min\{\delta_r, \epsilon / | A_r | \}$ if $A_r \neq 0$, and $| z - z' | < \min\{\delta_r\}$ if $A_r = 0$, then z, z' lie either in the same square s_μ or in adjoining squares s_μ, s_ν. In the former case

$$| f(z) - f(z') | < | A_\mu | \, | z - z' | + \epsilon\delta_\mu < (a + 1)\epsilon;$$

in the latter case, let ζ, ζ' be the points on which the line joining z to z' meets the boundaries of s_μ, s_ν respectively; then ζ, ζ' (which may coincide) lie in the same square and so

$$| f(z) - f(z') | = | f(z) - f(\zeta) + f(\zeta) - f(\zeta') + f(\zeta') - f(z') | < 3(a + 1)\epsilon.$$

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1.4 We prove the sufficiency of the condition 1.1 by appeal to Morera's theorem. Let C be any simple closed contour in S, of length L and bounding a region R. We denote by $s_1, s_2, \ldots, s_\mu$ the squares of the subdivision s_r which are completely contained in the closed region R, and by $s_{\mu+1}, s_{\mu+2}, \ldots, s_{\mu+\nu}$ the subsquares which are intersected by C. Further, let σ_r be the contour formed by the parts of the boundary of s_r contained in R together with the arcs c_r of C which are contained in s_r, and choose any point z_r in the common part of s_r and R.

Writing $\epsilon_r(z) = f(z) - f(z_r) - A_r(z - z_r)$ we have

$$\left| \int_{s_r} f(z)dz \right| = \left| \int_{s_r} \epsilon_r(z)dz \right| < 4\epsilon\delta_r^2, \ 1 \leqq r \leqq \mu,$$

and $\left| \int_{\sigma_r} f(z)dz \right| < \epsilon\delta_r(4\delta_r + l_r)$, where l_r is the length of c_r,

$$< 4\epsilon\delta_r^2 + a\epsilon l_r, \qquad \mu + 1 \leqq r \leqq \mu + \nu.$$

Hence $\left| \int_C f(z)dz \right| < 4\epsilon \sum_{r=1}^{\mu+\nu} \delta_r^2 + aL\epsilon \leqq \epsilon(4a^2 + aL)$;

but ϵ is arbitrary, and so $\int_C f(z)dz = 0$, whence by Morera's theorem, $f(z)$ is differentiable in the interior of S.

THEOREM 1*. A function $f(z)$ is differentiable in the interior of a square if

1.5 there is a constant κ, and corresponding to any positive ϵ there is a subdivision of the square into a finite number of squares s_r with sides δ_r, such that for each r there is a number A_r, a positive integer p_r, and a point z_r in s_r satisfying

1.51 $$| f(z) - f(z_r) - A_r(z - z_r)^{p_r} | < \kappa\epsilon\delta_r$$

for all points z inside and on the boundary of the square s_r.

The proof of Theorem 1* is the same as the proof of Theorem 1, § 1.4. We observe that if a function $f(z)$ satisfies 1.5 for all sufficiently small values of ϵ then it satisfies 1.5 for all values of ϵ, for a subdivision in which 1.51 holds for some one ϵ, is *a fortiori* a subdivision in which 1.51 holds for any greater ϵ, leaving A_r and δ_r unchanged.

Theorem 1* appears to be of no intrinsic interest and is introduced with a view to its application in the following rather curious result.

THEOREM 2. A function $f(z)$ is defined in a square S, of side a. Corresponding to any point z_0 in S and any $\epsilon > 0$ there are numbers $A = A(\epsilon, z_0)$, $\delta = \delta(\epsilon, z_0) > 0$ and an integer $p = p(z_0) > 1$ such that

$$| f(z) - f(z_0) - A(z - z_0)^p | < \epsilon\delta$$

at all points z of S which lie in the circle $| z - z_0 | = \delta$.

If, as $\epsilon \to 0$, $\delta(\epsilon, z)$ is bounded by M uniformly in z, then $f(z)$ is constant inside and on the boundary of S.

2.1 We prove first that $f(z)$ satisfies the conditions 1.5, with $\kappa = \max\{M/a, 4\sqrt{2}\}$, *inside* and on the boundary of S. Choose ϵ_0 so that $\delta(\epsilon, z) < M$ for all $\epsilon < \epsilon_0$ and all z in S. If there is an ϵ, less than ϵ_0, for which 1.5 is *not* satisfied, we may by repeated subdivision determine a point z_0 (inside or on the boundary of S) and a square T_n, of side $a/2^n$, which contains z_0 and for which 1.5 is not satisfied for this ϵ. Corresponding to this ϵ there is a δ such that

$$| f(z) - f(z_0) - A(z - z_0)^p | < \epsilon\delta$$

at all points z in the circle $| z - z_0 | = \delta$.

