COTORSION PAIRS AND MODEL STRUCTURES ON Ch(R)

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Abstract   We show that if the given cotorsion pair $(\mathcal{A}, \mathcal{B})$ in the category of modules is complete and hereditary, then both of the induced cotorsion pairs in the category of complexes are complete. We also give a cofibrantly generated model structure that can be regarded as a generalization of the projective model structure.

1. Introduction

For an associative ring R with 1, the category Ch(R) of complexes has two well-known model category structures with weak equivalences being the homology isomorphisms. The 'projective' model structure is characterized by having the fibrations being all epimorphisms and the cofibrations being the monomorphisms with dg-projective cokernels. The dual 'injective' model structure has the cofibrations being the monomorphisms and the fibrations being the epimorphisms with dg-injective kernels. In 2004, Gillespie [**17**] used Hovey's Theorem 2.2 [**21**], which relates complete cotorsion pairs in Ch(R) to model structures on Ch(R) to get the 'flat' model structure. Gillespie [**17**] proved in a general way that any hereditary cotorsion pair in R-Mod induces two cotorsion pairs in Ch(R) for which Hovey's Theorem can apply *if* both of the induced cotorsion pairs are complete. Thus, Gillespie put the open question of whether or not the induced cotorsion pairs are complete when the original cotorsion pair is complete. Gillespie followed a method

analogous to one in [**6**] to show that the cotorsion pairs in $\mathrm{Ch}(R)$ induced by the flat cotorsion pair are both complete. In the current paper we give a positive answer to this question when the original cotorsion pair is complete and hereditary as well. Later, in [**18**], Gillespie generalized flat model structure on $\mathrm{Ch}(R)$ to the category of complexes of quasi-coherent sheaves on a quasi-compact, semi-separated scheme X. Furthermore, he pointed out that the flat model structure on $\mathrm{Ch}(R)$ is cofibrantly generated [**18**, Corollary 7.2]. The cofibrantly generated property is very important for a model structure and is studied in [**18**, **19**, **21**]. So we are also motivated to consider the cofibrantly generated property of a model structure on $\mathrm{Ch}(R)$ in this paper.

The paper is structured as follows. Section 2 provides relevant definitions and notation which will be used throughout the paper. In §3, we will show that if the given cotorsion pair $(\mathcal{A}, \mathcal{B})$ in the category of modules is complete and hereditary, then both of the induced cotorsion pairs in the category of complexes are complete, which gives a positive answer to the open question of Gillespie [**17**]. In §4, we show that if $(\mathcal{A}, \mathcal{B}) = (\mathcal{D}, \mathcal{D}^{\perp})$ with $\mathcal{D}$ the class of all modules of projective dimension less than or equal to n, n a fixed nonnegative integer, then the associated model structure on $\mathrm{Ch}(R)$ is cofibrantly generated. This model structure can in fact be regarded as a generalization of the projective one.

2. Preliminaries

In this section, we give some relevant definitions and some notation for later use.

Cotorsion pairs were invented by Salce [**23**] in the category of abelian groups, and were rediscovered by Enochs and coauthors in the 1990s. Given a class $\mathcal{H}$ of objects in an abelian category $\mathcal{C}$, we will denote by $\mathcal{H}^{\perp}$ (respectively, $^{\perp}\mathcal{H}$) the right orthogonal (respectively, left orthogonal) class of objects X such that $\mathrm{Ext}^1(H, X) = 0$ (respectively, $\mathrm{Ext}^1(X, H) = 0$) for every $H \in \mathcal{H}$. Now using this notation, we recall from [**15**] that a pair of classes of objects $(\mathcal{A}, \mathcal{B})$ is said to be a *cotorsion pair* if $\mathcal{A}^{\perp} = \mathcal{B}$ and $^{\perp}\mathcal{B} = \mathcal{A}$. A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is said to be *cogenerated* by a set $\mathcal{S} \subseteq \mathcal{A}$ whenever $B \in \mathcal{B}$ if and only if $\mathrm{Ext}^1(S, B) = 0$ for all $S \in \mathcal{S}$. A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is called *hereditary*, if the class $\mathcal{A}$ is closed under taking kernels of epimorphisms, or, equivalently, the class $\mathcal{B}$ is closed under taking cokernels of monomorphisms. Also recall that a cotorsion pair $(\mathcal{A}, \mathcal{B})$ is said to be *complete* if it has enough injectives and projectives, that is, for any object X of $\mathcal{C}$ there are exact sequences $0 \longrightarrow X \longrightarrow B \longrightarrow A \longrightarrow 0$ and $0 \longrightarrow B' \longrightarrow A' \longrightarrow X \longrightarrow 0$, respectively, with $B, B' \in \mathcal{B}$ and $A, A' \in \mathcal{A}$. In the paper we will mainly consider cotorsion pairs in the category R-Mod of R-modules and the category $\mathrm{Ch}(R)$ of complexes of R-modules.

Recall from [**13**] that if $\mathcal{H}$ is a class of objects in an abelian category $\mathcal{C}$ and $X \in \mathcal{C}$, then an $\mathcal{H}$-*pre-envelope* of X is a morphism $f \colon X \to H$ with $H \in \mathcal{H}$, such that the triangle

$$
\begin{array}{ccc}
H' & & \\
\uparrow & \nwarrow & \\
X & \xrightarrow{\ f\ } & H
\end{array}
$$

can be completed for each morphism $X \to H'$ with $H' \in \mathcal{H}$. An $\mathcal{H}$-pre-envelope $f \colon X \to H$ is called *special* if $\mathrm{Ext}^1(\mathrm{Coker}(f), G) = 0$ for all $G \in \mathcal{H}$. If the triangle

can be completed only by isomorphisms, then f is called an $\mathcal{H}$-*envelope*. (Special) $\mathcal{H}$-*precovers* and $\mathcal{H}$-*covers* are defined dually.

Recall that a *model category* is a category that we shall assume has all limits and colimits, together with three subcategories (the *model structure*), called the *weak equivalences*, *cofibrations* and *fibrations*, that must satisfy various axioms. Since our results are about model categories, we assume that the reader is familiar with and interested in model categories. However, if one believes [**21**, Theorem 2.2] then one really does not need to know anything about model categories to understand the paper. A nice introduction to the basic idea of a model category can be found in [**9**, **20**].

Throughout this paper, let R be an associative ring with 1, R-Mod the category of left R-modules and $\mathrm{Ch}(R)$ the category of complexes of left R-modules. Let $\mathrm{Hom}(A, B)$ denote the set of all morphisms from A to B and let $\mathrm{Ext}^i(A, B)$ denote the right-derived functors of Hom. To every complex

$$C = \cdots \xrightarrow{\partial^C_{m+1}} C_m \xrightarrow{\partial^C_m} C_{m-1} \xrightarrow{\partial^C_{m-1}} C_{m-2} \xrightarrow{\partial^C_{m-2}} \cdots$$

in $\mathrm{Ch}(R)$ we associate the numbers

$$\sup C = \sup\{l \mid C_l \neq 0\} \quad \text{and} \quad \inf C = \inf\{l \mid C_l \neq 0\}.$$

The complex C is called 'bounded above' when $\sup C < \infty$, 'bounded below' when $\inf C > -\infty$ and 'bounded' when it is bounded below and above. The mth cycle module is defined as $\mathrm{Ker}(\partial^C_m)$ and is denoted by $Z_m C$. The mth boundary module is $\mathrm{Im}(\partial^C_{m+1})$ and is denoted by $B_m C$. The mth homology module of C is the module $H_m(C) = Z_m C / B_m C$. Given a left R-module K, we will denote by $D^i(K)$ the complex $\cdots \longrightarrow 0 \longrightarrow K \xrightarrow{id} K \longrightarrow 0 \longrightarrow \cdots$ with K in the i and $(i-1)$th positions and $S^i(K)$ the complex $\cdots \longrightarrow 0 \longrightarrow K \longrightarrow 0 \longrightarrow \cdots$ with K in the ith position.

