Compositio Mathematica **125:** 221–253, 2001.

Cohomology of Compact Locally Symmetric Spaces

T. N. VENKATARAMANA
School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Colaba, Mumbai 400 005, India. e-mail: venky@math.tifr.res.in

(Received: 23 July 1999; accepted in final form: 28 December 1999)

Abstract. We obtain a necessary condition for a cohomology class on a compact locally symmetric space $S(\Gamma) = \Gamma\backslash X$ (a quotient of a symmetric space X of the non-compact type by a cocompact arithmetic subgroup Γ of isometries of X) to restrict non-trivially to a compact locally symmetric subspace $S_H(\Gamma) = \Delta\backslash Y$ of $\Gamma\backslash X$. The restriction is in a 'virtual' sense, i.e. it is the restriction of possibly a translate of the cohomology class under a Hecke correspondence. As a consequence we deduce that when X and Y are the unit balls in $\mathbb{C}^n$ and $\mathbb{C}^m$, then low degree cohomology classes on the variety $S(\Gamma)$ restrict non-trivially to the subvariety $S_H(\Gamma)$; this proves a conjecture of M. Harris and J-S. Li. We also deduce the non-vanishing of cup-products of cohomology classes for the variety $S(\Gamma)$.

Mathematics Subject Classifications (2000). 11F75, 22E40.

Key words. restriction maps, cohomology of arithmetic groups.

Introduction

(0.1) In this paper we are concerned with restriction of cohomology (with $\mathbb{C}$ coefficients) of a *compact* Shimura variety $\mathrm{Sh}(G, X)$ to a smaller Shimura variety $\mathrm{Sh}(H, Y)$. In [C-V] we gave an explicit criterion (depending only on the imbedding of $H(\mathbb{R})$ in $G(\mathbb{R})$) for *holomorphic* cohomology classes to vanish on the smaller Shimura variety. We exploited the fact that holomorphic forms on smooth projective varieties are harmonic for a suitable (Kahlerian) metric.

Here we give such a criterion even when the cohomology class in question is not holomorphic; the restriction of a (non-holomorphic) harmonic form ω on $\mathrm{Sh}(G, X)$ to $\mathrm{Sh}(H, Y)$ does not appear to be harmonic in general (here, the form ω is harmonic with respect to the natural metric on $\mathrm{Sh}(G, X)$ arising from the Killing form on the group $G(\mathbb{R})$). However, as we will show (Theorem 2), it is still possible to give a criterion, based purely on the linear algebra of $G(\mathbb{R})$, for the restriction of ω to be non-zero. Here the restriction is in the sense of [Oda], [H-L], [C-V], [M-R]. We assume throughout this paper that the degree of the cohomology class does not exceed the complex dimension of $\mathrm{Sh}(H, Y)$.

As an application, we will prove that if $G = \mathrm{SU}(n, 1)$ and $H = \mathrm{SU}(n-1, 1)$, then the cohomology of $\mathrm{Sh}(G, X)$ in degrees $\leqslant n-1$, restricts injectively to that of $\mathrm{Sh}(H, Y)$. (See Theorem 4, (1)). This confirms a conjecture of Harris and Li (see [H-L]). We also show that if $G = \mathrm{SO}(n, 2)$ and $H = \mathrm{SO}(n-1, 2)$, then the cohomology of $\mathrm{Sh}(G, X)$ in degrees $\leqslant n-1$ restricts injectively to that of $\mathrm{Sh}(H, Y)$ (Theorem 4, (2)) (see [H-L] for similar statements).

The proof of our criterion is based on Theorem 1, which says that if we think of $\mathrm{Sh}^{K\cap H}(H, Y)$ as a cohomology class on $\mathrm{Sh}^K(G, X)$, then the span of the $G(\mathbb{A}_f)$-translates of this cycle class contains a nonzero $G(\mathbb{A}_f)$-invariant vector. Here $K \subset G(\mathbb{A}_f)$ is a compact open subgroup.

In Theorem 6, we refine the criterion of Theorem 2 mentioned above, in terms of the *'representation type'* $A_{\mathfrak{q}}$ of the cohomology class in question. Theorem 6 is the analogue of a condition obtained for *holomorphic* classes in [C-V]. The criterion of Theorem 6 can be used to obtain a condition for the vanishing of cup product of classes in $H^\bullet(\mathrm{Sh}(G, X))$ (Theorem 7). As an application (Theorems 8 and 9), we show that cup products of low degree classes on $\mathrm{Sh}(G, X)$ do not vanish (in a virtual sense), if the group G is $SU(n, 1)$ or $SO(2, n)$.

(0.2) We now describe the contents of the paper more precisely.

Let H and G be connected semisimple algebraic groups over $\mathbb{Q}$ all of whose $\mathbb{Q}$ simple factors are isotropic over $\mathbb{R}$. Let $f: H \to G$ be a morphism of $\mathbb{Q}$-algebraic groups with finite kernel. Let $\mathfrak{g}_0$ and $\mathfrak{h}_0$ denote, respectively, the real Lie algebras of $G(\mathbb{R})$ and $H(\mathbb{R})$. Let $\mathfrak{g}$, and $\mathfrak{h}$ respectively denote the complexifications of the real lie algebras $\mathfrak{g}_0$ and $\mathfrak{h}_0$.

By assumption, the map f has finite kernel and, hence, induces an *injection* $f: \mathfrak{h}_0 \to \mathfrak{g}_0$. Choose (as one may, by a Theorem of Mostow ([M])) a Cartan involution θ of $\mathfrak{g}_0$ whose restriction to $\mathfrak{h}_0$ is a Cartan involution on $\mathfrak{h}_0$. Let K_∞ (resp. K_∞^H) denote the set of points of $G(\mathbb{R})$ (resp. $H(\mathbb{R})$) fixed by θ. Then, K_∞ (resp. K_∞^H) is a maximal compact subgroup of $G(\mathbb{R})$ (resp. of $H(\mathbb{R})$). Moreover, it is easy to see that the group of real points of the kernel of the map $f: H \to G$ is contained in K_∞^H and that the inverse image of K_∞ under f is precisely K_∞^H. Write $X = G(\mathbb{R})/K_\infty$ and $Y = H(\mathbb{R})/K_\infty^H$. The map f induces a map of symmetric spaces $Y \to X$ which we again denote by f.

Let $\mathfrak{k}_0$ and $\mathfrak{k}_0^H$ denote the real Lie algebras of K_∞ and K_∞^H, respectively. With respect to the involution θ write the 'Cartan decompositions' $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ and $\mathfrak{h}_0 = \mathfrak{k}_0^H \oplus \mathfrak{p}_0^H$ of $\mathfrak{g}_0$ and $\mathfrak{h}_0$, respectively. Denote by $\mathfrak{k}$, $\mathfrak{k}^H$, $\mathfrak{p}$ and $\mathfrak{p}^H$, respectively, the complexifications of $\mathfrak{k}_0$, $\mathfrak{k}_0^H$, $\mathfrak{p}_0$ and $\mathfrak{p}_0^H$.

From now on, we will make the simplifying assumption that the groups $G(\mathbb{R})$ and $H(\mathbb{R})$ are *connected*. This is not, strictly speaking, necessary (because we may replace $G(\mathbb{R})$ by its connected component of identity), but it considerably simplifies the statements and notation.

Denote by $\mathbb{A}_f$ the ring of finite adèles over $\mathbb{Q}$. The natural inclusion of $\mathbb{Q}$ in $\mathbb{A}_f$ induces an imbedding of the group $G(\mathbb{Q})$ in the group $G(\mathbb{A}_f)$. Denote by G_f the *closure* of $G(\mathbb{Q})$ in $G(\mathbb{A}_f)$. Define H_f similarly.

Let K be a compact open subgroup $G(\mathbb{A}_f)$ such that the group $\Gamma = G(\mathbb{Q}) \cap K$ is a torsion free subgroup of $G(\mathbb{Q})$. The groups Γ are the (torsion free) 'congruence arithmetic subgroups' of $G(\mathbb{Q})$. Denote by $S(\Gamma)$ the locally symmetric space

$\Gamma\backslash X$. Denote by $\Gamma \cap H$ the inverse image of $\Gamma \subset G(\mathbb{Q})$ under the map $f: H(\mathbb{Q}) \to G(\mathbb{Q})$. Write $S_H(\Gamma)$ for the locally symmetric space $\Gamma \cap H\backslash Y$ (note that $\Gamma \cap H$ is not, in general, torsion free since it contains the finite group $\mathrm{Ker}(f) \cap H(\mathbb{Q})$, where $\mathrm{Ker}(f)$ is the kernel of f; however, this finite group is contained in the centre of $H(\mathbb{R})$ and hence acts trivially on the symmetric space Y; thus, $S_H(\Gamma)$ is still smooth, and is covered by Y). We get a smooth map

$$j = j(\Gamma): S_H(\Gamma) \to S(\Gamma), \tag{0}$$

for each torsion free congruence arithmetic subgroup Γ of $G(\mathbb{Q})$.

In the sequel, if V is a topological space, we write $H^\bullet(V)$ for the cohomology group $H^\bullet(V, \mathbb{C})$ of V with coefficients in $\mathbb{C}$.

From now on we will assume that G and H are *anisotropic* over $\mathbb{Q}$. Consequently (by a Theorem of Borel and Harish-Chandra), the spaces $S(\Gamma)$ are compact. By the Matsushima formula (see (1.2)), the space of harmonic forms on $S(\Gamma)$ may be identified with

$$H^\bullet(S(\Gamma)) = \mathrm{Hom}_{K_\infty}(\wedge^\bullet \mathfrak{p}, \mathcal{C}^\infty(\Gamma\backslash G(\mathbb{R}))(0)). \tag{1}$$

(We have already used the fact that $S(\Gamma)$ is compact in identifying the cohomology of $S(\Gamma)$ with the space of harmonic forms.) Here, $\mathcal{C}^\infty(\Gamma\backslash G(\mathbb{R}))(0)$ is the space of $\mathbb{C}$-valued smooth functions on the quotient $\Gamma\backslash G(\mathbb{R})$, which are annihilated by the Casimir of $\mathfrak{g}$.

Now Y is imbedded in X. Let $\widehat{Y}$ and $\widehat{X}$ be the compact duals of Y and X. There is a natural metric on the dual symmetric space $\widehat{X}$ which is invariant under the action of a maximal compact subgroup of the group $G(\mathbb{C})$ of complex points of the group G (see (1.3)) under which, the space of harmonic forms on $\widehat{X}$ may be identified with

$$H^\bullet(\widehat{X}) = \mathrm{Hom}_{K_\infty}(\wedge^\bullet \mathfrak{p}, \mathbb{C}). \tag{2}$$

From (1) and (2), we obtain a natural inclusion of $H^\bullet(\widehat{X}) \subset H^\bullet(S(\Gamma))$, by identifying $\mathbb{C}$ with the space of constant functions on the quotient $\Gamma\backslash G(\mathbb{R})$.

Suppose that Γ' is a congruence arithmetic subgroup of $G(\mathbb{Q})$ contained in Γ. Then there is a natural inclusion $i(\Gamma, \Gamma')$ of $H^\bullet(S(\Gamma))$ in $H^\bullet(S(\Gamma'))$, induced by the finite covering map $S(\Gamma') \to S(\Gamma)$. Thus we get a direct system of cohomology groups $H^\bullet(S(\Gamma))$ indexed by congruence arithmetic subgroups of $G(\mathbb{Q})$ and maps $i(\Gamma, \Gamma')$ for every $\Gamma' \subset \Gamma$. Consider the direct limit of the spaces $H^\bullet(S(\Gamma))$ as Γ varies through congruence arithmetic subgroups of $G(\mathbb{Q})$. We denote this direct limit by $H^\bullet(\mathrm{Sh}^0 G)$ (the notation seemingly implies that the direct limit is the cohomology of a suitable topological space $\mathrm{Sh}^0 G$; however, in the present paper, we will only use properties of the direct limit, and hence for our purposes, $H^\bullet(\mathrm{Sh}^0 G)$ is merely an abbreviation for the *direct limit* of the cohomology groups $H^\bullet(S(\Gamma))$.

The Matsushima formula (1) then takes the form (see Equation (6) of (1.2))

$$H^\bullet(\mathrm{Sh}^0 G) = \mathrm{Hom}_{K_\infty}(\wedge^\bullet \mathfrak{p}, \mathcal{C}^\infty(G(\mathbb{Q})\backslash(G(\mathbb{R}) \times G_f))(0)). \tag{3}$$

(recall that G_f is the closure of $G(\mathbb{Q})$ in $G(\mathbb{A}_f)$; in (3), $G(\mathbb{Q}) \subset G(\mathbb{R}) \times G_f$ where the

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diagonal map of $G(\mathbb{Q})$ in the product imbeds $G(\mathbb{Q})$ as a cocompact discrete subgroup of $G(\mathbb{R}) \times G_f$. In (3), $\mathcal{C}^\infty(G(\mathbb{Q})\backslash(G(\mathbb{R}) \times G_f))(0)$ denotes the space of 'smooth' functions (i.e. smooth on $G(\mathbb{R})$ and invariant under an open compact subgroup of the totally disconnected group G_f) on the quotient $G(\mathbb{Q})\backslash(G(\mathbb{R}) \times G_f)$ which are annihilated by the Casimir of $\mathfrak{g}$. From (2) and (3) we get a natural imbedding of $H^\bullet(\widehat{X})$ in $H^\bullet(\mathrm{Sh}^0 G)$ identifying $\mathbb{C}$ with the space of constant functions on the quotient $G(\mathbb{Q})\backslash(G(\mathbb{R}) \times G_f)$.

Now (3) shows that there is a natural action of the group G_f on the direct limit $H^\bullet(\mathrm{Sh}^0 G)$, since G_f acts by right shifts on the space

$$\mathcal{C}^\infty(G(\mathbb{Q})\backslash(G(\mathbb{R}) \times G_f)(0)$$

Given $g \in G_f$ denote again by g the map on $H^\bullet(\mathrm{Sh}^0 G)$ induced by right translation by g on the space $\mathcal{C}^\infty(G(\mathbb{Q})\backslash(G(\mathbb{R}) \times G_f)(0)$. The space $H^\bullet(\widehat{X})$ may be identified with the space of G_f-invariants of $H^\bullet(\mathrm{Sh}^0 G)$. If $\overline{\Gamma}$ denotes the closure of Γ in $G_f \subset G(\mathbb{A}_f)$ then $H^\bullet(S(\Gamma))$ may be identified with the space of $\overline{\Gamma}$-invariant vectors in $H^\bullet(\mathrm{Sh}^0 G)$ (see Sections (1.2) and (1.3) for the proofs of these assertions).