If $\delta = \delta(\epsilon, z_0)$ exceeds $a\sqrt{2}$ then the circle $| z - z_0 | = \delta$ completely contains the square S, and so, since $\delta < M$,

$$| f(z) - f(z_0) - A(z - z_0)^p | < \epsilon\delta < (M/a)\epsilon a$$

at all points z of S, which contradicts the hypothesis that 1.5 is not satisfied for this value of ϵ.

If $\delta \leqq a\sqrt{2}$, and if m is the *least* integer such that $2^m \geqq a\sqrt{2}/\delta$, so that the circle $| z - z_0 | = \delta$ completely contains the square T_{m+1} of side $c = a/2^{m+1}$, then in T_{m+1}

$$| f(z) - f(z_0) - A(z - z_0)^p | < \epsilon\delta < \epsilon a\sqrt{2}/2^{m-1} = 4\sqrt{2}\epsilon c,$$

so that 1.5 is satisfied for T_{m+1} for the given ϵ — a contradiction. Hence $f(z)$ satisfies 1.5 for all $\epsilon < \epsilon_0$, and therefore $f(z)$ is differentiable in the interior of S.

2.2 Let z_0 be an interior point of S, ϵ_n a null sequence, and $\delta_n = \delta(\epsilon_n, z_0)$; then either δ_n takes arbitrarily small values or δ_n has a lower bound $\lambda > 0$.

2.21 If $\delta_n \geqq \lambda > 0$ for all n, then

$$| f(z) - f(z_0) - (z - z_0)^p A(\epsilon_n, z_0) | < \epsilon_n\delta_n < M\epsilon_n$$

for all z in the circle $| z - z_0 | = \lambda$. Hence $A(\epsilon_n, z_0)$ converges to a

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limit $l(z_0)$, say, and $f(z)-f(z_0)=(z-z_0)^p l(z_0)$ at all points of the circle $|z-z_0|=\lambda$, so that

$$\{f(z)-f(z_0)\}/(z-z_0)=(z-z_0)^{p-1}l(z_0)\to 0$$

as $z\to z_0$, that is, $f'(z_0)=0$.

2.22 If δ_n takes arbitrarily small values, we can find ϵ_{n_r}, a sub-sequence of ϵ_n, such that δ_{n_r} is less than some assigned η for all r. Let ρ be the minimum distance of z_0 from the boundary of S, take $\eta<\rho$, and let γ be the circle, centre z_0, radius δ_{n_r}. Write

$$\epsilon(\omega)=f(\omega)-f(z_0)-A(\omega-z_0)^p;$$

then, since γ is completely contained in S, $|\epsilon(\omega)|<\epsilon_{n_r}\delta_{n_r}$ at all points of γ. Hence

$$|f'(z_0)|=\left|\frac{1}{2\pi i}\int_\gamma\frac{f(\omega)d\omega}{(\omega-z_0)^2}\right|=\frac{1}{2\pi}\left|\int_\gamma\frac{\epsilon(\omega)d\omega}{(\omega-z_0)^2}\right|\leqq\epsilon_{n_r}$$

and so $f'(z_0)=0$.

2.3 Accordingly $f'(z_0)=0$ at all points z_0 interior to S, so that $f(z)$ is constant in the interior of S. Since $f(z)$ satisfies 1.5 for S, it follows as in § 1.3 that $f(z)$ is continuous in the closed square, and so $f(z)$ is constant in the closed square.

A referee, to whom I am indebted for a number of valuable suggestions on the presentation of this note, drew my attention to the following generalisation of Theorem 2.

In the inequality 1.51 we may replace $A_r(z-z_r)^{pr}$ by $(z-z_r)^2 A_r(z,\epsilon)$, where, for each r, $A_r(z,\epsilon)$ is an analytic function of z, for all values of z_r and ϵ, and in Theorem 2, $A(z-z_0)^p$ may be replaced by $(z-z_0)^2h(z,z_0,\epsilon)$, where $h(z,z_0,\epsilon)$ is an analytic function of z for all values of ϵ and all z_0 in S.

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