In the following discussion, M and N denote complexes of left R-modules.

A homomorphism $\varphi \colon M \to N$ of degree m is a family $(\varphi_i)_{i \in \mathbb{Z}}$ of homomorphisms of R-modules $\varphi_i \colon M_i \to N_{i+m}$. All such homomorphisms form an abelian group, denoted $\mathcal{H}om(M, N)_m$; it is clearly isomorphic to $\prod_{i \in \mathbb{Z}} \mathrm{Hom}_R(M_i, N_{i+m})$. We will let $\mathcal{H}om(M, N)$ denote the complex of $\mathbb{Z}$-modules with mth component $\mathcal{H}om(M, N)_m$ and differential

$$(\partial(\varphi))_i = \partial^N_{i+m} \varphi_i - (-1)^m \varphi_{i-1} \partial^M_i.$$

A homomorphism $\varphi \in \mathcal{H}om(M, N)_m$ is called a *chain map* if $\partial(\varphi) = 0$, i.e. if

$$\partial^N_{i+m} \varphi_i = (-1)^m \varphi_{i-1} \partial^M_i \quad \text{for all } i \in \mathbb{Z}.$$

A chain map of degree 0 is called a *morphism*. A morphism $f\colon M \to N$ is called a *homology isomorphism* if the induced morphisms $H_m(f)\colon H_m(M) \to H_m(N)$ are isomorphisms for all $m \in \mathbb{Z}$. For more terminologies about complexes the reader can consult [**7**, **16**].

Given an ordinal number λ and a family $(X_\alpha)_{\alpha < \lambda}$ of subcomplexes of an R-complex X, recall that the family $(X_\alpha)_{\alpha < \lambda}$ is called a *continuous chain of subcomplexes* [**12**, Definition 2.8] if $X_\alpha \subseteq X_\beta$ whenever $\alpha \leqslant \beta < \lambda$ and if $X_\beta = \bigcup_{\alpha < \beta} X_\alpha$ whenever $\beta < \lambda$ is a limit ordinal. A family $(X_\alpha)_{\alpha \leqslant \lambda}$ is called a continuous chain if $(X_\alpha)_{\alpha < \lambda+1}$ is a continuous chain.

3. Cotorsion pairs and model structures on $\mathrm{Ch}(R)$

In this section, we mainly study the completeness of cotorsion pairs in $\mathrm{Ch}(R)$ induced by a cotorsion pair in R-Mod.

Recall from [**14**] that a complex P is said to be *dg-projective* if each P_m is projective and $\mathcal{H}om(P, E)$ is exact for any exact complex E. A *dg-injective* complex is defined dually. Gillespie [**17**, Definition 3.3] introduced the following definitions, which generalize the notions of *dg-projective* and *dg-injective* complexes.

Definition 3.1 (Gillespie [17, Definition 3.3]). Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair in R-Mod and let X be an R-complex.

1. X is called an $\mathcal{A}$ complex if it is exact and $Z_i X \in \mathcal{A}$ for all $i \in \mathbb{Z}$.

2. X is called a $\mathcal{B}$ complex if it is exact and $Z_i X \in \mathcal{B}$ for all $i \in \mathbb{Z}$.

3. X is called a dg-$\mathcal{A}$ complex if $X_i \in \mathcal{A}$ for all $i \in \mathbb{Z}$, and $\mathcal{H}om(X, B)$ is exact whenever B is a $\mathcal{B}$ complex.

4. X is called a dg-$\mathcal{B}$ complex if $X_i \in \mathcal{B}$ for all $i \in \mathbb{Z}$, and $\mathcal{H}om(A, X)$ is exact whenever A is an $\mathcal{A}$ complex.

We denote the class of $\mathcal{A}$ complexes by $\tilde{\mathcal{A}}$ and the class of dg-$\mathcal{A}$ complexes by $dg\tilde{\mathcal{A}}$. Similarly, the $\mathcal{B}$ complexes are denoted by $\tilde{\mathcal{B}}$ and the dg-$\tilde{\mathcal{B}}$ complexes are denoted by $dg\tilde{\mathcal{B}}$. Clearly, any dg-projective complex is in $dg\tilde{\mathcal{A}}$ and any dg-injective complex is in $dg\tilde{\mathcal{B}}$.

For later use we give the following lemma, which follows directly from [**17**, Proposition 3.6] and [**17**, Corollary 3.13].

Lemma 3.2. *If $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair in R-Mod, then $(\tilde{\mathcal{A}}, dg\tilde{\mathcal{B}})$ and $(dg\tilde{\mathcal{A}}, \tilde{\mathcal{B}})$ are cotorsion pairs in $\mathrm{Ch}(R)$.*

If $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair in R-Mod, then we call $(\tilde{\mathcal{A}}, dg\tilde{\mathcal{B}})$ and $(dg\tilde{\mathcal{A}}, \tilde{\mathcal{B}})$ induced cotorsion pairs by $(\mathcal{A}, \mathcal{B})$.

The next two lemmas play an important role in proving our main result.

Lemma 3.3. *Suppose that $(\mathcal{A}, \mathcal{B})$ is a complete and hereditary cotorsion pair in R-Mod and $0 \to X_1 \to X_2 \to X_3 \to 0$ a short exact sequence of R-modules. If $f_i\colon A_i \to X_i$ is a special $\mathcal{A}$-precover of X_i for $i = 1, 3$, then there exists a commutative diagram (see Figure 1) with exact rows and columns such that $f_2\colon A_2 \to X_2$ is a special $\mathcal{A}$-precover of X_2. Similarly, the dual version of this result holds.*
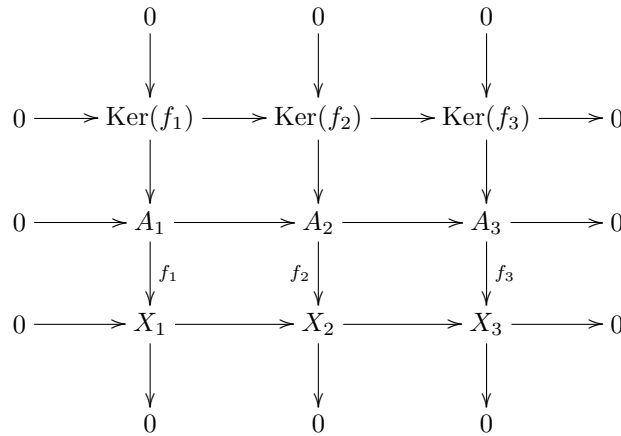
Figure 1.