Let m, n denote, respectively, the dimensions of the real manifolds $S_H(\Gamma)$ and $S(\Gamma)$. Write $n = m + (n - m)$. Now, closed differential forms on $\widehat{X}$ of degree m can be pulled back to the m-dimensional (oriented) submanifold $\widehat{Y}$ and integrated on $\widehat{Y}$. We thus get a linear form on $H^m(\widehat{X})$ which may be identified, by Poincaré duality for the cohomology of $S(\Gamma)$, with an element (the 'cycle class') $[\widehat{Y}]$ of $H^{n-m}(\widehat{X})$ (see (2.4)). Under the identification of $H^\bullet(\widehat{X})$ with the space of G_f -invariants in $H^\bullet(\mathrm{Sh}^0 G)$, we thus obtain an element in $H^{n-m}(\mathrm{Sh}^0 G)$ which we again denote by $[\widehat{Y}]$.

Similarly, closed differential forms of degree m on $S(\Gamma)$ can be pulled back to the m-dimensional manifold $S_H(\Gamma)$ under the map $j(\Gamma) : S_H(\Gamma) \to S(\Gamma)$ and integrated over $S_H(\Gamma)$. We thus get a linear form on $H^m(S(\Gamma))$, which may be identified, by Poincaré duality for the cohomology of $S(\Gamma)$, with an element of $H^{n-m}(S(\Gamma))$. We denote this element (the 'special cycle class') by $\xi_\Gamma = [S_H(\Gamma)] \in H^{n-m}(S(\Gamma))$ (see (2.5)). Note that by the last sentence in the preceding paragraph, the latter cohomology group may be identified with a subspace of the direct limit $H^{n-m}(\mathrm{Sh}^0 G)$. Thus, we view $[S_H(\Gamma)]$ as an element of $H^{n-m}(\mathrm{Sh}^0 G)$.

We are now in a position to state our first main result.

THEOREM 1. *Denote by V_Γ the $\mathbb{C}$-span of G_f-translates of the cycle class $\xi_\Gamma = [S_H(\Gamma)]$ in the direct limit cohomology group $H^{n-m}(\mathrm{Sh}^0 G)$. Then the space of G_f-invariants in V_Γ (is at most one dimensional and) is generated by $[\widehat{Y}]$.*

Theorem 1 is proved (see Section 2) by using the complete reducibility of the direct limit $H^\bullet(\mathrm{Sh}^0 G)$ as a module over G_f. This is shown to imply that the space of G_f-invariants in V_Γ is spanned by the projection η of the cycle class ξ_Γ. By integrating forms in $H^{n-m}(\widehat{X})$ over η we obtain that these integrals are proportional to the integrals over $\widehat{Y}$. This implies Theorem 1.

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(0.3) We will now define the 'Restriction map' which we mentioned briefly at the beginning of the introduction (see (0.1)). Return now to the Equation (0). Take the direct limit as Γ varies over all the congruence arithmetic subgroups Γ of $G(\mathbb{Q})$, in the Equation (0). We then get a map

$$j^*: H^\bullet(\mathrm{Sh}^0 G) \to H^\bullet(\mathrm{Sh}^0 H).$$

Now an element $g \in G_f$ acts on $H^\bullet(\mathrm{Sh}^0 G)$ as explained in the paragraphs preceding Theorem 1. Denote by $j_g^* = j^* \circ g$ the composite map. Define the *'restriction map'* denoted Res, as the product

$$\mathrm{Res} = \prod j_g^* : H^\bullet(\mathrm{Sh}^0 G) \to \prod H^\bullet(\mathrm{Sh}^0 H), \tag{4}$$

where the product is over all the elements $g \in G_f$.

Our main result is a *necessary* condition for the vanishing of the restriction map defined in the foregoing. It is an easy consequence of Theorem 1. In the following, if $\beta, \delta \in H^\bullet(\mathrm{Sh}^0 G)$ then $\beta \wedge \delta$ denotes the cup product of these two classes.

THEOREM 2. *If* $\alpha \in H^m(\mathrm{Sh}^0 G)$*, then* $\mathrm{Res}(\alpha) = 0$ *only if* $\alpha \wedge [\widehat{Y}] = 0$.

Theorem 2 is proved by observing that the vanishing of the restriction of a class α implies the vanishing of the cup-product $j_g^*(\alpha) \wedge \xi_\Gamma$ for every $g \in G(\mathbb{Q})$. This in turn implies (see Section (3)) the vanishing of $\alpha \wedge g(\xi_\Gamma)$ for every $g \in G_f$. By taking a suitable linear combination of these $g(\xi_\Gamma)$ and by using Theorem 1, we obtain Theorem 2.

(0.4) Suppose now that X is an *irreducible* Hermitian symmetric domain, that Y is also Hermitian symmetric and that the map $f: Y \to X$ is a holomorphic imbedding. If G is the semisimple part of a reductive group G^* satisfying the axioms in [D] and Z is the centre of G^*, then the space $\mathrm{Sh}^0 G =_{\mathrm{def}} G(\mathbb{Q}\backslash(X \times G_f)$ is a connected component of (the space of $\mathbb{C}$-points of) the Shimura Variety $\mathrm{Sh}(G^*, X)$ given by $\mathrm{Sh}(G^*, X)(\mathbb{C}) =_{\mathrm{def}} G^*(\mathbb{Q}\backslash(X \times G(\mathbb{A}_f)$ where $X = G^*(\mathbb{R}/Z(\mathbb{R})K_\infty$.

If Y has codimension one in X, then $S_H(\Gamma)$ is a divisor in $S(\Gamma)$. Moreover, it is known that $S(\Gamma)$ is a smooth projective variety ([B-B]). It can also be proved that $[\widehat{Y}]$ (upto ± 1) is a Lefschetz class on $S(\Gamma)$ for each torsion free (cocompact) Γ as in the foregoing. From Theorem 2 and Lefschetz's theorem on hyperplane sections we obtain

THEOREM 3. *If* $\dim(X) = 1 + \dim(Y)$ *and* X *is an irreducible Hermitian symmetric domain, then*

$$\mathrm{Res}: H^k(\mathrm{Sh}^0 G) \to \prod H^k(\mathrm{Sh}^0 H)$$

is injective for $k \leqslant \dim(Y)(= d = D - 1)$.

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As an application, we obtain

THEOREM 4. (1) *If* $G(\mathbb{R}) = \mathrm{SU}(n, 1)$ *upto compact factors and* $H(\mathbb{R}) = \mathrm{SU}(n-1, 1)$ *upto compact factors, then*

$$\mathrm{Res}\colon H^k(\mathrm{Sh}^0 G) \to \prod H^k(\mathrm{Sh}^0 H)$$

is injective for all $k \leqslant n-1$.

(2) *If* $G(\mathbb{R}) = \mathrm{Spin}(2, n)$ *and* $H(\mathbb{R}) = \mathrm{Spin}(2, n-1)$ *up to compact factors then*

$$\mathrm{Res}\colon H^k(\mathrm{Sh}^0 G) \to \prod H^k(\mathrm{Sh}^0 H)$$

is injective for all $k \leqslant n-1$.

(3) *If* $G(\mathbb{R}) = \mathrm{Sp}_2$, $H(\mathbb{R}) = \mathrm{Sp}_1 \times \mathrm{Sp}_1$*, then*

$$\mathrm{Res}\colon H^2(\mathrm{Sh}^0 G)) \to \prod H^2(\mathrm{Sh}^0 H))$$

is injective.

Remarks. In Theorem 4 the imbeddings of the non-compact factors of $H(\mathbb{R})$ in those of $G(\mathbb{R})$ are the natural ones (e.g. as in Section 3 of [C-V]). For example, in (1) of Theorem 4, $\mathrm{SU}(n-1, 1)$ is that subgroup of $\mathrm{SU}(n, 1)$ which leaves invariant the first basis vector ε_1. Here $\varepsilon_1, \cdots, \varepsilon_n, \varepsilon_{n+1}$ is the standard basis of $\mathbb{C}^{n+1}$ and the latter is the standard representation of $\mathrm{SU}(n, 1)$. The other imbeddings (2) and (3) of Theorem 4 are defined similarly. We do not specify the $\mathbb{Q}$-structures and the $\mathbb{Q}$-imbeddings involved, because the statement and the proof of Theorem 4 do not use the specific nature of the $\mathbb{Q}$-imbedding.

The parts (1) and (2) of Theorem 4 were conjectured by Harris and Li ([H-L]) and proved by them for a number of degrees k (see [H-L] for precise statements). In fact our approach was suggested by an attempt to answer their question: does the linear span of the divisors $\{g[\mathrm{Sh}(H)]; g \in G(\mathbb{A}_f)\}$ contain a very ample divisor? As we have remarked in the paragraph preceding the statement of Theorem 3, the answer is yes, and the ample divisor may be taken to be a divisor (upto ± 1) on the compact dual $\widehat{X}$.

The analogue of (3) when G is *isotropic* over $\mathbb{Q}$, i.e., G is the split Sp_2, was proved by Weissauer ([W]); he showed that $H^2(S(\Gamma))$ injects into (a direct sum of) the cohomology of a product of modular curves.

Theorem 2 also yields the following:

THEOREM 5. *Suppose* $G(\mathbb{R}) = \mathrm{SU}(n, 1)$ *and* $H(\mathbb{R}) = \mathrm{SU}(p, 1)$ *upto compact factors (with* $p \leqslant n-1$*). Then*

$$\mathrm{Res}\colon H^k(\mathrm{Sh}^0 G) \to \prod_{g \in G(\mathbb{Q})} H^k(\mathrm{Sh}^0 H)$$

is injective for all $k \leqslant p$.

We stress that in Theorem 5 we do not assume that G (over $\mathbb{Q}$) contains a $\mathbb{Q}$-subgroup, whose real points form a group isomorphic (upto compact factors) to $SU(k, 1)$ for *every* $k \leqslant n-1$ but only that G (over $\mathbb{Q}$) contains a $\mathbb{Q}$-subgroup, the group of whose real points – up to compact factors – is isomorphic to $SU(p, 1)$ for the integer p in question. Here again the imbedding of the real group $SU(p, 1)$ in the real group $SU(n, 1)$ is the natural one: if $\varepsilon_1, \cdots, \varepsilon_n, \varepsilon_{n+1}$ is the standard basis of the standard representation $\mathbb{C}^{n+1}$ of $SU(n, 1)$, then $SU(p, 1)$ is the subgroup of $SU(n, 1)$ which leaves the first $n-p$ vectors of this standard basis invariant. We do not need to specify the $\mathbb{Q}$-imbedding of the group H in the group G, since the statement (and the proof) of Theorem 5 does not depend on the $\mathbb{Q}$-imbedding.

(0.5) So far, our criteria have been independent of the 'infinity type' of the cohomology class α but dependent only on the degree of the cohomology class. We can split the cohomology of $\mathrm{Sh}^0 G$ in terms of the 'representation type' at infinity and obtain more precise information on the vanishing of the restriction map.

Rewrite the Matsushima formula (cf. Lemme (3.5) (ii) of [Cl 3])

$$H^\bullet(\mathrm{Sh}^0 G) = \oplus m(\pi_\infty \otimes \pi_f) H^\bullet(\mathfrak{g}, K_\infty, \pi_\infty) \otimes \pi_f. \tag{5}$$

Here, $H^\bullet(\mathfrak{g}, K_\infty, \pi_\infty)$ denotes the relative Lie algebra cohomology, π_∞ is (the space of K_∞-finite vectors in) a unitary irreducible representation of $G(\mathbb{R})$ and π_f a unitary irreducible admissible representation of G_f. The number $m(\pi_\infty \otimes \pi_f)$ is the multiplicity with which the representation $\pi_\infty \otimes \pi_f$ of $G(\mathbb{R}) \times G_f$ occurs in the space of square integrable (with respect to the Haar measure) functions on the compact quotient $G(\mathbb{Q}) \backslash (G(\mathbb{R}) \times G_f)$.

Assume now that X is Hermitian Symmetric.

The representations π_∞ with non-trivial $(\mathfrak{g}, K_\infty)$-cohomology (such representations will be referred to as *cohomological*) are classified as the modules $A_\mathfrak{q}$ (see, e.g., [V-Z]) in terms of certain θ-stable parabolic subalgebras $\mathfrak{q}$ of the complex Lie algebra $\mathfrak{g}$ ((recall that θ is the Cartan involution on $\mathfrak{g}$ induced by the pair $(G(\mathbb{R}), K_\infty)$). Given a parabolic subalgebra $\mathfrak{q}$ as in Section (6.1), let $\pi_\infty = A_\mathfrak{q}$ be the associated cohomological representation. Let $\mathfrak{u}$ be its nilradical, denote by k the dimension of the space $\mathfrak{u} \cap \mathfrak{p}$, and $V(\mathfrak{q})$ the K_∞-span of the top exterior power of $\mathfrak{u} \cap \mathfrak{p}$ in $\wedge^k \mathfrak{p}$. Denote the top exterior power of $\mathfrak{u} \cap \mathfrak{p}$ by $e(\mathfrak{q})$. This is a line in $\wedge^k \mathfrak{p}$. Let $\mathfrak{u}^-$ be the 'opposite' of $\mathfrak{u}$ (see Section (6.1)). It can be shown that $\dim[(\mathfrak{u} \cap \mathfrak{p}^+) + (\mathfrak{u}^- \cap \mathfrak{p}^+)]$ is also $k = \dim[\mathfrak{u} \cap \mathfrak{p}]$. Write $V^+(\mathfrak{q})$ for the K_∞-span of the top exterior power $e^+(\mathfrak{q})$ of $(\mathfrak{u} \cap \mathfrak{p}^+) + (\mathfrak{u}^- \cap \mathfrak{p}^+)$.

We will say that a cohomology class $\alpha \in H^k(\mathrm{Sh}^0 G)$ is of type $A_\mathfrak{q}$ if under the Matsushima isomorphism (5), α lies in the component corresponding to $\pi_\infty = A_\mathfrak{q}$ on the right hand side of (5). If, further, the degree k of the class α is the dimension $\mathfrak{u} \cap \mathfrak{p}$ as in the last paragraph, then we will say that α is *strongly primitive* of type $A_\mathfrak{q}$.

From now on, we will assume that X and Y are Hermitian symmetric and that $f: Y \to X$ is holomorphic. For an integer k, denote by $E(G, H, k)$ the K_∞-span of the kth exterior power $\wedge^k \mathfrak{p}_H^+$ in the space $\wedge^k \mathfrak{p}^+$.