Proof. See [**1**, Theorem 3.1] or [**8**, Theorem 3] for the proof of the first part. The second part of this result can be proved dually.   □

Lemma 3.4. *Suppose that* $(\mathcal{A}, \mathcal{B})$ *is a complete and hereditary cotorsion pair in* R-Mod. *Then every exact complex admits a special* $\tilde{\mathcal{A}}$-*precover and a special* $\tilde{\mathcal{B}}$-*pre-envelope.*

Proof. Let E be an exact complex. Then we have short exact sequences of R-modules:

$$0 \longrightarrow Z_i E \longrightarrow E_i \longrightarrow Z_{i-1} E \longrightarrow 0.$$

On the one hand, there exists a special $\mathcal{A}$-precover $f'_k \colon A'_k \to Z_k E$ of the module $Z_k E$ for each $k \in \mathbb{Z}$ by hypothesis, and on the other hand it follows from Lemma 3.3 that for each $i \in \mathbb{Z}$ there exists a commutative diagram (see Figure 2) with exact rows and columns such that $f_i \colon A_i \to E_i$ is a special $\mathcal{A}$-precover of E_i. Now if we put

$$A =: \cdots \longrightarrow A_{i+1} \xrightarrow{\partial^A_{i+1}} A_i \xrightarrow{\partial^A_i} A_{i-1} \longrightarrow \cdots$$

with $\partial^A_i = \mu_{i-1}\nu_{i-1}$, then the morphism $f = (f_i)_{i \in \mathbb{Z}} \colon A \to E$ is a special $\tilde{\mathcal{A}}$-precover of the complex E. In fact, by the construction above, A is in $\tilde{\mathcal{A}}$, the morphism f is epic and $\mathrm{Ker}(f) \in \tilde{\mathcal{B}}$. We get from [**17**, Proposition 3.6] and [**17**, Lemma 3.10] that $\mathrm{Ext}^1(X, Y) = 0$ for any $X \in \tilde{\mathcal{A}}$ and any $Y \in \tilde{\mathcal{B}}$, in particular, $\mathrm{Ext}^1(X, \mathrm{Ker}(f)) = 0$ for any $X \in \tilde{\mathcal{A}}$. Thus, the sequence

$$0 \longrightarrow \mathrm{Hom}(X, \mathrm{Ker}(f)) \longrightarrow \mathrm{Hom}(X, A) \longrightarrow \mathrm{Hom}(X, E) \longrightarrow 0$$

is exact for any $X \in \tilde{\mathcal{A}}$, and so $f \colon A \to E$ is a special $\tilde{\mathcal{A}}$-precover of the complex E.

Dually, we can prove any exact complex admits a special $\tilde{\mathcal{B}}$-pre-envelope.   □

There are many cotorsion pairs in R-Mod to satisfy the complete and hereditary properties [**2**, **22**, **24**]. So the following theorem, which is our main result in this section, gives a positive answer to the open question of Gillespie [**17**], as mentioned in § 1.

Figure 2.

Theorem 3.5. *Suppose that $(\mathcal{A}, \mathcal{B})$ is a complete cotorsion pair in R-Mod. If the cotorsion pair $(\mathcal{A}, \mathcal{B})$ is hereditary, then the induced cotorsion pairs $(\tilde{\mathcal{A}}, dg\tilde{\mathcal{B}})$ and $(dg\tilde{\mathcal{A}}, \tilde{\mathcal{B}})$ in $\mathrm{Ch}(R)$ are both complete.*

Proof. We need only to show the cotorsion pair $(\tilde{\mathcal{A}}, dg\tilde{\mathcal{B}})$ is complete, because the completeness of the cotorsion pair $(dg\tilde{\mathcal{A}}, \tilde{\mathcal{B}})$ can be proved dually.

Let X be any complex. Then by [**16**, Theorem 2.2.4] there exists an exact sequence

$$0 \longrightarrow I \xrightarrow{g} E \xrightarrow{f} X \longrightarrow 0$$

such that $f\colon E \to X$ is a special exact precover of X, that is, E is exact and I is dg-injective, where $g\colon I \to E$ is a natural injection. By Lemma 3.4, we have an exact sequence

$$0 \longrightarrow B \longrightarrow A \xrightarrow{\alpha} E \longrightarrow 0$$

with $\alpha\colon A \to E$ a special $\tilde{\mathcal{A}}$-precover of E, that is, A is in $\tilde{\mathcal{A}}$ and B is in $\tilde{\mathcal{B}}$. Now consider the pullback diagram (Figure 3) of morphisms $\alpha\colon A \to E$ and $g\colon I \to E$. By [**17**, Lemma 3.10], we have $B \in dg\tilde{\mathcal{B}}$. Moreover, since I is obviously in $dg\tilde{\mathcal{B}}$ and the class $dg\tilde{\mathcal{B}}$ is closed under extensions, we get $U \in dg\tilde{\mathcal{B}}$. This implies that the above morphism $A \to X$ is a special $\tilde{\mathcal{A}}$-precover of X. Thus, $(\tilde{\mathcal{A}}, dg\tilde{\mathcal{B}})$ has enough projectives. On the other hand, the category $\mathrm{Ch}(R)$ of complexes of modules has enough projectives and enough injectives, and so the cotorsion pair $(\tilde{\mathcal{A}}, dg\tilde{\mathcal{B}})$ has enough injectives by [**15**, Proposition 1.1.5]. This completes the proof. $\qquad\square$

If we denote by $\mathcal{E}$ the class of all exact complexes of R-modules, it has been proved that the class $^{\perp}\mathcal{E}$ is that of all dg-projective complexes and the class $\mathcal{E}^{\perp}$ of all dg-injective complexes [**14**]. The existence of dg-projective precovers and dg-injective pre-envelopes has been studied and proved [**3, 14, 16**]. These results have played an important role in extending the notions of projective and injective dimensions from modules to unbounded

Figure 3.

complexes [**5**]. Here we will apply the techniques developed in the present paper to give very short and easy proofs of the existence of such precovers and pre-envelopes.

Corollary 3.6. *The cotorsion pairs* $(^\perp\mathcal{E}, \mathcal{E})$ *and* $(\mathcal{E}, \mathcal{E}^\perp)$ *are both complete. In particular, every complex has a special dg-projective precover and a special dg-injective pre-envelope.*

Proof. Let $(\mathcal{A}, \mathcal{B}) = (\mathcal{P}, \mathcal{M})$, where $\mathcal{M}$ denotes the class of all R-modules and $\mathcal{P}$ denotes the class of all projective R-modules. Then $(\mathcal{A}, \mathcal{B})$ is clearly complete and hereditary. Thus, $(^\perp\mathcal{E}, \mathcal{E}) = (dg\tilde{\mathcal{A}}, \tilde{\mathcal{B}})$ is complete by Theorem 3.5, and so every complex has a special dg-projective precover by [**15**, Proposition 1.2.6].

Dually, if we let $(\mathcal{A}, \mathcal{B}) = (\mathcal{M}, \mathcal{I})$, where $\mathcal{M}$ denotes the class of all R-modules and $\mathcal{I}$ the class of all injective R-modules, then $(\mathcal{A}, \mathcal{B})$ is clearly complete and hereditary. Thus, $(\mathcal{E}, \mathcal{E}^\perp) = (\tilde{\mathcal{A}}, dg\tilde{\mathcal{B}})$ is complete by Theorem 3.5, and so every complex has a special dg-injective pre-envelope by [**15**, Proposition 1.2.6]. $\square$

Let $(\mathcal{F}, \mathcal{C})$ be the flat cotorsion pair in R-Mod. Since the cotorsion pair $(\mathcal{F}, \mathcal{C})$ is complete and hereditary, the following result is easily seen by Theorem 3.5.

Corollary 3.7 (Gillespie [17]). *Let* $(\mathcal{F}, \mathcal{C})$ *be the flat cotorsion pair in* R-Mod. *Then the cotorsion pairs* $(\tilde{\mathcal{F}}, dg\tilde{\mathcal{C}})$ *and* $(dg\tilde{\mathcal{F}}, \tilde{\mathcal{C}})$ *are complete.*

In [**17**], Gillespie obtained the flat model structure on Ch(R) from [**21**, Theorem 2.2] and from the above facts. In the general case, we have the following result.

Corollary 3.8. *Let* $(\mathcal{A}, \mathcal{B})$ *be a complete cotorsion pair in* R-Mod. *If* $(\mathcal{A}, \mathcal{B})$ *is hereditary, then there is an associated model structure on* Ch(R) *where the weak equivalences are homology isomorphisms, the cofibrations (respectively, trivial cofibrations) are the monomorphisms whose cokernels are in* $dg\tilde{\mathcal{A}}$ *(respectively,* $\tilde{\mathcal{A}}$*) and the fibrations (respectively, trivial fibrations) are the epimorphisms whose kernels are in* $dg\tilde{\mathcal{B}}$ *(respectively,* $\tilde{\mathcal{B}}$*). In particular,* $dg\tilde{\mathcal{A}}$ *is the class of cofibrant objects and* $dg\tilde{\mathcal{B}}$ *is the class of fibrant objects.*

Proof. It follows from [**17**, Corollary 3.13] that $\tilde{\mathcal{A}} = dg\tilde{\mathcal{A}} \cap \mathcal{E}$ and $\tilde{\mathcal{B}} = dg\tilde{\mathcal{B}} \cap \mathcal{E}$, since the cotorsion pair $(\mathcal{A}, \mathcal{B})$ of modules is hereditary, where $\mathcal{E}$ denotes the class of all exact complexes. As we have shown in Theorem 3.5, both of the induced cotorsion pairs $(\tilde{\mathcal{A}}, dg\tilde{\mathcal{B}})$ and $(dg\tilde{\mathcal{A}}, \tilde{\mathcal{B}})$ are complete; thus, one can get the associated model structure by using the converse of [**21**, Theorem 2.2] (taking $\mathcal{P}$ to be the class of all short exact sequences in the theorem) along with [**21**, Definition 5.1]. $\square$

4. A cofibrantly generated model structure on $\mathrm{Ch}(R)$

Given a fixed non-negative integer n, let $\mathcal{D}$ be the class of all modules with projective dimension less than or equal to n. By [**2**, Theorem 4.2] (see also [**24**]), the pair $(\mathcal{D}, \mathcal{D}^{\perp})$ forms a complete cotorsion pair in R-Mod. Since it is clearly hereditary, the following result is easily seen by proving Theorem 3.5.

Corollary 4.1. *The cotorsion pairs $(\tilde{\mathcal{D}}, \widetilde{dg\mathcal{D}^{\perp}})$ and $(dg\tilde{\mathcal{D}}, \widetilde{\mathcal{D}^{\perp}})$ are complete.*

The main contribution of this section is the following result.

Theorem 4.2. *Let $(\mathcal{A}, \mathcal{B}) = (\mathcal{D}, \mathcal{D}^{\perp})$. Then the associated model structure on $\mathrm{Ch}(R)$ as in Corollary 3.8 is cofibrantly generated.*

The above result also extends the projective model structure given in [**20**, §2.3] (see also [**21**, Example 3.3]) and mentioned in §1. We will give its proof at the end of this section.

In order to prove Theorem 4.2, we will need a stronger version of Corollary 4.1, that is, we need to show that both the cotorsion pairs induced by $(\mathcal{D}, \mathcal{D}^{\perp})$ are cogenerated by sets in $\mathrm{Ch}(R)$. (Note that in a Grothendieck category with a projective generator, if a cotorsion pair $(\mathcal{A}, \mathcal{B})$ is cogenerated by a set, then it is complete [**15**, Corollary 3.1.6]. This important result was originally stated and proved for the category of modules [**10, 11**].) Firstly, we generalize slightly the result given in [**24**, Lemma 3.6] as follows. We give its proof here for completeness, since we use it frequently.

Lemma 4.3. *Let R be a ring, let $\aleph = \mathrm{card}(R) + \aleph_0$ be an infinite cardinal number and let $M \in \mathcal{D}$ be an R-module. Then, for any submodule $A \leqslant M$ with $\mathrm{card}(A) \leqslant \aleph$, there exists a submodule N of M such that $A \leqslant N$, $\mathrm{card}(N) \leqslant \aleph$, $N \in \mathcal{D}$ and $M/N \in \mathcal{D}$.*

Proof. Let

$$0 \longrightarrow P_n \xrightarrow{f_n} P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \longrightarrow 0$$

be a projective resolution of M. By the Kaplansky Theorem [**4**, Theorem 26.1], each $P_l = \bigoplus_{\alpha < \lambda_l} P_{l\alpha}$, where $P_{l\alpha}$ is countably generated for all $0 \leqslant l \leqslant n$ and $\alpha < \lambda_l$.

Since $\mathrm{card}(A) \leqslant \aleph$, there is a subset $F_0 \subseteq \lambda_0$ with $\mathrm{card}(F_0) \leqslant \aleph$ such that $A \subseteq f_0(\bigoplus_{j \in F_0} P_{0j})$. Similarly, there is a subset $F_1 \subseteq \lambda_1$ with $\mathrm{card}(F_1) \leqslant \aleph$ such that

$$\mathrm{Ker}(f_0|_{\bigoplus_{j \in F_0} P_{0j}}) \subseteq f_1\left(\bigoplus_{j \in F_1} P_{1j}\right),$$

etc. Finally, there is a subset $F_n \subseteq \lambda_n$ with $\mathrm{card}(F_n) \leqslant \aleph$ such that

$$\mathrm{Ker}(f_{n-1}|_{\bigoplus_{j \in F_{n-1}} P_{n-1,j}}) \subseteq f_n\bigg(\bigoplus_{j \in F_n} P_{nj}\bigg).$$

Now, there is a subset $F_{n-1} \subseteq F'_{n-1} \subseteq \lambda_{n-1}$ with $\mathrm{card}(F'_{n-1}) \leqslant \aleph$ such that

$$f_n\bigg(\bigoplus_{j \in F_n} P_{nj}\bigg) \subseteq \bigoplus_{j \in F'_{n-1}} P_{n-1,j},$$

etc. Finally, there is a subset $F_0 \subseteq F'_0 \subseteq \lambda_0$ with $\mathrm{card}(F'_0) \leqslant \aleph$ such that $f_1(\bigoplus_{j \in F'_1} P_{1j}) \subseteq \bigoplus_{j \in F'_0} P_{0,j}$. Continuously using this back-and-forth procedure, we obtain for each $0 \leqslant l \leqslant n$ a subset $H_l = F_l \cup F'_l \cup F''_l \cup \ldots$ of λ_l with $\mathrm{card}(H_l) \leqslant \aleph$ such that the restricted sequence