THEOREM 6. *With the foregoing notation, suppose that* α *is a strongly primitive class of type* $A_\mathfrak{q}$ *of degree* k. *Suppose that* $\mathrm{Res}(\alpha) = 0$. *Then,*

(1) $V(\mathfrak{q})^* \wedge [\widehat{Y}] = 0$.
(2) *The intersection* $V^+(\mathfrak{q}) \cap E(G, H, k) = 0$.

Remark. Note that $[\widehat{Y}]$ is an element of $H^\bullet(\widehat{X}) = \overset{\bullet}{\wedge} \mathfrak{p}^*$. Here, $\mathfrak{p}^*$ is the dual of $\mathfrak{p}$. In (1) of Theorem 6, $V(\mathfrak{q})^* \subset \overset{k}{\wedge} \mathfrak{p}^*$ similarly denotes the dual of $V(\mathfrak{q})$ and the Equation (1) holds in the exterior algebra $\overset{\bullet}{\wedge} \mathfrak{p}^*$. Similarly, Equation (2) of Theorem 6 holds in the exterior algebra $\overset{\bullet}{\wedge} \mathfrak{p}^+$.

Remark. If α is *holomorphic*, then (1) of Theorem 1 is *equivalent* to the condition

$$\text{all the } G_f\text{-translates of } \alpha \text{ vanish on } \mathrm{Sh}^0 H. \tag{0}$$

The equivalence of (0) and of (2) of Theorem 6 is exactly the criterion obtained in [C-V] for *holomorphic* cohomology classes α.

0.6. CUP PRODUCTS

Note that if α and β are cohomology classes on $S(\Gamma)$, then the restriction to the diagonal of the class $\alpha \otimes \beta$ on the product $S(\Gamma) \times S(\Gamma)$ is the cup product $\alpha \wedge \beta$. Denote by $\widehat{\Delta}$ the *diagonal* in the product $\widehat{X} \times \widehat{X}$. Let $[\widehat{\Delta}]$ be the associated cycle class in $H^\bullet(\widehat{X} \times \widehat{X})$.

By using the criterion of Theorem 6 to the situation of cup products we obtain the following.

THEOREM 7. *Let* α *and* α' *be strongly primitive cohomology classes in* $H^\bullet(\mathrm{Sh}^0 G)$ *of degrees* k *and* k', *and of type* $A_\mathfrak{q}$ *and* $A_{\mathfrak{q}'}$, *respectively. If* $g(\alpha) \wedge \alpha' = 0$ *for all* $g \in G_f$ *and if* $k + k' \leqslant D$ *then* $(\alpha \otimes \alpha') \wedge [\widehat{\Delta}] = 0$. *Further, we have*

$$E(G \times G, G, k + k') \cap (V^+(\mathfrak{q}) \otimes V^+(\mathfrak{q}')) = 0 \tag{1}$$

and

$$V^+(\mathfrak{q}) \wedge V^+(\mathfrak{q}') = 0. \tag{2}$$

(Here D is the complex dimension of the Hermitian symmetric domain X.) Moreover, these two conditions (1) and (2) are equivalent.

Remark. Note that if α and α' are holomorphic classes, then ([C-V], Section 1) $V^+(\mathfrak{q})$ and $V^+(\mathfrak{q}')$ are irreducible and $e(\mathfrak{q}) \wedge \kappa_0 e(\mathfrak{q}')$ generates $V^+(\mathfrak{q}) \wedge V^+(\mathfrak{q}')$ as a

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K_∞-module, (here κ_0 is the longest element of the Weyl group of (K_∞, T) and T is a fixed maximal torus in K_∞)). Then the condition $V^+(\mathfrak{q}) \wedge V^+(\mathfrak{q}') = 0$ of Theorem 7, is equivalent to the condition

$$e(\mathfrak{q}) \wedge \kappa_0 e(\mathfrak{q}') = 0,$$

i.e. the intersection of $\mathfrak{u} \cap \mathfrak{p}^+$ and $\kappa_0(\mathfrak{u}') \cap \mathfrak{p}^+$ in $\mathfrak{p}^+$ is non-zero. This is exactly the criterion of [C1] for the vanishing of the cup product $g^*\alpha \wedge \alpha'$ for all $g \in G_f$.
As applications, we have

THEOREM 8. *Let* $G(\mathbf{R}) = \mathrm{SU}(n, 1)$ *up to compact factors. Let* α, α' *be non-zero cohomology classes of degrees* k, k' *in* $H^\bullet(\mathrm{Sh}^0 G)$ *with* $k + k' \leqslant n$. *Then there exists a* $g \in G_f$ *such that* $g(\alpha) \wedge \alpha' \neq 0$.

Remark. In the holomorphic case, this is proved in [S1] and [C2].

THEOREM 9. *Let* $G(\mathbf{R}) = \mathrm{Spin}(n, 2)$ *upto compact factors. Let* α, α' *be non-zero cohomology classes of degrees* k, k' *with* $k, k' < [n2]$ *(where* $[n2]$ *is the integral part). Then there exists a* $g \in G_f$ *such that* $g(\alpha) \wedge \alpha' \neq 0$.

Remark. Related results are proved in [Ku].

1. The Structure of $H^\bullet(\mathrm{Sh}^0 G)$ as a G_f Module

1.1. THE STRUCTURE OF $H^\bullet(\mathrm{Sh}^0 G)$ AS AN ALGEBRA

As in the introduction, let G be a connected semisimple algebraic group defined over $\mathbb{Q}$. Therefore, G is an almost direct product of $\mathbb{Q}$-simple connected $\mathbb{Q}$-subgroups G_i of G: $G = G_1 G_2 \cdots G_a$. Assume that each $\mathbb{Q}$ simple factor G_i is isotropic over $\mathbb{R}$ (i.e. that $G_i(\mathbb{R})$ is non-compact for each i). We will assume that $G(\mathbb{R})$ is *connected*. Let K_∞ be a maximal compact subgroup of the group $G(\mathbb{R})$ of real points of G. Form the symmetric space $X = G(\mathbb{R})/K_\infty$ as in the introduction.

Let $\mathbb{A}_f$ be the ring of finite adéles over $\mathbb{Q}$. The natural imbedding of $\mathbb{Q}$ in the finite adeles $\mathbb{A}_f$ induces an imbedding of $G(\mathbb{Q})$ in the group $G(\mathbb{A}_f)$. Denote by G_f the closure of $G(\mathbb{Q})$ in the group $G(\mathbb{A}_f)$. Since $G(\mathbb{A}_f)$ is totally disconnected, so is G_f and a fundamental system of neighbourhoods of identity in G_f is provided by the intersections $K = K_0 \cap G_f$ where K_0 runs through compact open subgroups of $G(\mathbb{A}_f)$. Note that the group K is a compact open subgroup of G_f and that the intersection $\Gamma = K \cap G(\mathbb{Q})$ is a congruence arithmetic subgroup of $G(\mathbb{Q})$. Conversely, by definition, a congruence arithmetic subgroup Γ of $G(\mathbb{Q})$ is an intersection $K \cap G(\mathbb{Q})$ for a compact open subgroup K of G_f. Then the density of $G(\mathbb{Q})$ in G_f immediately implies that the closure of Γ in G_f is precisely K. We now assume that K is so chosen that the group $\Gamma = K \cap G(\mathbb{Q})$ is torsion-free. Then, Γ acts properly discontinuously, without fixed points on the symmetric space X and thus $S(\Gamma) = \Gamma\backslash X$ is a manifold, covered by X.

If M is a manifold, denote by $\Omega^\bullet(M)$ the space of smooth differential forms on M (i.e. sections of the exterior powers of the *complexified* cotangent bundle of M). This is an algebra under wedge product (which is not commutative in general). It contains as a subalgebra, the subspace $\Omega_0^\bullet(M)$ of *closed* differential forms. The space of exact forms is a two sided ideal in the algebra of closed forms. Thus we get, by de-Rham's Theorem, an algebra homomorphism from $\Omega_0^\bullet(M)$ onto the cohomology algebra $H^\bullet(M)$ of M (with coefficients in $\mathbb{C}$), whose kernel is the space of exact forms.

Let $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{p}$ be, respectively, the complexified lie algebra of G, of K_∞, and the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form. Then the space $\Omega^\bullet(\Gamma\backslash X)$ may be naturally identified with

$$\mathrm{Hom}_{K_\infty}(\wedge^\bullet\mathfrak{p}, \mathcal{C}^\infty(\Gamma\backslash G(\mathbb{R})))$$

(e.g., see [R], Chapter 7, Section 3, Equation (7.14)). The latter is nothing but the invariants of K_∞ in the tensor product $\wedge^\bullet\mathfrak{p}^* \otimes \mathcal{C}^\infty(\Gamma\backslash G(\mathbb{R}))$. The latter space is a tensor product of the exterior algebra of $\mathfrak{p}^*$ (the dual of $\mathfrak{p}$) and the algebra of smooth functions on the quotient $\Gamma\backslash G(\mathbb{R})$. Therefore the tensor product (and the space of K_∞-invariants in it) gets a natural structure of an algebra and the foregoing isomorphism

$$\Omega^\bullet(\Gamma\backslash X) = \mathrm{Hom}_{K_\infty}(\wedge^\bullet\mathfrak{p}, \mathcal{C}^\infty(\Gamma\backslash G(\mathbb{R}))) \tag{1}$$

is an isomorphism of algebras.

Now the density of $G(\mathbb{Q})$ in G_f shows that

$$S(\Gamma) = \Gamma\backslash X = G(\mathbb{Q})\backslash(X \times G_f)/K \stackrel{\text{def}}{=} S_K^0,$$

where K is an open compact subgroup of G_f and $\Gamma = K \cap G(\mathbb{Q})$. Under the identification of $S(\Gamma)$ with S_K^0, the isomorphism (1) becomes

$$\begin{aligned}\Omega^\bullet&(G(\mathbb{Q})\backslash(X \times G_f)/K) \\ &= \mathrm{Hom}_{K_\infty}(\wedge^\bullet\mathfrak{p}, \mathcal{C}^\infty(G(\mathbb{Q})\backslash(G(\mathbb{R}) \times G_f)/K))\end{aligned} \tag{2}$$

as an isomorphism of algebras. We often use both the descriptions $S(\Gamma)$ and S_K^0 interchangeably, according to our convenience. Since these two are isomorphic, there is no cause for confusion.

By taking direct limits in Equation (2) as K varies through compact open subgroups of G_f we obtain

$$\begin{aligned}\lim\ &\Omega^\bullet(G(\mathbb{Q})\backslash X \times G_f/K) \\ &= \mathrm{Hom}_{K_\infty}(\wedge^\bullet\mathfrak{p}, \mathcal{C}^\infty(G(\mathbb{Q})\backslash(G(\mathbb{R}) \times G_f))),\end{aligned} \tag{3}$$

as an isomorphism of algebras. Here lim denotes the direct limit as K varies through open compact subgroups of G_f. We will denote the direct limit on the left side of Equation (3) by $\Omega^\bullet(\mathrm{Sh}^0 G)$.

1.2. THE ACTION OF G_f ON $H^\bullet(\mathrm{Sh}^0 G)$.

Fix $g \in G_f$. We get an isomorphism S_K^0 with $S^0_{g^{-1}Kg}$ induced by right translation by $g \in G_f$ on the quotient $G(\mathbb{Q})\backslash(G(\mathbb{R}) \times G_f)$. Thus, translation by g yields an algebra isomorphism of $\Omega^\bullet(S_K^0)$ with $\Omega^\bullet(S^0_{g^{-1}Kg})$. By taking direct limits over K we get an action of the the group G_f on $\Omega^\bullet(\mathrm{Sh}^0 G)$; under the isomorphism (3) of Section (1.1), this action is the same as right translations on the space $\mathcal{C}^\infty$ of smooth functions on the right-hand side of (3). Note that G_f acts by algebra automorphisms on $\Omega^\bullet(\mathrm{Sh}^0 G)$, as is evident from the fact that g from $\Omega^\bullet(S_K^0)$ onto $\Omega^\bullet(S^0_{g^{-1}Kg})$ is an algebra isomorphism. It is clear that the action of G_f commutes with the differentials of the complex $\Omega^\bullet(\mathrm{Sh}^0 G)$.

Now, as in the introduction, $H^\bullet(\mathrm{Sh}^0 G)$ is defined as the direct limit of $H^\bullet(S_K^0)$ as K varies over compact open subgroups of G_f. Thus, $H^\bullet(\mathrm{Sh}^0 G)$ is a quotient algebra of $\Omega_0^\bullet(\mathrm{Sh}^0 G)$, the algebra of closed differential forms in $\Omega^\bullet(\mathrm{Sh}^0 G)$. The group G_f acts on this algebra $H^\bullet(\mathrm{Sh}^0 G)$, by algebra automorphisms since it commutes with the differentials of $\Omega^\bullet(\mathrm{Sh}^0 G)$.

Let $K \subset G_f$ be a compact open subgroup. By taking K invariants in (3) (in the following, the space of K invariants in a $\mathbb{C}$-vector space W on which K acts will be denoted by W^K), we obtain that

$$(\Omega^\bullet(\mathrm{Sh}^0 G))^K = \mathrm{Hom}_{K_\infty}(\wedge^\bullet \mathfrak{p}, \mathcal{C}^\infty(G(\mathbb{Q})\backslash(G(\mathbb{R}) \times G_f)/K)). \tag{4}$$

By Equation (2), the space on the right-hand side of (4) is precisely $\Omega^\bullet(S_K^0)$. By observing that the differentials in the complex $\Omega^\bullet(\mathrm{Sh}^0 G)$ commute with the action of G_f and by averaging with respect to K, we obtain from Equation (4), that

$$H^\bullet(S_K^0) = (H^\bullet(\mathrm{Sh}^0 G))^K. \tag{5}$$

From now on, we assume that G is anisotropic *over* $\mathbb{Q}$.

There is a $G(\mathbb{R})$-invariant metric on $X = G(\mathbb{R})/K_\infty$ which, at the tangent space $\mathfrak{p}_0$ at the identity coset eK_∞ coincides with the restriction of the Killing form. This descends to a metric on $S(\Gamma) = S_K^0$. By the Matsushima formula (see [B-W]), the space of Harmonic forms on $S(\Gamma)$ for this metric is given by

$$\mathrm{Hom}_{K_\infty}(\wedge^\bullet \mathfrak{p}, \mathcal{C}^\infty(\Gamma\backslash G(\mathbb{R}))(0))$$

where (as explained in the introduction) $\mathcal{C}^\infty(\Gamma\backslash G(\mathbb{R}))(0)$ denotes the space of smooth functions on $\Gamma\backslash G(\mathbb{R})$ killed by the Casimir of $\mathfrak{g}$.

Note that this space of harmonic differential forms is not an algebra in general, since a wedge product of harmonic forms need not be harmonic. However, by our assumption that G is anisotropic over $\mathbb{Q}$, we obtain – by Hodge Theory on $S(\Gamma)$ – that this space maps isomorphically onto the cohomology $H^\bullet(S(\Gamma)) =$

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$H^\bullet(S^0_K)$. By taking direct limits as K varies through compact open subgroups of G_f we obtain an isomorphism

$$H^\bullet(\mathrm{Sh}^0 G) = \mathrm{Hom}_{K_\infty}(\wedge^\bullet \mathfrak{p}, \mathcal{C}^\infty(G(\mathbb{Q})\backslash(G(\mathbb{R}) \times G_f))(0)). \tag{6}$$

As we have said before, the right-hand side of (6) is not an algebra in general, but is a space of closed forms in $\Omega^\bullet(\mathrm{Sh}^0 G)$, which maps isomorphically onto the cohomology algebra $H^\bullet(\mathrm{Sh}^0 G)$.

As before, the under the isomorphism (6), the G_f action on $H^\bullet(\mathrm{Sh}^0 G)$ is induced by right translation by G_f on the space $\mathcal{C}^\infty(G(\mathbb{Q})\backslash(G(\mathbb{R}) \times G_f)$.

In particular, the space of G_f-invariants in $H^\bullet(\mathrm{Sh}^0 G)$ is isomorphic to $\mathrm{Hom}_{K_\infty}(\wedge^\bullet \mathfrak{p}, \mathbb{C})$. This is because the space of G_f invariant functions on the space $G(\mathbb{Q})\backslash(G(\mathbb{R}) \times G_f)$ are just the constant functions (because of the density of $G(\mathbb{Q})$ in $G(\mathbb{R})$; this is weak approximation). Moreover, the constant functions are certainly killed by the Casimir of $\mathfrak{g}$.

1.3. THE COMPACT DUAL OF X

Consider the *real* Lie subalgebra $\mathfrak{g}_u = \mathfrak{k}_0 \oplus i\mathfrak{p}_0$ where i is a square root of -1. Let $G(\mathbb{C})$ be the group of complex points of G. The analytic subgroup of $G(\mathbb{C})$ whose Lie algebra is $\mathfrak{g}_u$ is denoted G_u. As is well known, G_u is a maximal compact subgroup of $G(\mathbb{C})$. Clearly, K_∞ is a subgroup of G_u. Form the quotient $\widehat{X} = G_u/K_\infty$. This is a compact symmetric space, which is the 'compact dual' of X.

The Killing form on $\mathfrak{g}_u$ is *negative definite* and its negative gives a K_∞-invariant metric on the space $i\mathfrak{p}_0$ (which may be identified with the tangent space at the identity coset eK_∞ of $\widehat{X} = G_u/K_\infty$). By translation, we get a G_u invariant metric on $\widehat{X}$. The space of complex harmonic differential forms on $\widehat{X}$ with respect to this metric is by a theorem of E. Cartan, given by $\mathrm{Hom}_{K_\infty}(\wedge^\bullet \mathfrak{p}, \mathbb{C})$. Note however, that this latter space is indeed an algebra under wedge product, and it maps isomorphically onto the cohomology of the compact manifold $\widehat{X}$:

$$H^\bullet(\widehat{X}) = \mathrm{Hom}_{K_\infty}(\wedge^\bullet \mathfrak{p}, \mathbb{C}). \tag{7}$$

Note that the isomorphism (7) is an isomorphism of *algebras*.

Now, the space $\mathrm{Hom}_{K_\infty}(\wedge^\bullet \mathfrak{p}, \mathbb{C})$ is a *subalgebra* of the algebra of closed (these classes are even harmonic, by the Matsushima Formula) differential forms in $\Omega^\bullet(\Gamma\backslash X)$ for every torsion free cocompact Γ as in the end of (1.2). The closed differential forms map onto the cohomology algebra $H^\bullet(S(\Gamma))$ (and the harmonic ones map bijectively onto $H^\bullet(S(\Gamma)) = H^\bullet(S^0_K)$ where K is the closure of Γ in the group G_f).

Thus, the cohomology algebra $H^\bullet(\widehat{X})$ may be thought of as a subalgebra of the algebra $H^\bullet(S^0_K)$. The latter, by Equation (5) of Section (1.2), is the subalgebra of K-invariants in the direct limit cohomology algebra $H^\bullet(\mathrm{Sh}^0 G)$. The isomorphisms (6) and (7) (of the Sections (1.2) and (1.3) respectively) show that under this isomorphism, $H^\bullet(\widehat{X})$ is precisely the space of G_f-invariants of $H^\bullet(\mathrm{Sh}^0 G)$.

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1.4. A CANONICAL GENERATOR OF $H^n(S^0_K)$

Let (as in the introduction) n be the dimension of the *real* manifold X (or of the real manifold S^0_K). Our assumption on the connectedness of $G(\mathbb{R})$ implies that K_∞, being a maximal compact subgroup of $G(\mathbb{R})$, is connected and acts trivially on the top exterior power of $\mathfrak{p}_0$, the tangent space at the identity coset of $X = G(\mathbb{R})/K_\infty$. Hence, by translation of an orientation of $\mathfrak{p}_0$ everywhere on X, we see that X is orientable and that the orientation is preserved by $G(\mathbb{R})$. By the same reasoning, since $G(\mathbb{C})$ and its maximal compact G_u are connected, and K_∞ is connected, it follows that $\widehat{X} = G_u/K_\infty$ is also orientable, with an orientation preserved by G_u. In particular, $\widehat{X}$ is a compact oreintable manifold, and hence its top degree cohomology is one dimensional: $H^n(\widehat{X}) = \mathbb{C}\omega_G$. Here, we have fixed a generator ω_G of $H^n(\widehat{X})$ once and for all.

The orientability of $S(\Gamma)$ implies that its top degree cohomology is also one dimensional: $H^n(S(\Gamma)) = \mathbb{C}$. Now, by the first sentence of the last paragraph of Section (1.3), there is a natural *imbedding* of $H^n(\widehat{X})$ in $H^n(S(\Gamma))$: thus, the (image of the) generator ω_G of $H^n(\widehat{X})$ chosen in the foregoing paragraph, is a generator of $H^n(S(\Gamma))$. We will always use in the sequel, this generator ω_G of $H^n(S(\Gamma))$. Note that ω_G is *not*, in general, the unit volume form on $S(\Gamma)$ because the volume of $S(\Gamma)$ with respect to ω_G goes to infinity as K (the closure of Γ in G_f) becomes smaller.

Thus $H^n(S(\Gamma)) = \mathbb{C}\omega_G = H^n(\widehat{X})$. By taking direct limits we obtain $H^n(\mathrm{Sh}^0 G) = \mathbb{C}\omega_G = H^n(\widehat{X})$.

We will now collect together our conclusions on the structure of the direct limit $H^\bullet(\mathrm{Sh}^0 G)$ as a G_f module.

(1.5) PROPOSITION. *Let G be a semisimple group defined and anisotropic over $\mathbb{Q}$ satisfying the assumptions of (1.1). Let X be the symmetric space associated to $G(\mathbb{R})$. Then the following hold.*

(1) *As a module over the group G_f, The direct limit $H^\bullet(\mathrm{Sh}^0 G)$ is a direct sum of irreducible smooth G_f-modules π_f, with each π_f occurring with a finite multiplicity $m(\pi_f)$:*

$$H^\bullet(\mathrm{Sh}^0 G) = \bigoplus m(\pi_f)\pi_f.$$

(2) *$H^\bullet(\mathrm{Sh}^0 G)$ is smooth and admissible as a G_f module. Further, G_f acts by algebra automorphisms of the cohomology algebra $H^\bullet(\mathrm{Sh}^0 G)$.*

(3) *The cohomology algebra $H^\bullet(\widehat{X})$ of the compact dual $\widehat{X}$ of X, under the imbedding (7) defined by the Matsushima formula, is isomorphic (as an algebra) to the subalgebra of G_f-invariants in $H^\bullet(\mathrm{Sh}^0 G)$.*

(1.6) *Remark.* From now on, we will (as we may, thanks to (3) of Proposition (1.5)) think of $H^\bullet(\widehat{X})$ as a subalgebra of $H^\bullet(\mathrm{Sh}^0 G)$. We will (as we may, because of

Equation (5) of (1.2)) also think of $H^\bullet(S(\Gamma))$ as the subalgebra of K-invariants in $H^\bullet(\mathrm{Sh}^0 G)$ where K is the closure of Γ in G_f. This shows that $H^\bullet(\widehat{X})$ is a subalgebra of $H^\bullet(S(\Gamma))$ for every (torsion free, cocompact) congruence arithmetic subgroup Γ of $G(\mathbb{Q})$.

Proof of Proposition (1.5). (1) is well known and follows from the Matsushima Formula. For a reference see [C 3], Lemme (3.15), (ii) (in [C 3] (1) is stated for the group $G(\mathbb{A}_f)$ but the same proof goes through word for word, by replacing the group $G(\mathbb{A}_f)$ by G_f).

To prove (2), let $K \subset G_f$ be a compact open subgroup. We may assume (by replacing K by a smaller open subgroup if necessary) that $\Gamma = G(\mathbb{Q}) \cap K$ is torsion free. Then, by Equation (5) of Section (1.2), the space of K invariants in $H^\bullet(\mathrm{Sh}^0 G)$ is precisely $H^\bullet(S_K^0) = H^\bullet(S(\Gamma))$ and the latter is finite-dimensional because $S(\Gamma)$ is a compact manifold. This proves the admissibility of $H^\bullet(\mathrm{Sh}^0 G)$ (recall that a G_f-module W is admissible if the space of K invariants in W is finite-dimensional for each open compact subgroup K of G_f). As $H^\bullet(\mathrm{Sh}^0 G)$ is the direct limit of $H^\bullet(S_K^0)$, it follows that $H^\bullet(\mathrm{Sh}^0 G)$ is a smooth G_f -module as well (a G_f-module is smooth if every vector in the module is fixed by some open compact subgroup of G_f).

Note that (3) simply restates the conclusion of the last paragraph of Section (1.3).

2. The Cycle Class $[S_H(\Gamma)]$ and the Restriction Map

2.1. NOTATION

Let H be a connected semisimple algebraic group over $\mathbb{Q}$ (all of whose $\mathbb{Q}$-simple factors are isotropic over $\mathbb{R}$). Assume that $H(\mathbb{R})$ is connected. Let $f: H \to G$ be a morphism of algebraic $\mathbb{Q}$-groups with finite kernel. Thus f induces an injection of the Lie algebra $\mathfrak{h}_0$ of $H(\mathbb{R})$ into $\mathfrak{g}_0$. Assume that f and H have the following properties: $H(\mathbb{R}) \cap f^{-1}(K_\infty) = K_\infty^H$ is a maximal compact subgroup of $H(\mathbb{R})$; further, The Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ when restricted to the subalgebra $\mathfrak{h}_0$ gives the Cartan decomposition $\mathfrak{h}_0 = \mathfrak{k}_0^H \oplus \mathfrak{p}_0^H$ where $\mathfrak{k}_0^H$ is the Lie algebra of K_∞^H and $\mathfrak{p}_0^H$ is the intersection $\mathfrak{p} \cap \mathfrak{h}_0$; now $Y \stackrel{\mathrm{def}}{=} H(\mathbb{R})/K_\infty^H$ is a symmetric space and $f: Y \to X$ is an imbedding of symmetric spaces.

2.2. THE RESTRICTION MAP

Let Γ be a torsion free congruence arithmetic subgroup of $G(\mathbb{Q})$. Denote by $\Gamma \cap H$ the inverse image of Γ in $H(\mathbb{Q})$ under the map $f: H(\mathbb{Q}) \to G(\mathbb{Q})$. Write $S_H(\Gamma) = \Gamma \cap H \backslash Y$. As explained in the introduction, $S_H(\Gamma)$ is a manifold covered by Y. The map f induces a smooth map $j = j(\Gamma): S_H(\Gamma) \to S(\Gamma)$. This induces a homomorphism j^* of the cohomology algebras:

$$j(\Gamma)^*: H^\bullet(S(\Gamma)) \to H^\bullet(S_H(\Gamma)). \tag{1}$$

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By taking direct limit of both sides of Equation (1) as Γ varies through torsion free congruence arithmetic subgroups of $G(\mathbb{Q})$, we obtain a homomorphism of algebras

$$j^*: H^\bullet(\mathrm{Sh}^0 G) \to H^\bullet(\mathrm{Sh}^0 H). \tag{2}$$

Fix $g \in G_f$. Then g acts on $H^\bullet(\mathrm{Sh}^0 G)$ as in (1.2), by a linear transformation which we again denote by g. Then the composite $j_g^* = j^* \circ g$ is a homomorphism from $H^\bullet(\mathrm{Sh}^0 G)$ into $H^\bullet(\mathrm{Sh}^0 H)$. The product over all $g \in G_f$ of these maps j_g^*, is referred to as the *restriction map* and is denoted Res:

$$\mathrm{Res} = \prod j_g^*: H^\bullet(\mathrm{Sh}^0 G) \to \prod H^\bullet(\mathrm{Sh}^0 H). \tag{3}$$

Maps similar to Res have been considered in [Oda], [H-L], [K-R], [C-V].

2.3. NOTATION

Let $\mathfrak{h}$ be the complexification of $\mathfrak{h}_0$ (define similarly, the complex vector spaces $\mathfrak{k}^H$ and $\mathfrak{p}^H$). Let $\mathfrak{h}_u = \mathfrak{k}_0^H \oplus i\mathfrak{p}_0^H$. Then the map $f: H \to G$ induces an injective Lie algebra homomorphism $\mathfrak{h}_u \to \mathfrak{g}_u$. Let $H(\mathbb{C})$ be the group of complex points of H (with Lie algebra $\mathfrak{h}$). Then, the analytic subgroup H_u of $H(\mathbb{C})$ with Lie algebra $\mathfrak{h}_u$ is a maximal compact subgroup of $H(\mathbb{C})$ which clearly contains K_∞^H. Form the 'compact dual' $\widehat{Y} = H_u/K_\infty^H$. The map f induces a natural imbedding of the compact dual symmetric spaces $\widehat{j}: \widehat{Y} \to \widehat{X}$.