$$0 \longrightarrow \bigoplus_{j \in H_n} P_{nj} \xrightarrow{f_n} \bigoplus_{j \in H_{n-1}} P_{n-1,j} \longrightarrow \cdots \longrightarrow \bigoplus_{j \in H_1} P_{1j} \xrightarrow{f_1} \bigoplus_{j \in H_0} P_{0j} \xrightarrow{f_0} N \longrightarrow 0$$

is exact, and $A \subseteq N$. By construction, the factor sequence

$$0 \longrightarrow \bigoplus_{j \notin H_n} P_{nj} \xrightarrow{\bar{f}_n} \bigoplus_{j \notin H_{n-1}} P_{n-1,j} \longrightarrow \cdots \longrightarrow \bigoplus_{j \notin H_1} P_{1j} \xrightarrow{\bar{f}_1} \bigoplus_{j \notin H_0} P_{0j} \xrightarrow{\bar{f}_0} M/N \longrightarrow 0$$

is also exact. So $N, M/N \in \mathcal{D}$. Clearly, we have $\mathrm{card}(N) \leqslant \aleph$ since $\mathrm{card}(\bigoplus_{j \in H_0} P_{0j}) \leqslant \aleph \cdot \aleph_0 = \aleph$. $\qquad\square$

Recall from [**16**] that an R-complex N has projective dimension at most n if and only if N is exact and any R-module $Z_i N$ has projective dimension at most n in R-Mod for all $i \in \mathbb{Z}$. Thus, the class $\tilde{\mathcal{D}}$ is exactly the class of all R-complexes of projective dimension at most n. For a complex X, we define its cardinality to be $\mathrm{card}(X) = \mathrm{card}(\bigoplus_{i \in \mathbb{Z}} X_i)$. The idea of the next lemma derives from [**3**].

Lemma 4.4. *Let R be a ring, and let $\aleph = \mathrm{card}(R) + \aleph_0$ be an infinite cardinal number. Then, for any complex $Q \in \tilde{\mathcal{D}}$ and any element $x \in Q_k$ ($k \in \mathbb{Z}$ is arbitrary), there exists a subcomplex L of Q with $L \in \tilde{\mathcal{D}}$ such that $x \in L_k$, $\mathrm{card}(L) \leqslant \aleph$ and Q/L is also in $\tilde{\mathcal{D}}$.*

Proof. Let us suppose (without loss of generality) that $k > 0$ and $x \in Q_k$. Consider then the following exact complex

$$\cdots \longrightarrow A^1_{k+2} \xrightarrow{\partial_{k+2}} A^1_{k+1} \xrightarrow{\partial_{k+1}} Rx \xrightarrow{\partial_k} \partial_k(Rx) \xrightarrow{\partial_{k-1}} 0, \qquad (S1)$$

where A^1_i is a submodule of Q_i constructed as follows: $\mathrm{card}(Rx) \leqslant \aleph$ since $\mathrm{card}(R) \leqslant \aleph$, so we can find $A^1_{k+1} \leqslant Q_{k+1}$ such that $\mathrm{card}(A^1_{k+1}) \leqslant \aleph$ and $\partial_{k+1}(A^1_{k+1}) = \mathrm{Ker}(\partial_k|_{Rx})$. Then $A^1_{k+2} \leqslant Q_{k+2}$, $\mathrm{card}(A^1_{k+2}) \leqslant \aleph$, and $\partial_{k+2}(A^1_{k+2}) = \mathrm{Ker}(\partial_{k+1}|_{A^1_{k+1}})$, and we repeat the argument.

Now, we have $\mathrm{Ker}(\partial_k|_{Rx}) \leqslant \mathrm{Ker}(\partial_k)$, so we know by Lemma 4.3 that $\mathrm{Ker}(\partial_k|_{Rx})$ can be embedded into a submodule S_k^2 of $\mathrm{Ker}(\partial_k)$ with $\mathrm{card}(S_k^2) \leqslant \aleph$, $S_k^2 \in \mathcal{D}$ and $\mathrm{Ker}(\partial_k)/S_k^2 \in \mathcal{D}$. Then consider the exact complex

$$\cdots \longrightarrow A_{k+2}^2 \xrightarrow{\partial_{k+2}} A_{k+1}^2 \xrightarrow{\partial_{k+1}} Rx + S_k^2 \xrightarrow{\partial_k} \partial_k(Rx) \xrightarrow{\partial_{k-1}} 0, \qquad (S2)$$

where the A_i^2 are taken as above. It is obvious that $\mathrm{Ker}(\partial_k|_{Rx+S_k^2}) = S_k^2$, which is in $\mathrm{Ker}(\partial_k)$, and that $\mathrm{card}(Rx + S_k^2) \leqslant \aleph + \aleph = \aleph$.

Note that $\partial_k(Rx) \subseteq \mathrm{Ker}(\partial_{k-1})$, so we can embed $\partial_k(Rx)$ into a submodule S_{k-1}^3 of $\mathrm{Ker}(\partial_{k-1})$ in such a way that $\mathrm{card}(S_{k-1}^3) \leqslant \aleph$ ($\mathrm{card}(\partial_k(Rx)) \leqslant \aleph$), $S_{k-1}^3 \in \mathcal{D}$ and $\mathrm{Ker}(\partial_{k-1})/S_{k-1}^3 \in \mathcal{D}$, and then take the exact complex

$$\cdots \longrightarrow A_{k+2}^3 \xrightarrow{\partial_{k+2}} A_{k+1}^3 \xrightarrow{\partial_{k+1}} A_k^3 \xrightarrow{\partial_k} S_{k-1}^3 \xrightarrow{\partial_{k-1}} 0. \qquad (S3)$$

We see again that $\mathrm{Ker}(\partial_{k-1}|_{S_{k-1}^3}) = S_{k-1}^3$.

We turn over and find $S_k^4 \leqslant \mathrm{Ker}(\partial_k)$ with $\mathrm{card}(S_k^4) \leqslant \aleph$, $S_k^4 \in \mathcal{D}$, $\mathrm{Ker}(\partial_k)/S_k^4 \in \mathcal{D}$ and $S_k^4 \supseteq \mathrm{Ker}(\partial_k|_{A_k^3})$, and then construct $A_i^4 \leqslant Q_i$ ($\mathrm{card}(A_i^4) \leqslant \aleph$, for all $i > k$) such that

$$\cdots \longrightarrow A_{k+2}^4 \xrightarrow{\partial_{k+2}} A_{k+1}^4 \xrightarrow{\partial_{k+1}} A_k^3 + S_k^4 \xrightarrow{\partial_k} S_{k-1}^3 \xrightarrow{\partial_{k-1}} 0 \qquad (S4)$$

is exact. Once more $\mathrm{Ker}(\partial_k|_{A_k^3+S_k^4}) = S_k^4 \leqslant \mathrm{Ker}(\partial_k)$. We then find $S_{k+1}^5 \leqslant \mathrm{Ker}(\partial_{k+1})$ with $\mathrm{card}(S_{k+1}^5) \leqslant \aleph$, $S_{k+1}^5 \in \mathcal{D}$, $\mathrm{Ker}(\partial_{k+1})/S_{k+1}^5 \in \mathcal{D}$ and $\mathrm{Ker}(\partial_{k+1}|_{A_{k+1}^4}) \subseteq S_{k+1}^5$, and consider the exact complex

$$\cdots \longrightarrow A_{k+2}^5 \xrightarrow{\partial_{k+2}} A_{k+1}^4 + S_{k+1}^5 \xrightarrow{\partial_{k+1}} A_k^3 + S_k^4 \xrightarrow{\partial_k} S_{k-1}^3 \xrightarrow{\partial_{k-1}} 0, \qquad (S5)$$

in which $\mathrm{Ker}(\partial_{k+1}|_{A_{k+1}^4+S_{k+1}^5}) = S_{k+1}^5 \leqslant \mathrm{Ker}(\partial_{k+1})$.