2.4. THE CYCLE CLASS $[\widehat{Y}]$

Consider the imbedding $\widehat{j}: \widehat{Y} \to \widehat{X}$. Let m be the real dimension of Y. Now by the definition of ω_H, it generates the cohomology group $H^m(\widehat{Y})$. Therefore, given $\alpha \in H^m(\widehat{X})$, its pullback $\widehat{j}^*(\alpha)$ is of the form $\lambda(\alpha)\omega_H$ for some scalar $\lambda(\alpha)$. Thus, λ defines a linear form on $H^m(\widehat{X})$.

Given $\beta \in H^{n-m}(\widehat{X})$ we get a linear form λ_β on the space $H^m(\widehat{X})$ given by $\lambda_\beta(\alpha)\omega_G = \alpha \wedge \beta$ for all $\alpha \in H^m(\widehat{X})$. By Poincaré duality for the cohomology of $\widehat{X}$, it follows that *every* linear form on $H^m(\widehat{X})$ is of the form λ_β for some $\beta \in H^{n-m}(\widehat{X})$. Therefore the linear form λ of the previous paragraph is of the form $\lambda_{[\widehat{Y}]}$ for some element $[\widehat{Y}] \in H^{n-m}(\widehat{X})$. We will refer to this element $[\widehat{Y}]$ as the cycle class associated to the cycle $\widehat{Y}$ in $\widehat{X}$.

2.5. DEFINITION OF THE CYCLE CLASS $[S_H(\Gamma)]$

We will now fix the congruence arithmetic subgroup Γ of $G(\mathbb{Q})$. As in (1.4), our assumption that $H(\mathbb{R})$ is connected ensures that the $S_H(\Gamma)$ are all orientable. Let m be the (real) dimension of the manifold Y (or of $S_H(\Gamma)$). Then, by replacing G by H throughout in (1.4), we obtain a canonical generator-denoted ω_H – of the (one-dimensional) top degree cohomology $H^m(S_H(\Gamma)) = \mathbb{C}$ of $S_H(\Gamma)$. This

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ω_H is, in fact, a generator of the cohomology of the compact dual $\widehat{Y}$, and there is a natural isomorphism

$$\mathbb{C}\omega_H = H^m(\widehat{Y}) = H^m(S_H(\Gamma)) \tag{4}$$

(only in the top degree m) as explained in the last paragraph of (1.3).

Recall that the dimensions of X and Y are n and m respectively. Consider the cup-product pairing

$$H^m(S(\Gamma)) \times H^{n-m}(S(\Gamma)) \to H^n(S(\Gamma)).$$

The pairing is non-degenerate by Poincaré duality. From (1.4) we get that $H^n(S(\Gamma))$ is generated by ω_G. Given $\beta \in H^{n-m}(S(\Gamma))$, we get a linear form λ_β defined by

$$\alpha \wedge \beta = \lambda_\beta \omega_G \tag{5}$$

for all $\alpha \in H^m(S(\Gamma))$. By Poincaré duality for the cohomology of $S(\Gamma)$, *every* linear form on $H^m(S(\Gamma))$ is of the form λ_η for some $\eta \in H^{n-m}(S(\Gamma))$.

We will now define a linear form λ on $H^m(S(\Gamma))$. From Equation (1) of (2.2) we get a map $j(\Gamma)^*: H^m(S(\Gamma)) \to H^m(S_H(\Gamma))$. From Equation (4), the latter space is isomorphic to $\mathbb{C}\omega_H$. Define $\lambda(\alpha)$ by the formula

$$\lambda(\alpha)\omega_H = j(\Gamma)^*(\alpha). \tag{5}$$

Thus, λ is a linear form on $H^m(S(\Gamma))$. By the conclusion of the preceding paragraph, there is an element (denoted $[S_H(\Gamma)]$) in $H^{n-m}(S(\Gamma))$ such that $\lambda = \lambda_{[S_H(\Gamma)]}$. This is the *cycle class* associated to the special cycle $S_H(\Gamma)$ of the manifold $S(\Gamma)$.

Remark. If the level Γ were fixed once and for all, then the cycle class could have been defined in the usual way (by fixing an arbitrary generator of $S(\Gamma)$). But, since the levels are varying, one needs to be careful in choosing generators of the top degree cohomology of $S_H(\Gamma)$ and $S(\Gamma)$. One can do this here because of the canonical generators ω_H and ω_G arising from the cohomology of the compact duals $\widehat{Y}$ and $\widehat{X}$.

(2.6) *Remark.* Let M and N be compact orientable manifolds of dimensions m and n respectively. Fix (non-zero) generators ω_M and ω_N of $H^m(M)$ and $H^n(N)$ respectively. Let $j: M \to N$ be a smooth map. We then get a linear form $\alpha \mapsto \lambda(\alpha)$ on $H^m(N)$ defined by $j^*(\alpha) = \lambda(\alpha)\omega_N$. By Poincaré duality, there exists an element $[M] \in H^{n-m}(N)$ such that the wedge product $\alpha \wedge [M] = \lambda(\alpha)\omega_N$.

If $k \leqslant n$ is any integer and $\beta \in H^k(N)$ is such that $j^*(\beta) = 0$, then $\beta \wedge [M] = 0$. To prove this, we may assume that $k \leqslant m$ (for otherwise, $\beta \wedge [M]$ is of degree $k + (n-m) > n$ and is zero anyway). Let $\alpha \in H^{m-k}(N)$. Then, $j^*(\alpha \wedge \beta) = \lambda\omega_M$ for some scalar λ. Then, by the definition of the cycle class $[M]$, we get $\alpha \wedge \beta \wedge [M] = \lambda\omega_N$. However, $j^*(\alpha \wedge \beta) = 0$ since $j^*(\beta) = 0$. Therefore, $\lambda = 0$ and we get, $\alpha \wedge \beta \wedge [M] = 0$, for *every* α. by Poincaré duality for $H^\bullet(M)$, we then get $\beta \wedge [M] = 0$.

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(2.7) *Remark*. Let $H^\bullet(\widehat{X}) \subset H^\bullet(\mathrm{Sh}^0 G)$ and $H^\bullet(\widehat{Y}) \subset H^\bullet(\mathrm{Sh}^0 H)$ be the natural imbeddings as in (1.3). Let $\widehat{j}^*: H^\bullet(\widehat{X}) \to H^\bullet(\widehat{Y})$ and $j^*: H^\bullet(\mathrm{Sh}^0 G) \to H^\bullet(\mathrm{Sh}^0 H)$ be the restriction maps. Then, for all $\alpha \in H^\bullet(\widehat{X})$ we have

$$\widehat{j}^*(\alpha) = j^*(\alpha). \tag{7}$$

3. Proof of Theorem 1

Let $V = V_\Gamma$ be the $\mathbb{C}$-span of G_f-translates of the cycle class $\xi = \xi_\Gamma = [S_H(\Gamma)]$. Now, $\xi \in H^{n-m}(S(\Gamma))$ (recall that m and n are the dimensions of the spaces Y and X, respectively). Thus, V is a submodule of the G_f-module $H^{n-m}(\mathrm{Sh}^0 G)$. By Proposition (1.5), $H^{n-m}(\mathrm{Sh}^0 G)$ is completely reducible as a G_f-module. Hence so is the submodule V.

Now, V is a cyclic module containing ξ as a cyclic vector. Hence the space of G_f-invariant linear forms on V is at most one dimensional (since such a linear form is determined completely by its value on the vector ξ). Now complete reducibility of V implies that the space V^0 of G_f-invariant vectors in V is also at most one dimensional. Write $V = V^0 \oplus V^1$ where V^1 is a G_f-invariant subspace of V supplementary to V^0. Since V^1 has no invariant vectors, complete reducibility of V^1 shows that it has no G_f-invariant linear forms on it either. Write $\xi = \eta + \xi^1$ where $\eta \in V^0$ and $\xi^1 \in V^1$. *We will show that*

$$\eta = [\widehat{Y}]. \tag{1}$$

This will prove Theorem 1 because η is a generator of the space V^0 of G_f-invariants in V and is a linear combination of G_f-translates of the cycle class ξ (because every element of V is).

To prove the Equation (1), we proceed as follows. Let

$$\alpha \in H^m(\widehat{X}) \subset H^m(\mathrm{Sh}^0 G)$$

be an arbitrary, but fixed, vector. Consider the wedge product $v \wedge \alpha$ for $v \in V$. Now $H^n(\widehat{X})$ maps isomorphically onto $H^n(\mathrm{Sh}^0 G) = \mathbb{C}\omega_G$ (see the end of (1.4); ω_G is the canonical generator chosen in (1.4)). Note that $v \wedge \alpha \in H^n(\mathrm{Sh}^0 G)$ for all $v \in V$. Thus we may write $v \wedge \alpha = \lambda_\alpha(v)\omega_G$. We first show that λ_α is a G_f-invariant linear form on V. Fix $g \in G_f$ and $v \in V$. Then,

$$g(v \wedge \alpha) = g(v) \wedge g(\alpha), \tag{2}$$

because g acts by algebra automorphisms ((1.2)). Since $v \wedge \alpha$ is a class of degree n and $H^n(\mathrm{Sh}^0 G)$ is $H^n(\widehat{X})$, it follows that $g(v \wedge \alpha) = v \wedge \alpha$. Moreover, $\alpha \in H^m(\widehat{X})$ is G_f-invariant. Thus we get $v \wedge \alpha = g(v) \wedge \alpha$ from Equation (2). By the definition of λ_α, this means that $\lambda_\alpha(v) = \lambda_\alpha(g(v))$. Thus, λ_α is a G_f-invariant linear form on V. Therefore, it vanishes on V^1 (recall that V^1 has no invariant forms). Con-

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sequently, $\lambda_\alpha(\xi) = \lambda_\alpha(\eta)$ and by the definition of λ_α, we obtain

$$\eta \wedge \alpha = \xi \wedge \alpha. \tag{3}$$

We will now show that

$$\alpha \wedge \xi = \alpha \wedge [\widehat{Y}]. \tag{4}$$

By the definition of $[\widehat{Y}]$, $\alpha \wedge [\widehat{Y}] = \mu\omega_G$ where $\mu \in \mathbb{C}$ is such that $\widehat{j}^*(\alpha) = \mu\omega_H$.

Let β be an arbitrary element of $H^m(S(\Gamma))$ and write $j^*(\beta) = \lambda\omega_H \in H^m(S_H(\Gamma))$ where λ is a scalar (which depends on the class β, of course). Then by the definition of the cycle class ξ, we get $\beta \wedge \xi = \lambda\omega_G$. By substituting α for the arbitrary element β we get $j^*(\alpha) = \lambda\omega_H$ with $\alpha \wedge \xi = \lambda\omega_G$.

But by Equation (7) of (2.7), we get that $j^*(\gamma) = \widehat{j}^*(\gamma)$ for all elements $\gamma \in H^m(\widehat{X})$. In particular, the foregoing paragraphs imply that $\lambda = \mu$. This is equivalent to (4).

Since the degrees of α and ξ are m and $n - m$, respectively, we obtain that

$$\alpha \wedge \xi = (-1)^{m(n-m)}\xi \wedge \alpha; \alpha \wedge \eta = (-1)^{m(n-m)}\eta \wedge \alpha. \tag{5}$$

Now the Equations (3), (4) and (5) imply

$$\alpha \wedge [\widehat{Y}] = \alpha \wedge \eta$$

for all $\alpha \in H^m(\widehat{X})$. By Poincaré duality for the cohomology of $\widehat{X}$, we then obtain (1).

This completes the proof of Theorem 1.

4. Proof of Theorem 2

4.1. NOTATION

Let $\xi_\Gamma \in H^{n-m}(S(\Gamma))$ be the cycle class $[S_H(\Gamma)]$ as before. Suppose that Γ' is a congruence arithmetic subgroup of $G(\mathbb{Q})$ which is a *normal* subgroup of Γ. Let $\xi_{\Gamma'} = [S_H(\Gamma')] \in H^{n-m}(S(\Gamma'))$ be the cycle class for the level Γ'. We will view all these cohomology groups as subgroups of $H^{n-m}(\mathrm{Sh}^0 G)$.

Let K and K' be the closures in G_f of Γ and Γ', respectively. Then the imbedding of Γ in K induces a map of finite groups $\Gamma/\Gamma' \to K/K'$ which is an isomorphism since Γ is dense in K.

Recall that G_f operates on $H^{n-m}(\mathrm{Sh}^0 G)$ as in (1.2).

LEMMA (4.2). *With the notation of (4.1), we have*

$$\sum \theta(\xi_{\Gamma'}) = m\xi_\Gamma$$

where m is the order of the group $K/K' = \Gamma/\Gamma'$.

Here, the sum is over all the elements of the finite group K/K'. The action of K' on $\xi_{\Gamma'} \in H^{n-m}(S^0_{K'})$ is trivial by the identification of the latter cohomology group as

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the space K' invariants of $H^{n-m}(\mathrm{Sh}^0 G)$. The equation of the Lemma (makes sense and) holds in the cohomology group $H^{n-m}(\mathrm{Sh}^0 G)$.

Proof of Lemma (4.2). Let $\alpha \in H^m(S(\Gamma))$. Form the wedge product $(\sum \theta(\xi_{\Gamma'})) \wedge \alpha$. Now, the action by θ preserves cup products in $H^\bullet(S(\Gamma'))$. Note that $\theta(\xi_{\Gamma'}) \wedge \alpha \in H^n(S(\Gamma'))$. Further, $H^n(S(\Gamma')) = H^n(\widehat{X}) = \mathbb{C}\omega_G$ and θ acts trivially on $H^n(\widehat{X})$. Therefore, $\theta(\xi_{\Gamma'}) \wedge \alpha = \xi_{\Gamma'} \wedge \theta^{-1}(\alpha)$. But, since $\alpha \in H^m(S(\Gamma))$, the class α is invariant under the action of K/K'. Therefore, $\theta(\xi_{\Gamma'}) \wedge \alpha = \xi_{\Gamma'} \wedge \alpha$ for all θ and we get

$$\left(\sum \theta(\xi_{\Gamma'})\right) \wedge \alpha = m\xi_{\Gamma'} \wedge \alpha.$$

To prove the Lemma, we must then show that $\xi_{\Gamma'} \wedge \alpha = \xi_\Gamma \wedge \alpha$. By the definition of the cycle class ξ_Γ (see (2.4)), $\xi_\Gamma \wedge \alpha$ is a multiple $\lambda\omega_G$ of the canonical class ω_G where λ is defined by $j(\Gamma)^*(\alpha) = \lambda\omega_H$. We are thus reduced to showing that $j(\Gamma')^*(\alpha) = j(\Gamma)^*(\alpha)$ for all $\alpha \in H^m(S(\Gamma))$. This is immediate from the definition of the imbedding of $H^m(S(\Gamma))$ in $H^m(S(\Gamma'))$ as the space of K invariants in $H^m(S(\Gamma'))$.