The next step is to find $S_{k+2}^6 \leqslant \mathrm{Ker}(\partial_{k+2})$ such that $\mathrm{card}(S_{k+2}^6) \leqslant \aleph$, $S_{k+2}^6 \in \mathcal{D}$, $\mathrm{Ker}(\partial_{k+2})/S_{k+2}^6 \in \mathcal{D}$ and that $\mathrm{Ker}(\partial_{k+2}|_{A_{k+2}^5}) \subseteq S_{k+2}^6$, and then consider the exact complex

$$\cdots \longrightarrow A_{k+3}^6 \xrightarrow{\partial_{k+3}} A_{k+2}^5 + S_{k+2}^6 \xrightarrow{\partial_{k+2}} A_{k+1}^4 + S_{k+1}^5 \xrightarrow{\partial_{k+1}} A_k^3 + S_k^4 \xrightarrow{\partial_k} S_{k-1}^3 \xrightarrow{\partial_{k-1}} 0, \qquad (S6)$$

in which $\mathrm{Ker}(\partial_{k+2}|_{A_{k+2}^5+S_{k+2}^6}) = S_{k+2}^6 \leqslant \mathrm{Ker}(\partial_{k+2})$.

Therefore, we prove by induction that for any $m \geqslant 4$ we can construct an exact complex

$$\cdots \xrightarrow{\partial_{k+m-2}} A_{k+m-3}^m \xrightarrow{\partial_{k+m-3}} T_{k+m-4}^m \xrightarrow{\partial_{k+m-4}} T_{k+m-5}^m \xrightarrow{\partial_{k+m-5}}$$
$$\cdots \xrightarrow{\partial_{k+1}} T_k^m \xrightarrow{\partial_k} T_{k-1}^m \longrightarrow 0 \qquad (Sm)$$

such that $\mathrm{Ker}(\partial_{k+m-j}|_{T_{k+m-j}^m})$ is a submodule of $\mathrm{Ker}(\partial_{k+m-j})$ for $j \geqslant 4$ and that all of the terms have cardinality less than or equal to $\aleph$.

If we take the direct limit $L = \varinjlim(Sm)$ with $m \geqslant 1$, then we see that the complex L is exact and that $\mathrm{Ker}(\partial_i|_{L_i}) \in \mathcal{D}$ is a submodule of $\mathrm{Ker}(\partial_i)$ for all $i \geqslant k - 1$. Furthermore,

$\text{card}(L_i) \leqslant \aleph$ for any $i \geqslant k-1$, so $\text{card}(L) \leqslant \aleph_0 \cdot \aleph = \aleph$. We finally consider the complex L to be

$$L = \cdots \xrightarrow{\partial_{k+2}} L_{k+1} \xrightarrow{\partial_{k+1}} L_k \xrightarrow{\partial_k} L_{k-1} \xrightarrow{\partial_{k-1}} 0 \longrightarrow 0 \cdots,$$

which is a subcomplex of Q, $x \in L_k$, $\text{Ker}(\partial_i|_{L_i}) \in \mathcal{D}$ and $\text{Ker}(\partial_i)/\text{Ker}(\partial_i|_{L_i}) \in \mathcal{D}$ by the above construction for all $i \geqslant k-1$ since each $\text{Ker}(\partial_i) \in \mathcal{D}$. Therefore, the subcomplex L of Q has projective dimension at most n.

To finish the proof, we need only argue that $Q/L = (Q_i/L_i, \bar{\partial}_i)$ has projective dimension at most n. An easy computation shows that $\text{Ker}(\bar{\partial}_i) = \text{Ker}(\partial_i)/\text{Ker}(\partial_i|_{L_i})$, but by construction $\text{Ker}(\partial_i)/\text{Ker}(\partial_i|_{L_i})$ is in $\mathcal{D}$ for any $i \geqslant k-1$, and $\text{Ker}(\bar{\partial}_i) = \text{Ker}(\partial_i)/\text{Ker}(\partial_i|_{L_i}) = \text{Ker}(\partial_i)$ also has projective dimension at most n for all $i < k-1$ by the hypothesis. Clearly, Q/L is exact, since both Q and L are exact. So Q/L has projective dimension at most n. $\qquad\square$

Theorem 4.5. *The cotorsion pair* $(\tilde{\mathcal{D}}, \widetilde{dg\mathcal{D}^{\perp}})$ *is cogenerated by a set.*

Proof. Assume that $Q \in \tilde{\mathcal{D}}$ is a complex and $x \in Q_i$. Then by Lemma 4.4 we know that we can find a subcomplex $L_0 \in \tilde{\mathcal{D}}$ of Q such that $x \in (L_0)_i$, $\text{card}(L_0) \leqslant \aleph$ and that the quotient complex Q/L_0 has projective dimension at most n. Furthermore, we let

$$0 \longrightarrow \bigoplus_{i \in \mathbb{Z}} D^{i+1}\left(\bigoplus_{j \in X_{ni}} P_{ni}^j\right) \longrightarrow \bigoplus_{i \in \mathbb{Z}} D^{i+1}\left(\bigoplus_{j \in X_{n-1,i}} P_{n-1,i}^j\right) \longrightarrow \cdots$$

$$\cdots \longrightarrow \bigoplus_{i \in \mathbb{Z}} D^{i+1}\left(\bigoplus_{j \in X_{1i}} P_{1i}^j\right) \longrightarrow \bigoplus_{i \in \mathbb{Z}} D^{i+1}\left(\bigoplus_{j \in X_{0i}} P_{0i}^j\right) \longrightarrow Q \longrightarrow 0$$

be a projective resolution of the complex Q, where

$$0 \longrightarrow \bigoplus_{j \in X_{ni}} P_{ni}^j \longrightarrow \bigoplus_{j \in X_{n-1,i}} P_{n-1,i}^j \longrightarrow \cdots \longrightarrow \bigoplus_{j \in X_{1i}} P_{1i}^j \longrightarrow \bigoplus_{j \in X_{0i}} P_{0i}^j \longrightarrow Z_i Q \longrightarrow 0$$

is a projective resolution of $Z_i Q$ and P_{ki}^j is countably generated for all $0 \leqslant k \leqslant n$, $j \in X_{ki}$ and for each $i \in \mathbb{Z}$. By the proofs of Lemmas 4.3 and 4.4, for each $i \in \mathbb{Z}$ there exists a projective resolution of $Z_i L_0$ of the form

$$0 \longrightarrow \bigoplus_{j \in X_{ni}^0} P_{ni}^j \longrightarrow \bigoplus_{j \in X_{n-1,i}^0} P_{n-1,i}^j \longrightarrow \cdots \longrightarrow \bigoplus_{j \in X_{1i}^0} P_{1i}^j \longrightarrow \bigoplus_{j \in X_{0i}^0} P_{0i}^j \longrightarrow Z_i L_0 \longrightarrow 0,$$

where $X_{k,i}^0 \subseteq X_{k,i}$ for all $k \in \{0,1,2,\ldots,n\}$, $\text{card}(X_{k,i}^0) \leqslant \aleph$, and that

$$0 \longrightarrow \bigoplus_{j \in X_{n,i} \backslash X_{ni}^0} P_{ni}^j \longrightarrow \bigoplus_{j \in X_{n-1,i} \backslash X_{n-1,i}^0} P_{n-1,i}^j \longrightarrow \cdots$$

$$\cdots \longrightarrow \bigoplus_{j \in X_{1,i} \backslash X_{1i}^0} P_{1i}^j \longrightarrow \bigoplus_{j \in X_{0,i} \backslash X_{0i}^0} P_{0i}^j \longrightarrow Z_i(Q/L_0) \longrightarrow 0$$

is a projective resolution of $Z_i(Q/L_0)$ for each $i \in \mathbb{Z}$. By the Horseshoe Lemma, we get the following projective resolutions of complexes L_0 and Q/L_0:

$$0 \longrightarrow \bigoplus_{i\in\mathbb{Z}} D^{i+1}\bigg(\bigoplus_{j\in X^0_{ni}} P^j_{ni} \bigg) \longrightarrow \bigoplus_{i\in\mathbb{Z}} D^{i+1}\bigg(\bigoplus_{j\in X^0_{n-1,i}} P^j_{n-1,i} \bigg) \longrightarrow \cdots$$

$$\cdots \longrightarrow \bigoplus_{i\in\mathbb{Z}} D^{i+1}\bigg(\bigoplus_{j\in X^0_{1,i}} P^j_{1i} \bigg) \longrightarrow \bigoplus_{i\in\mathbb{Z}} D^{i+1}\bigg(\bigoplus_{j\in X^0_{0i}} P^j_{0i} \bigg) \longrightarrow L_0 \longrightarrow 0,$$

$$0 \longrightarrow \bigoplus_{i\in\mathbb{Z}} D^{i+1}\bigg(\bigoplus_{j\in X_{ni}\setminus X^0_{ni}} P^j_{ni} \bigg) \longrightarrow \bigoplus_{i\in\mathbb{Z}} D^{i+1}\bigg(\bigoplus_{j\in X_{n-1,i}\setminus X^0_{n-1,i}} P^j_{n-1,i} \bigg) \longrightarrow \cdots$$

$$\cdots \longrightarrow \bigoplus_{i\in\mathbb{Z}} D^{i+1}\bigg(\bigoplus_{j\in X_{1i}\setminus X^0_{1i}} P^j_{1i} \bigg) \longrightarrow \bigoplus_{i\in\mathbb{Z}} D^{i+1}\bigg(\bigoplus_{j\in X_{0i}\setminus X^0_{0i}} P^j_{0i} \bigg) \longrightarrow Q/L_0 \longrightarrow 0.$$

Take then any element $y+(L_0)_j \in (Q/L_0)_j$ and, as in the case of L_0, find a subcomplex $L_1/L_0 \leqslant Q/L_0$ such that

$$\operatorname{card}(L_1/L_0) \leqslant \aleph, \quad y + (L_0)_j \in (L_1/L_0)_j \quad \text{and} \quad L_1/L_0, Q/L_1 \in \tilde{\mathcal{D}}.$$

By construction, L_1/L_0 has a projective resolution

$$0 \longrightarrow \bigoplus_{i\in\mathbb{Z}} D^{i+1}\bigg(\bigoplus_{j\in Y^1_{ni}} P^j_{ni} \bigg) \longrightarrow \bigoplus_{i\in\mathbb{Z}} D^{i+1}\bigg(\bigoplus_{j\in Y^1_{n-1,i}} P^j_{n-1,i} \bigg) \longrightarrow \cdots$$

$$\cdots \longrightarrow \bigoplus_{i\in\mathbb{Z}} D^{i+1}\bigg(\bigoplus_{j\in Y^1_{1,i}} P^j_{1i} \bigg) \longrightarrow \bigoplus_{i\in\mathbb{Z}} D^{i+1}\bigg(\bigoplus_{j\in Y^1_{0i}} P^j_{0i} \bigg) \longrightarrow L_1/L_0 \longrightarrow 0$$

with

$$0 \longrightarrow \bigoplus_{j\in Y^1_{ni}} P^j_{ni} \longrightarrow \bigoplus_{j\in Y^1_{n-1,i}} P^j_{n-1,i} \cdots \longrightarrow \bigoplus_{j\in Y^1_{1i}} P^j_{1i} \longrightarrow \bigoplus_{j\in Y^1_{0i}} P^j_{0i} \longrightarrow Z_i(L_1/L_0) \longrightarrow 0$$

a projective resolution of $Z_i(L_1/L_0)$ for each $i \in \mathbb{Z}$, where $Y^1_{k,i} \subseteq X_{k,i} \setminus X^0_{k,i}, \operatorname{card}(Y^1_{k,i}) \leqslant \aleph$, for all $k \in \{0, 1, 2, \ldots, n\}$. It is easy to see that $L_1 \in \tilde{\mathcal{D}}$ since $\tilde{\mathcal{D}}$ is closed under extensions. If we set $X^1_{k,i} = X^0_{k,i} \cup Y^1_{k,i}$ for each $i \in \mathbb{Z}$ and $k \in \{0, 1, \ldots, n\}$, then the Horseshoe Lemma yields a projective resolution

$$0 \longrightarrow \bigoplus_{i\in\mathbb{Z}} D^{i+1}\bigg(\bigoplus_{j\in X^1_{ni}} P^j_{ni} \bigg) \longrightarrow \bigoplus_{i\in\mathbb{Z}} D^{i+1}\bigg(\bigoplus_{j\in X^1_{n-1,i}} P^j_{n-1,i} \bigg) \longrightarrow \cdots$$

$$\cdots \longrightarrow \bigoplus_{i\in\mathbb{Z}} D^{i+1}\bigg(\bigoplus_{j\in X^1_{1,i}} P^j_{1i} \bigg) \longrightarrow \bigoplus_{i\in\mathbb{Z}} D^{i+1}\bigg(\bigoplus_{j\in X^1_{0i}} P^j_{0i} \bigg) \longrightarrow L_1 \longrightarrow 0$$

of L_1 (note that also here $\mathrm{card}(X^1_{k,i}) \leqslant \aleph$ for all $k \in \{0, 1, \ldots, n\}$ and $i \in \mathbb{Z}$). Hence,

$$0 \longrightarrow \bigoplus_{i \in \mathbb{Z}} D^{i+1}\left(\bigoplus_{j \in X_{ni} \backslash X^1_{ni}} P^j_{ni} \right) \longrightarrow \bigoplus_{i \in \mathbb{Z}} D^{i+1}\left(\bigoplus_{j \in X_{n-1,i} \backslash X^1_{n-1,i}} P^j_{n-1,i} \right) \longrightarrow \cdots$$

$$\cdots \longrightarrow \bigoplus_{i \in \mathbb{Z}} D^{i+1}\left(\bigoplus_{j \in X_{1i} \backslash X^1_{1i}} P^j_{1i} \right) \longrightarrow \bigoplus_{i \in \mathbb{Z}} D^{i+1}\left(\bigoplus_{j \in X_{0i} \backslash X^1_{0i}} P^j_{0i} \right) \longrightarrow Q/L_1 \longrightarrow 0$$

is a projective resolution of Q/L_1.