(4.3) *Proof of Theorem 2.* Suppose that $\alpha \in H^k(\mathrm{Sh}^0 G)$ with $\mathrm{Res}(\alpha) = 0$. Let $g \in G_f$ be arbitrary but fixed. Now, $\alpha \in H^k(S(\Gamma)) = H^k(S_K^0)$ for some congruence arithmetic subgroup Γ of $G(\mathbb{Q})$. Here, K is the closure of Γ in G_f. Thus, $g(\alpha) \in H^k(S(\Gamma') = H^k(S_{K'}^0)$ for some congruence arithmetic subgroup Γ' of $G(\mathbb{Q})$. We may assume, by replacing Γ' by a smaller subgroup if necessary, that Γ' is a normal subgroup of finite index in Γ. Let $\xi_{\Gamma'} = [S_H(\Gamma')]$ be the cycle class corresponding to Γ'. Define ξ_Γ similarly.

We have the map $j(\Gamma') : S_H(\Gamma') \to S(\Gamma')$ and $j_g^* = j(\Gamma')^* \circ g$. Since Res is a product of j_g^* and $\mathrm{Res}(\alpha) = 0$ it follows that $j(\Gamma')^* \circ g(\alpha) = 0$. By Remark (2.6), we obtain that $g(\alpha) \wedge [S_H(\Gamma')] = 0$, i.e,

$$g(\alpha) \wedge \xi_{\Gamma'} = 0. \tag{1}$$

Now Equation (1) still holds if we replace g by $\theta^{-1}g$ where θ is any element of K since the action of θ on $H^k(\mathrm{Sh}^0 G)$ leaves the subspace $H^k(S(\Gamma')) = (H^k(\mathrm{Sh}^0 G))^{K'}$ stable (recall that K normalises K'). Thus

$$\theta^{-1}g(\alpha) \wedge \xi_{\Gamma'} = 0 = g(\alpha) \wedge \theta(\xi_{\Gamma'}). \tag{2}$$

By summing over all $\theta \in K/K'$ in Equation (2) and using Lemma (4.1), we obtain that $g(\alpha) \wedge \xi_\Gamma = 0$. This is equivalent to saying that $\alpha \wedge g(\xi_\Gamma) = 0$ for all $g \in G_f$. By choosing a suitable linear combination and by using Theorem 1 we obtain that $\alpha \wedge [\widehat{Y}] = 0$. This is Theorem 2.

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5. Hermitian Symmetric Domains

5.1. NOTATION

In this section, we will assume that X and Y are Hermitian symmetric domains and that the imbedding $f\colon Y \to X$ is holomorphic. The complex tangent space $\mathfrak{p}$ at the identity coset eK_∞ of $X = G(\mathbb{R})/K_\infty$ then decomposes into a direct sum $\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$ where $\mathfrak{p}^+$(resp. $\mathfrak{p}^-$) denotes the holomorphic (resp. anti-holomorphic) tangent space at the identity coset. We have similarly a decomposition $\mathfrak{p}_H = \mathfrak{p}_H^+ \oplus \mathfrak{p}_H^-$, with $\mathfrak{p}_H^+ \subset \mathfrak{p}^+$ (and similarly $\mathfrak{p}_H^- \subset \mathfrak{p}^-$.

Denote by D and d the *complex* dimensions of the complex manifolds X and Y. In our earlier notation (of Section 1), $n = 2D$ and $m = 2d$.

Let $\kappa \in \mathrm{sym}^2(\mathfrak{p}_0)^*$ denote the restriction to $\mathfrak{p}_0$ of the Killing form on $\mathfrak{g}_0 = \mathrm{Lie}(G(\mathbb{R}))$. Then, κ defines a positive definite symmetric bilinear form on $\mathfrak{p}_0$ which is K_∞ invariant. It can be extended to a $\mathbb{C}$-linear symmetric bilinear form $\kappa_{\mathbb{C}}$ on $\mathfrak{p} = \mathfrak{p}_0 \otimes \mathbb{C}$.

The connected component C of identity of the centre of K_∞ acts nontrivially on $\mathfrak{p}$. Indeed, there exists an element $J \in C$ such that under the adjoint action, J acts by the scalar $\pm i$ on $\mathfrak{p}^\pm$ (see [He], Chapter (VII), Theorem (4.5); there our element J is denoted s_0). We may thus write $\mathfrak{p}^\pm$ as the set of elements $x - (\pm i)J(x)$ with $x \in \mathfrak{p}_0$.

Denote by $v \mapsto \overline{v}$ the complex conjugation on $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}$ which leaves $\mathfrak{g}_0$ pointwise fixed and acts by complex conjugation on the coefficients $\mathbb{C}$. Then it is clear from the last paragraph that the complex conjugation maps $\mathfrak{p}^+$ to $\mathfrak{p}^-$. If $v = x - iJ(x) \in \mathfrak{p}^+$ with $x \in \mathfrak{p}_0$ then

$$\kappa_{\mathbb{C}}(v, \overline{v}) = \kappa(x, x) + \kappa(J(x), J(x)).$$

This shows that the Hermitian form $h : (z, w) \mapsto \kappa_{\mathbb{C}}(z, \overline{w})$ (with $z, w \in \mathfrak{p}^+$) is positive definite on $\mathfrak{p}^+$.

5.2. THE CLASS L

By definition, the element J of the centre of K_∞ (defined in Section (5.1)) acts by -1 on the tensor space $\mathfrak{p}^+ \otimes \mathfrak{p}^+$ (and similarly on $\mathfrak{p}^- \otimes \mathfrak{p}^-$). In particular, there are no K_∞-invariant vectors in the space $\mathfrak{p}^+ \otimes \mathfrak{p}^+$ and in $\mathfrak{p}^- \otimes \mathfrak{p}^-$. Thus, $\kappa_{\mathbb{C}} \in (\mathfrak{p}^+)^* \otimes (\mathfrak{p}^-)^* \subset \mathrm{sym}^2(\mathfrak{p}^*)$. However, the tensor representation $(\mathfrak{p}^+)^* \otimes (\mathfrak{p}^-)^*$ also occurs in

$$\overset{2}{\wedge}\, \mathfrak{p}^* = \overset{2}{\wedge}\, (\mathfrak{p}^+)^* \oplus \overset{2}{\wedge}\, (\mathfrak{p}^-)^* \oplus (\mathfrak{p}^+)^* \otimes (\mathfrak{p}^-)^*.$$

Thus $\kappa_{\mathbb{C}}$ gives an element of

$$\mathrm{Hom}_{K_\infty}(\wedge^2 \mathfrak{p}, \mathbb{C}) = H^2(\widehat{X}).$$

We denote the element of $H^2(\widehat{X})$ thus obtained by L_G. When the group G is fixed, we

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will denote L_G by L. Note that for every torsion free cocompact Γ as in (1.2), this element L lies in $H^2(S(\Gamma))$.

Let (e) be *any* orthonormal basis of $(\mathfrak{p}^+)^*$ under the dual of the Hermitian form h defined on $\mathfrak{p}^+$ in (5.1). Then,

$$L = \sum e \wedge \bar{e} \in \wedge^2 \mathfrak{p}^*. \tag{1}$$

5.3. THE LEFSCHETZ THEOREM FOR L

We note that ([B-B]) the spaces $S(\Gamma)$ are smooth projective varieties. It is also known ([B-B]) that the class L is (upto a scalar multiple) the class of an ample divisor on $S(\Gamma)$. It is then a consequence of the Lefschetz hyperplane section Theorem for $S(\Gamma)$ that if $k \leqslant D$, then the map $\alpha \mapsto \alpha \wedge L^j$ on the space $H^j(S(\Gamma))$ is injective for all $j \leqslant D - k$. The injectivity can also be proved directly, as in Section 9, pages 60-61 of [A]).

5.4. NOTATION

We now take for Z the variety $S^0_K G$ for some open compact subgroup $K \subset G_f$ such that $K \cap G(\mathbb{Q})$ is neat. Then one has a finite map $j: S^0_{H\cap K} \to S^0_K$. As in (1.3), let $\xi = [S^0_{K\cap H}]$ denote the cycle class of the cycle $S^0_{K\cap H}$ in $H^{2D-2d}(S^0_K)$.

If we assume that $X = G(\mathbb{R})/K_\infty$ is an irreducible Hermitian symmetric domain, then K_∞ acts irreducibly on $\mathbf{p}^+$. Now

$$H^2(\widehat{X}, \mathbb{C}) = \mathrm{Hom}_{K_\infty}(\overset{2}{\wedge}(\mathbf{p}^+ \oplus \mathbf{p}^-), \mathbb{C}) = \mathrm{Hom}_{K_\infty}(\mathbf{p}^+ \otimes \mathbf{p}^-, \mathbb{C}) = \mathbb{C},$$

since $\mathbf{p}^-$ is the dual of $\mathbf{p}^+$ as a K_∞-representation. We know that $H^2(\widehat{X}, \mathbb{C})$ contains $L' = -L$. Hence the space of G_f-invariants in $H^2(\mathrm{Sh}^0(G, X))$ is $\mathbb{C}L$.

(5.5) *Proof of Theorems 3 and 4.* Now by assumption $S^0_{H\cap K}$ is a divisor in S^0_K. Therefore $\widehat{Y}$ is a divisor in $\widehat{X}$ and so, by the last paragraph, $[\widehat{Y}]$ is a non-zero multiple of L. By Theorem 1 and 2, we get that if $\alpha \in H^k(\mathrm{Sh}^0 G)$ is such that Res $(\alpha) = 0$ then $\alpha \wedge [\widehat{Y}] = 0$. That is, $\alpha \wedge L = 0$. Since $d + 1 = D$ and the degree of α is $k \leqslant d$, the Lefschetz hyperplane section Theorem tells us that $\alpha = 0$. This proves Theorem 3.

Theorem 4 follows immediately since the pairs (G, H) of Theorem 4 satisfy the hypotheses of Theorem 3.

(5.6) *Proof of Theorem 5.* If $H(\mathbb{R}) = \mathrm{SU}(p, 1)$, upto compact factors, then its compact dual is $\widehat{Y} = \mathbb{P}^p \subset \mathbb{P}^n = \widehat{X}$. Hence $[\widehat{Y}] = L^{n-p}$. If Res $(\alpha) = 0$ for some $\alpha \in H^m(\mathrm{Sh}^0 G)$ with $m \leqslant p$, then by Theorem 2, $\alpha \wedge [\widehat{Y}] = 0$. But $[\widehat{Y}] = L^{n-p}$ and therefore $\alpha \wedge L^{n-p} = 0$ (and the degree of α is $m \leqslant p$). By the Lefschetz hyperplane section Theorem, we then have $\alpha = 0$. Therefore, we have proved Theorem 5.

6. Schubert Cells and the Cohomology of $\widehat{X}$

6.1. NOTATION

Let $T \subset K_\infty$ be a maximal torus. Then, as is well known (and is easy to prove), T is a maximal torus in $G(\mathbb{R})$ as well. Let $\mathfrak{t}_0 = \mathrm{Lie}(T)$, $\mathfrak{t} = \mathfrak{t}_0 \otimes_\mathbb{R} \mathbb{C}$, $\Phi(\mathfrak{k}, T) =$ roots of $T_\mathbb{C}$ in $\mathfrak{k}$, $\Phi(\mathfrak{g}, T) =$ roots of $T_\mathbb{C}$ in $\mathfrak{g}$, $\Phi^+(\mathfrak{k}, T) =$ a system of positive roots on $\mathfrak{k}$ fixed once and for all, $\Phi^+(\mathfrak{g}, T) = \Phi^+(\mathfrak{k}, T) \cup \Phi(\mathfrak{p}^+, T)$, where $\Phi(\mathfrak{p}^+, T) =$ the roots of $T(\mathbb{C})$ in $\mathfrak{p}^+$.

Given $X \in i$ Lie (T) such that $\gamma(X) \geqslant 0$ for all 'positive compact roots' $\gamma \in \Phi^+(\mathfrak{k}, T)$, we set

$$\mathfrak{q} = \mathfrak{q}(X) = \mathfrak{g}^X \oplus_{\gamma(X)>0} \mathfrak{g}_\gamma, \quad \ell = \mathfrak{g}^X, \quad \mathfrak{u} = \oplus_{\gamma(X)>0} \mathfrak{g}_\gamma$$

where $\mathfrak{g}^X =$ centralizer of X in $\mathfrak{g}$. Then $\mathfrak{q}(X)$ is stable under the Cartan involution θ for the pair $(\mathfrak{g}_0, \mathfrak{k}_0)$. We have $\mathfrak{u} \cap \mathfrak{p} = \mathfrak{u} \cap \mathfrak{p}^+ \oplus \mathfrak{u} \cap \mathfrak{p}^-$. Let $p = \dim(\mathfrak{u} \cap \mathfrak{p}^+)$ and $q = \dim(\mathfrak{u} \cap \mathfrak{p}^-)$, and let $k = p + q$.

The complex conjugation on $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C}$ leaving $\mathfrak{g}_0$ fixed pointwise, acts by (-1) on $i\mathfrak{t}_0$, and hence converts positive roots into negative roots and takes $\mathfrak{p}^-$ into $(\mathfrak{p})^+$; denote the complex conjugation by $v \mapsto \overline{v}$ on $\mathfrak{g}$ and again by $w \mapsto \overline{w}$ on the exterior algebra (or the tensor algebra) of $\mathfrak{g}$. Then we set

$$e^+(\mathfrak{q}) = \overset{p}{\wedge} (\mathfrak{u} \cap \mathfrak{p}^+) \wedge \overset{q}{\wedge} \overline{(\mathfrak{u} \cap \mathfrak{p}^+)} \subset \overset{k}{\wedge} \mathfrak{p}^+ \tag{1}$$

and $V^+(\mathfrak{q}) =$ the smallest $K_\infty-$stable subspace of $\overset{k}{\wedge} \mathfrak{p}^+$ containing $e^+(\mathfrak{q})$ (note that if $\mathfrak{u} \cap \mathfrak{p}^- = 0$ i.e. $q = 0$, then $\mathfrak{q}$ is 'holomorphic' and $V^+(\mathfrak{q}) = V(\mathfrak{q}) = K_\infty-$span of $e(\mathfrak{q})$, is irreducible as a $K_\infty-$module). However, $V^+(\mathfrak{q})$ need not be irreducible in general.

Note also that $\mathfrak{u} \cap \overline{\mathfrak{u}} = 0$ (this is because the element X acts by strictly positive eigenvalues on $\mathfrak{u}$ and strictly negative eigenvalues on $\mathfrak{u}^-$) Therefore, $e^+(\mathfrak{q}) \neq 0$. Write $V(\mathfrak{q})$ for the $K_\infty-$span of $e(\mathfrak{q})$. Then $V(\mathfrak{q}) \subset \overset{p}{\wedge} \mathfrak{p}^+ \otimes \overset{q}{\wedge} \mathfrak{p}^-$ and $V(\mathfrak{q})$ is a $K_\infty-$irreducible subspace of $\overset{k}{\wedge} \mathfrak{p} (k = p + q)$ (the irreducibility can be proved by observing that the line $e(\mathfrak{q})$ is stabilised by the Borel subalgebra $\mathfrak{b}_K$ of the $\mathfrak{k}$ which is the sum of $\mathfrak{t}$ and all the root spaces corresponding to the positive *compact* roots occurring in $\mathfrak{k}$).

If $(v_i)_{i \in I}$ is a basis of $\mathfrak{u} \cap \mathfrak{p}$, then complete it to a basis $(v_j) j \in J$ of $\mathfrak{p}$. Let $(v_j^*) j \in J$ be the dual basis in $\mathfrak{p}^*$. Define $e(\mathfrak{q}^*$ as the top exterior of the span of v_i^* $(i \in I)$. Let $V(\mathfrak{q})^*$ be the K_∞ span of $(e(\mathfrak{q})^*$. Then $V(\mathfrak{q})^*$ is indeed the dual of $V(\mathfrak{q})$. Moreover, it is known that the multiplicity of $V(\mathfrak{q})$ in $\wedge^k \mathfrak{p}$ is exactly one ([V-Z], Section 6). Thus, $V(\mathfrak{q})^*$ is independent of the basis chosen.

By a Theorem of Kostant ([Kos]), the K_∞-representation $\wedge \mathfrak{p}^+$ is multiplicity free. In particular,

$$V^+(\mathfrak{q}) = E_1 \oplus E_2 \oplus \cdots \oplus E_\ell \tag{2}$$

with each E_i irreducible as a K_∞-module, and $E_i \neq E_j$ if $i \neq j$.

The exterior algebra $\overset{\bullet}{\wedge} \mathfrak{p}$ is stable under complex conjugation and $V(\mathfrak{q}) \wedge \mathfrak{p}$. Moreover, $k \leqslant D = \dim(\mathfrak{p}^+) < 2D = \dim(\mathfrak{p})$. We may form the wedge of $V(\mathfrak{q})$

and $\overline{V(\mathfrak{q})}$ in $\wedge\mathfrak{p}$:

$$V(\mathfrak{q}) \wedge \overline{V(\mathfrak{q})} \subset \wedge\mathfrak{p}.$$

Now $V(\mathfrak{q}) \otimes \overline{V(\mathfrak{q})}$ contains $e(\mathfrak{q}) \otimes \overline{e(\mathfrak{q})}$ as a cyclic vector. For, first consider the translates of $e(\mathfrak{q}) \otimes \overline{e(\mathfrak{q})}$ under $B_K \subset K(\mathbb{C})$, where B_K is the Borel subgroup of $K(\mathbb{C})$ with Lie algebra $\mathfrak{b}_K = \mathfrak{t}\oplus_{\gamma\in\Phi^+(\mathfrak{k},T)}\mathfrak{g}_\gamma$. Since B_K leaves $e(\mathfrak{q})$ stable and $\overline{e(\mathfrak{q})}$ is $\overline{B}_K$ invariant, we obtain all the vectors of the form $e(\mathfrak{q}) \otimes B_K\overline{B}_K\overline{e(\mathfrak{q})}$ in the $K(\mathbb{C})$-span of $e(\mathfrak{q}) \otimes \overline{e(\mathfrak{q})}$. Now $B_K\overline{B}_K$ is a Zariski open set in $K(\mathbb{C})$ since $\overline{B}_K$ is opposite to B_K. Hence $e(\mathfrak{q}) \otimes \overline{V(\mathfrak{q})}$ lies in the $K_\mathbb{C}$-span. Now translating elements of $e(\mathfrak{q}) \otimes \overline{V(\mathfrak{q})}$ by elements of $K_\mathbb{C}$, we get: $e(\mathfrak{q}) \otimes \overline{e(\mathfrak{q})}$ generates $V(\mathfrak{q}) \otimes \overline{V(\mathfrak{q})}$. Therefore $e(\mathfrak{q}) \wedge \overline{e(\mathfrak{q})}$ generates $V(\mathfrak{q}) \wedge \overline{V(\mathfrak{q})}$.

From (1) we obtain $e(\mathfrak{q}) = e_h(\mathfrak{q}) \wedge e_{ah}(\mathfrak{q})$, where $e_h(\mathfrak{q})$ (resp. $e_{ah}(\mathfrak{q})$) is the holomorphic part $\overset{p}{\wedge}(\mathfrak{u} \cap \mathfrak{p}^+)$ of $e(\mathfrak{q})$ (resp. the antiholomorphic part $\overset{q}{\wedge}(\mathfrak{u} \cap \mathfrak{p}^-)$). Therefore $e(\mathfrak{q}) \wedge \overline{e(\mathfrak{q})}$ is the same as

$$(e_h(\mathfrak{q}) \wedge \overline{e_{ah}(\mathfrak{q})}) \wedge (\overline{e_h(\mathbf{q})} \wedge e_{ah}(\mathbf{q})).$$

The first vector $e_h(\mathfrak{q}) \wedge \overline{e_{ah}(\mathfrak{q})}$ is precisely $e^+(\mathfrak{q}) \subset V^+(\mathfrak{q}) \subset \overset{k}{\wedge}\mathfrak{p}^+$. By taking the K_∞-spans we obtain:

$$V(\mathfrak{q}) \wedge \overline{V(\mathfrak{q})} \subset V^+(\mathfrak{q}) \otimes \overline{V^+(\mathfrak{q})}. \tag{3}$$

6.2. PRELIMINARY RESULTS

We wish to compute the integral

$$\int_{K_\infty} k(e^+(\mathfrak{q}) \otimes \overline{e^+(\mathfrak{q})})\mathrm{d}\mu(k)$$

(μ is a Haar measure on K_∞) which has values in $\overset{k}{\wedge}\mathfrak{p}^+ \otimes \overset{k}{\wedge}\mathfrak{p}^-$. Write

$$e^+(\mathfrak{q}) = \xi_1 + \cdots + \xi_\ell$$

according to the decomposition (2). By orthogonality $(E_i \otimes \overline{E}_j)^K = 0$ if $i \neq j$. Therefore we get

$$\int_{K_\infty} k(e^+(\mathfrak{q}) \otimes \overline{e^+(\mathfrak{q})})\mathrm{d}\mu(k) = \sum_{i=1}^{\ell} \int_{K_\infty} k(\xi_i \otimes \overline{\xi}_i)\mathrm{d}\mu(k). \tag{4}$$

Since $e^+(\mathfrak{q})$ is a cyclic vector for $V^+(\mathfrak{q})$, it follows from (2) that $\xi_i \neq 0$ for each i.

Suppose $\rho : K_\infty \to \mathrm{U}(n)$ is a unitary irreducible representation of K_∞ on an n–dimensional vector space $\mathbb{C}^n = V$, let $v \in V - \{0\}$. Write $v = c_1\varepsilon_1 + \cdots + c_n\varepsilon_n$ for the standard unitary basis $\varepsilon_1, \ldots, \varepsilon_n$ of V. Let $\varepsilon_1^*, \ldots, \varepsilon_n^*$ be a dual basis of V^*, the dual of V. Consider the tensor representation $V \otimes V^*$ of K_∞. Write $k(v \otimes w)$ for the action of $k \in K_\infty$ on an element $v \otimes w \in V \otimes V^*$. If v is as above,

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write $v^* = \overline{c_1}\varepsilon_1^* + \cdots + \overline{c_n}\varepsilon_n^*$. Then,

$$\int_{K_\infty} k(v \otimes v^*)\mathrm{d}\mu(k) = \sum_{v,j} c_i \overline{c}_j \int_{K_\infty} k(\varepsilon_i \otimes \varepsilon_j^*)\mathrm{d}\mu(k).$$

It follows from Schur's Lemma that

$$\int_{K_\infty} \rho(k)(\varepsilon_i \otimes \varepsilon_j^*)\mathrm{d}\mu(k) = \frac{\delta_{ij}}{n}\left(\sum_{r=1}^{n} \varepsilon_r \otimes \varepsilon_r^*\right) \tag{5}$$

and

$$\int_{K_\infty} \rho(k)(v \otimes v^*)\mathrm{d}\mu(k) = \left(\sum \mid c_i \mid^2\right)\frac{I_n}{n} = \frac{\mid v \mid^2}{n}\lambda_{(\mathbb{C}^n)^*}, \tag{5$'$}$$

where $\lambda_{(\mathbb{C}^n)^*} = \sum_{i=1}^{n} \varepsilon_i \otimes \varepsilon_i^*$.

Now let $\lambda_{E_i^*} \in E_i \otimes \overline{E}_i \subset \overset{k}{\wedge}\,\mathfrak{p}^+ \otimes \overset{k}{\wedge}\,\mathfrak{p}^-$ be the tensor corresponding to the space E_i. From (4) and (5) we obtain

$$\int_{K_\infty} k(e^+(\mathfrak{q}) \otimes \overline{e^+(\mathfrak{q})})\mathrm{d}\mu(k) = \sum_{i=1}^{\ell}\left(\frac{\mid \xi_i \mid^2}{\dim E_i}\right)\lambda_{E_i^*}. \tag{6}$$

We note that $\mid \xi_i \mid^2 / \dim E_i = b_i > 0$ for each i. Now

$$e(\mathfrak{q}) \wedge \overline{e(\mathfrak{q})} \in \overset{\bullet}{\wedge}\,\mathfrak{p} = \wedge\mathfrak{p}^+ \otimes \wedge\mathfrak{p}^- \quad \text{and} \quad e(\mathfrak{q}) \wedge \overline{e(\mathfrak{q})} = e^+(\mathfrak{q}) \otimes \overline{e^+(\mathfrak{q})}.$$

Therefore, we get

$$\int_{K_\infty} k(e(\mathfrak{q}) \wedge \overline{e(\mathfrak{q})})\mathrm{d}\mu(k) = \sum_{i=1}^{\ell} b_i \lambda_{E_i^*}, \tag{7}$$

where each b_i is *strictly* positive. We emphasize that (7) is not entirely formal, and depends on the multiplicity one result of Kostant in (2).

Consider the 'wedging map' $V(\mathfrak{q}) \otimes \overline{V(\mathfrak{q})} \to V(\mathfrak{q}) \wedge \overline{V(\mathfrak{q})} \subset \wedge\mathfrak{p}$. By Schur's Lemma, $(V(\mathfrak{q}) \otimes \overline{V(\mathfrak{q})})^{K_\infty} = \mathbb{C}$ and (7) shows that $(V(\mathfrak{q}) \wedge \overline{V(\mathfrak{q})})^{K_\infty} \neq 0$. Therefore the wedging map is an isomorphism on K_∞-fixed vectors. From (5') and (7) it follows that for any $\xi \in V(\mathfrak{q}) - (0)$

$$\int_{K_\infty} k(\xi \wedge \overline{\xi})\mathrm{d}\mu(k) = \frac{\mid \xi \mid^2}{\mid e(\mathfrak{q}) \mid^2} \sum b_i \lambda_{E_i^*} \tag{8}$$

for some K_∞-fixed inner product on $V(\mathfrak{q})$ (which is unique upto multiples anyway); here again, $b_i > 0$ for each i.

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We may replace in (8) $\xi \in V(\mathfrak{q})$ by $\xi \in V(\mathfrak{q})^*$ where $V(\mathfrak{q})^*$ is the dual of $V(\mathfrak{q})$. We then get analogously, for $\xi \in V(\mathfrak{q})^*$,

$$\int_{K_\infty} k(\xi \wedge \overline{\xi})\mathrm{d}\mu(k) = \frac{|\ \xi\ |^2}{|\ e(\mathfrak{q})^*\ |^2}\sum b_i\lambda_{E_i}. \tag{8*}$$

with $b_i > 0$ for each i.

CHARACTERISATION AND PROPERTIES OF SCHUBERT CELLS

(6.3) It will be convenient to think of elements of $H^\bullet(\widehat{X})$ as elements of $\overset{\bullet}{\wedge}\mathfrak{p}^*$. The Killing form κ is negative definite on $i\mathfrak{p}_0$ and hence identifies $\mathfrak{p}$ with its dual. Under this, $(\mathfrak{p})^+$ and $(\mathfrak{p})^-$ are dual to each other as K_∞ representations. Note that under the map $v \mapsto \overline{v}$ defined in (6.1), $\wedge^\bullet\mathfrak{p}^+$ maps onto $\wedge^\bullet\mathfrak{p}^-$. If $E \subset \wedge^k\mathfrak{p}^+$ is a K_∞ stable subspace and $\overline{E}$ is its image under the foregoing map, then $\overline{E} \subset \wedge^k\mathfrak{p}^-$ is the dual of E.

Now by a Theorem of Cartan (see Equation (7) of Section (1.3))

$$H^\bullet(\widehat{X}) = \mathrm{Hom}_{K_\infty}(\overset{\bullet}{\wedge}(\mathfrak{p}^+ \oplus \mathfrak{p}^-), \mathbb{C}).$$

By the Theorem of Kostant quoted in (2) the representation $\overset{\bullet}{\wedge}(\mathfrak{p})^+$ is multiplicity free and we may write

$$\overset{k}{\wedge}\mathfrak{p}^+ = \oplus_{E \in X_k} E \tag{9}$$

where each E is irreducible. The E_i which occur in the decomposition (2) are again elements of X_k. Define λ'_{E_i}s as the generator of the one dimensional vector space $(E_i^* \otimes (\overline{E}_i)^*)^{K_\infty} \subset (\overset{k}{\wedge}(\mathfrak{p}^+)^* \otimes \overset{k}{\wedge}(\mathfrak{p}^-)^*)^{K_\infty}$ which is given by

$$\lambda_E \overset{\mathrm{def}}{=} \sum e \wedge \overline{e} \tag{10}$$

where the sum runs over an orthonormal basis e of E^* (under the natural Hermitian inner product on E^* induced by the Killing form κ; see (5.1)).