We then find inductively a chain of subcomplexes $L_m \leqslant Q$ for all $m \in \mathbb{N}$ such that $L_m, L_{m+1}/L_m, Q/L_m \in \tilde{\mathcal{D}}$ and $\mathrm{card}(L_{m+1}/L_m) \leqslant \aleph$ for all $m \in \mathbb{N}$. Furthermore, the sequences

$$0 \longrightarrow \bigoplus_{i \in \mathbb{Z}} D^{i+1}\left(\bigoplus_{j \in Y^m_{ni}} P^j_{ni} \right) \longrightarrow \bigoplus_{i \in \mathbb{Z}} D^{i+1}\left(\bigoplus_{j \in Y^m_{n-1,i}} P^j_{n-1,i} \right) \longrightarrow \cdots$$

$$\cdots \longrightarrow \bigoplus_{i \in \mathbb{Z}} D^{i+1}\left(\bigoplus_{j \in Y^m_{1,i}} P^j_{1i} \right) \longrightarrow \bigoplus_{i \in \mathbb{Z}} D^{i+1}\left(\bigoplus_{j \in Y^m_{0i}} P^j_{0i} \right) \longrightarrow L_m/L_{m-1} \longrightarrow 0,$$

$$0 \longrightarrow \bigoplus_{i \in \mathbb{Z}} D^{i+1}\left(\bigoplus_{j \in X^m_{ni}} P^j_{ni} \right) \longrightarrow \bigoplus_{i \in \mathbb{Z}} D^{i+1}\left(\bigoplus_{j \in X^m_{n-1,i}} P^j_{n-1,i} \right) \longrightarrow \cdots$$

$$\cdots \longrightarrow \bigoplus_{i \in \mathbb{Z}} D^{i+1}\left(\bigoplus_{j \in X^m_{1,i}} P^j_{1i} \right) \longrightarrow \bigoplus_{i \in \mathbb{Z}} D^{i+1}\left(\bigoplus_{j \in X^m_{0i}} P^j_{0i} \right) \longrightarrow L_m \longrightarrow 0$$

and

$$0 \longrightarrow \bigoplus_{i \in \mathbb{Z}} D^{i+1}\left(\bigoplus_{j \in X_{ni} \backslash X^m_{ni}} P^j_{ni} \right) \longrightarrow \bigoplus_{i \in \mathbb{Z}} D^{i+1}\left(\bigoplus_{j \in X_{n-1,i} \backslash X^m_{n-1,i}} P^j_{n-1,i} \right) \longrightarrow \cdots$$

$$\cdots \longrightarrow \bigoplus_{i \in \mathbb{Z}} D^{i+1}\left(\bigoplus_{j \in X_{1i} \backslash X^m_{1,i}} P^j_{1i} \right) \longrightarrow \bigoplus_{i \in \mathbb{Z}} D^{i+1}\left(\bigoplus_{j \in X_{0i} \backslash X^m_{0i}} P^j_{0i} \right) \longrightarrow Q/L_m \longrightarrow 0$$

are projective resolutions of L_m/L_{m-1}, L_m and Q/L_m, respectively, where $X^m_{k,i} = X^{m-1}_{k,i} \cup Y^m_{k,i}$ for all $k \in \{0, 1, \ldots, n\}$ and all $i \in \mathbb{Z}$.

Let us define $L_{\omega_0} = \bigcup_{m \in \mathbb{N}} L_m$. It is clear that if we take $X^{\omega_0}_{k,i} = \bigcup_{m \in \mathbb{N}} X^m_{k,i}$ for each $i \in \mathbb{Z}$ and each $k \in \{0, 1, \ldots, n\}$, the sequences

$$0 \longrightarrow \bigoplus_{i \in \mathbb{Z}} D^{i+1}\left(\bigoplus_{j \in X^{\omega_0}_{ni}} P^j_{ni} \right) \longrightarrow \bigoplus_{i \in \mathbb{Z}} D^{i+1}\left(\bigoplus_{j \in X^{\omega_0}_{n-1,i}} P^j_{n-1,i} \right) \longrightarrow \cdots$$

$$\cdots \longrightarrow \bigoplus_{i \in \mathbb{Z}} D^{i+1}\left(\bigoplus_{j \in X^{\omega_0}_{1i}} P^j_{1i} \right) \longrightarrow \bigoplus_{i \in \mathbb{Z}} D^{i+1}\left(\bigoplus_{j \in X^{\omega_0}_{0i}} P^j_{0i} \right) \longrightarrow L_{\omega_0} \longrightarrow 0$$

and

$$0 \longrightarrow \bigoplus_{i\in\mathbb{Z}} D^{i+1}\bigg(\bigoplus_{j\in X_{ni}\setminus X_{ni}^{\omega_0}} P_{ni}^j \bigg) \longrightarrow \bigoplus_{i\in\mathbb{Z}} D^{i+1}\bigg(\bigoplus_{j\in X_{n-1,i}\setminus X_{n-1,i}^{\omega_0}} P_{n-1,i}^j \bigg) \longrightarrow \cdots$$

$$\cdots \longrightarrow \bigoplus_{i\in\mathbb{Z}} D^{i+1}\bigg(\bigoplus_{j\in X_{1i}\setminus X_{1,i}^{\omega_0}} P_{1i}^j \bigg) \longrightarrow \bigoplus_{i\in\mathbb{Z}} D^{i+1}\bigg(\bigoplus_{j\in X_{0i}\setminus X_{0i}^{\omega_0}} P_{0i}^j \bigg) \longrightarrow Q/L_{\omega_0} \longrightarrow 0$$

are projective resolutions of L_{ω_0} and Q/L_{ω_0}, respectively, and that $\mathrm{card}(X_{k,i}^{\omega_0}) \leqslant \aleph_0 \cdot \aleph = \aleph$ for all $i\in\mathbb{Z}$ and $k\in\{0,1,\ldots,n\}$. Thus, we can continue the argument and, by transfinite induction, find a continuous chain of subcomplexes of Q, $\{L_\alpha; \alpha<\lambda\}$, such that $Q = \bigcup_{\alpha<\lambda} L_\alpha$, $L_0, L_{\alpha+1}/L_\alpha \in \tilde{\mathcal{D}}$ and $\mathrm{card}(L_0) \leqslant \aleph$, $\mathrm{card}(L_{\alpha+1}/L_\alpha) \leqslant \aleph$ for all $\alpha+1<\lambda$.

Therefore, applying [**15**, Proposition 3.1.1], the cotorsion pair $(\tilde{\mathcal{D}}, \widetilde{dg\mathcal{D}^\perp})$ is cogenerated by any set of representatives of complexes $L \in \tilde{\mathcal{D}}$ such that $\mathrm{card}(L) \leqslant \aleph$. □

Proposition 4.6. *The cotorsion pair $(dg\tilde{\mathcal{D}}, \widetilde{\mathcal{D}^\perp})$ is cogenerated by a set.*

Proof. This follows from [**18**, Proposition 3.8] and the case that the cotorsion pair $(\mathcal{D}, \mathcal{D}^\perp)$ is cogenerated by a set [**24**, Theorem 3.7]. □

We now finish this section by giving the proof of Theorem 4.2 as follows.

Proof of Theorem 4.2. This follows from Theorem 4.5, Proposition 4.6, [**21**, Lemma 6.7] and [**21**, Lemma 6.8]. □

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