A theorem of Kostant ([Kos], Theorem (6.15)) says that these λ_E are proportional to the cycle classes corresponding to the Schubert Cells in $\widehat{X} = G(\mathbb{C})/P^-(\mathbb{C})$. Here, $P^-(\mathbb{C})$ is the connected subgroup of $G(\mathbb{C})$ whose Lie algebra is $\mathfrak{k} \oplus \mathfrak{p}^-$. We will abuse notation slightly and refer to λ_E as 'Schubert Cells'.

We will now gather together some properties of these 'Schubert Cells' λ_E. If $E^* \subset \wedge^r(\mathfrak{p}^+)^*$ and $F \subset \wedge^s(\mathfrak{p}^+)^*$, denote by $E^* \wedge F^*$ the span of vectors of the form $e \wedge f$ with $e \in E^*$ and $f \in F^*$.

(6.4) LEMMA. *Let E and F be irreducible K_∞ stable subspaces of $\wedge^r\mathfrak{p}^+$ and $\mathfrak{p}^+$ respectively. Then*

$$\lambda_E \wedge \lambda_F = (-1)^{rs}\sum c_\mu \lambda_{F_\mu}$$

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where the sum is over all the irreducible subspaces F_μ of $E \wedge F$ and each c_μ is strictly positive.

Proof. Equation (10) applied to λ_E and λ_F implies that $\lambda_E \wedge \lambda_F = \sum e \wedge \bar{e} \wedge f \wedge \bar{f}$ where the sum is over orthonormal bases (e) and (f) of E^* and F^*, respectively. Rewrite this as

$$\lambda_E \wedge \lambda_F = (-1)^{rs} \sum (e \wedge f) \wedge (\bar{e} \wedge \bar{f}). \tag{11}$$

Since $\lambda_E \wedge \lambda_F$ is K_∞-invariant, we may replace both sides of the Equation (11) by their integrals over K_∞. Write $E^* \wedge F^* = \oplus F_\mu^*$ where each $F_\mu \subset \wedge^k \mathfrak{p}^+$ is irreducible. Then, for every vector of the form $e \wedge f$, we get (as in (8^*) of (6.2))

$$\int_{K_\infty} k((e \wedge f) \wedge (\bar{e} \wedge \bar{f}))\mathrm{d}\mu(k) = \sum b_\mu \lambda_{F_\mu},$$

where $b_\mu \geqslant 0$ for each μ and is the norm of the projection of $e \wedge f$ to the component F_μ^* of $E^* \wedge F^*$. Therefore,

$$\int_{K_\infty} k((e \wedge f) \wedge (\bar{e} \wedge \bar{f}))\mathrm{d}\mu(k) = \sum c_\mu \lambda_{F_\mu},$$

where c_μ is the sum over all the b_μ as e and f vary.

Since (e) and (f) form bases of E^* and F^*, it follows that if F_μ be fixed, then for some e and f, the projection to F_μ^* of $e \wedge f$ is non-zero; therefore the corresponding b_μ is strictly positive. Hence, all the c_μ are strictly positive. The Lemma now follows from integrating both sides of Equation (11) over K_∞.

(6.5) LEMMA. *Let L be the element of* $\mathrm{Hom}_{K_\infty}(\mathfrak{p}^+ \wedge \mathfrak{p}^-, \mathbb{C})$ *defined in Section 5. Then, for every integer $k \leqslant D$ (with $D = \dim(X)$), we get*

$$L^k = (-1)^{k(k-1)/2} \sum c_i \lambda_{F_i},$$

where F_i runs over all the K_∞ irreducible subspaces of $\wedge^k \mathfrak{p}^+$ and each c_i is strictly positive.

Proof. The proof is by induction. Assume $k = 1$. Then, by the definition of the Hermitian inner product on $\mathfrak{p}$, we see that $L = \sum e \wedge \bar{e}$ where e runs over *any* orthonormal basis of $(\mathfrak{p}^+)^*$. Write $\mathfrak{p}^+$ as a sum of irreducible representations E_j for the action of K_∞ and pick an orthonormal basis (e_j) for each j. Choose for (e) the union of the bases (e_j) over all j. Then, we get (by (1) of Section (5.2))

$$L = \sum e \wedge \bar{e} = \sum \lambda_{E_j}$$

and the Lemma holds for $k = 1$.

Assume now that the Lemma holds for k and write

$$L^k = (-1)^{k(k-1)/2} \sum c_i \lambda_{E_i}$$

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as in the Lemma. Then,

$$L^{k+1} = (-1)^{k(k-1)/2} \sum c_i \lambda_{E_j} \wedge \lambda_{E_i},$$

where the sum is over all the irreducible $E_j \subset \mathfrak{p}^+$ and all the irreducible $E_i \subset \wedge^k \mathfrak{p}^+$. Now, by Lemma (6.4), we get

$$\lambda_{E_j} \wedge \lambda_{E_i} = (-1)^k \sum c_\mu \lambda_{F_\mu},$$

where the sum is over all the irreducible representations $F_\mu \subset E_i \wedge E_j$ and $c_\mu > 0$. Therefore,

$$L^{k+1} = (-1)^{k(k+1)/2} \sum c_i \sum c_\mu \lambda_{F_\mu}, \tag{12}$$

where the sum is over all i, all j and all irreducible $F_\mu \subset E_j \wedge E_i \subset \wedge^{k+1} \mathfrak{p}^+$ and $c_\mu > 0$. In Equation (12), first fix F_μ and sum over all the i and j such that $F_\mu \subset E_j \wedge E_i$. We then obtain Lemma (6.5) for L^{k+1}.

(6.6) *Remark*. Suppose we are given *any* K_∞-invariant metric on the real vector space $\mathfrak{g}_0$. This yields a K_∞ invariant $\mathbb{C}$-linear form $\mathfrak{p}^+ \otimes \mathfrak{p}^- \subset \wedge^2 \mathfrak{p}$ which we denote by L'. The lemma (6.5) applies to L' as well (the proof is exactly the same). We will use this remark later, where we take H for G and the restriction of L_G to $\widehat{Y}$ for L'.

(6.7) LEMMA. *Let $\beta \in H^{2k}(\widehat{X})$ be a non-negative linear combination of Schubert Cells λ_E and let L' be as in (6.6). If $\beta \wedge (L')^{D-k} = 0$ then $\beta = 0$.*

Proof. Suppose to the contrary, that β is a positive linear combination of some Schubert Cells λ_E. Let $\varepsilon = (-1)^{k(D-k)+((D-k)(D-k-1)/2)}$. Then, Lemmas (6.4), (6.5) and Remark (6.6) imply that $\varepsilon\beta \wedge (L')^{D-k}$ is a *non-negative* linear combination of Schubert Cells λ_{F_μ}. Thus, to prove Lemma (6.7), we may assume that β is a Schubert Cell λ_E. Then Lemma (6.5) shows that $\varepsilon\beta \wedge (L')^{D-k}$ is a strictly positive linear combination of λ_{F_μ} where F_μ is an irreducible subspace of $E \wedge \wedge^{D-k}\mathfrak{p}^+$. Thus, $\beta \wedge (L')^{D-k}$ vanishes if and only if $E \wedge \wedge^{D-k}\mathfrak{p}^+ = 0$. This is impossible since the wedge product pairing between $\wedge^k \mathfrak{p}^+$ and $\wedge^{D-k}\mathfrak{p}^+$ (with values in the one dimensional space $\wedge^d \mathfrak{p}^+$) is non-degenerate. Thus, $\beta = 0$.

(6.8) LEMMA. *Let $\lambda_E \in H^k(\widehat{X})$ be a 'Schubert Cell of $\widehat{X}$ as before. Let $\widehat{j} : \widehat{Y} \to \widehat{X}$ be the embedding of (1.5). Then, the restriction of λ_E to $\widehat{Y}$ is a non-negative sum of 'Schubert Cells' of $\widehat{Y}$: $\widehat{j}^*(\lambda_E) = \sum a_j \lambda_{F_j}$ where F_j runs through the set of irreducible representations of K_∞^H occurring in $\wedge^k(\mathfrak{p}_H^+)$ and $a_j \geqslant 0$ for each j. Moreover, the restriction $\widehat{j}^*(\lambda_E) = 0$ if and only if $E \cap E(G, H, k) = 0$.*

Proof. We will view the restriction $\widehat{j}^*(\lambda_E)$ as K_∞^H invariant linear form on $\wedge^k \mathfrak{p}_H^+ \otimes \wedge^k \mathfrak{p}_H^-$. If $v \in \wedge^k \mathfrak{p}_H^+$ and $w \in \wedge^k \mathfrak{p}_H^-$, then $\widehat{j}^*(\lambda_E)(v \otimes w) = \lambda_E(\pi_E(v) \otimes \pi_{\overline{E}}(w))$ where π_E denotes the K_∞ equivariant projection of $\wedge^k \mathfrak{p}^+$ to E (similarly define $\pi_{\overline{E}}$). Let $F_j \subset \wedge^k \mathfrak{p}_H^+$ be K_H^∞ irreducible. Then, it has (up to multiples) a unique K_∞^H invariant linear form on it which may be chosen to be λ_{F_j}. Thus, there is a scalar

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c_j such that for all $v \in F_j$ and $w \in \overline{F}_j$, we have $\widehat{j}^*(\lambda_E)(v \otimes w) = c_j \lambda_{F_j}(v \otimes w)$. Let v be arbitrary and choose $w = \overline{v}$. Then, $\widehat{j}^*(\lambda_E)(v \otimes w) = \lambda_E(v \otimes w) \geqslant 0$. Similarly, $\lambda_{F_j}(v \otimes w) \geqslant 0$ which shows that $c_j \geqslant 0$ for each j. If the restriction of λ_E vanishes, then the same equation shows that the K_∞-equivariant projection to E vanishes on the space $\wedge^k \mathfrak{p}_H^+$ and hence vanishes on its K_∞-span $E(G, H, k)$. By the multiplicity one theorem of Kostant, this is equivalent to saying that $E \cap E(G, H, k) = 0$. This completes the proof.

(6.9) LEMMA. *Let $\beta = \sum c_i \lambda_{E_i}$ be a cohomology class in $H^{2k}(\widehat{X})$ where E_i are certain irreducible subspaces of $\wedge^k \mathfrak{p}^+$ and for each E_i, the coefficient c_i is strictly positive. If the restriction $\widehat{j}^*(\beta \wedge L^{d-k}) = 0$ then $(\oplus E_i) \cap E(G, H, k) = 0$.*

Proof. We need only prove the Lemma when $k \leqslant d = \dim(Y)$ since otherwise the restriction of β is trivially zero and $E(G, H, k)$ also vanishes because $\wedge^k \mathfrak{p}_H^+ = 0$. Assume $k \leqslant d$. Suppose $\beta' =_{\mathrm{def}} \widehat{j}^*(\beta)$. Let $L' = \widehat{j}^*(L)$ be the restriction of $L \in H^2(\widehat{X})$ to $\widehat{Y}$. Let $\gamma = \beta' \wedge (L')^{d-k}$.

By (6.6) (applied to the space $\widehat{Y}$), $(-1)^{(d-k)(d-k-1)/2} L'^{d-k}$ is a strictly positive linear combination of λ_F where F runs through *all* K_∞^H irreducible subspaces of $\wedge^{d-k} \mathfrak{p}_H^+$. By Lemma (6.8), β' is a non-negative sum of Schubert cells on $\widehat{Y}$.

Hence Lemma (6.7) applied to $\widehat{Y}$ shows that $\beta' \wedge (L')^{d-k} = 0$ if and only if $\beta' = 0$. By the second part of Lemma (6.8), This happens if and only if $E_i \cap E(G, H, k) = 0$ for each E_i occurring in the expression of β as a non-negative linear combination of Schubert cells λ_{E_i}. The multiplicity one Theorem of Kostant then implies the conclusion of Lemma (6.9).

7. Proof of Theorem 6

(7.1) We prove Theorem 6. Suppose that $\alpha \in H^k(\mathrm{Sh}^0 G)$ is strongly primitive of type $A_{\mathfrak{q}}$. Suppose $\mathrm{Res}(\alpha) = 0$. Then, the criterion of Theorem 2 says that $\alpha \wedge [\widehat{Y}] = 0$.

Recall (from Equation (6) of Section (1.2)) a version of the Matsushima formula:

$$H^\bullet(\mathrm{Sh}^0 G), \mathbb{C}) = \mathrm{Hom}_K(\overset{\bullet}{\wedge}\, \mathfrak{p}, \mathcal{C}^\infty(G(\mathbf{Q}) \backslash (G(\mathbb{R}) \times G_f))(0).$$

We temporarily denote by $\mathcal{C}^\infty(0)$ the space of smooth functions on the quotient

$$G(\mathbb{Q}) \backslash (G(\mathbb{R}) \times G_f)$$

which are killed by the Casimir. Let $\{\xi_I; I\}$ be an orthonormal basis of $V(\mathfrak{q})^* \subset \overset{k}{\wedge}\, \mathfrak{p}^*$ (for the natural Hermitian metric h as in Section (5.1), extending the Killing form on $\mathfrak{p}_0$ to $\mathfrak{p} = \mathfrak{p}_0 \otimes \mathbb{C}$). Let

$$\alpha = \sum_I \varphi_I \xi_I \in H^m(\mathrm{Sh}^0 G) \subset \overset{m}{\wedge}\, \mathfrak{p}^* \otimes \mathcal{C}^\infty(0) \tag{1}$$

(the inclusion arising from the Matsushima formula).

Our class α is of type $\mathfrak{q}$, and

$$\alpha \in (V(\mathfrak{q})^* \otimes \mathcal{C}^\infty(0)) = \mathrm{Hom}_{K_\infty}(V(\mathfrak{q}), \mathcal{C}^\infty(0));$$

write $\alpha = \sum \varphi_I \xi_I$. Now, $V(\mathfrak{q})$ is irreducible. Hence the map $\alpha : V(\mathfrak{q}) \to \mathcal{C}^\infty(0)$ (being K_∞-equivariant) is *injective*. In particular, the φ_I are linearly independent.

Now $\alpha \wedge [\widehat{Y}] = 0$. Then (1) shows that

$$\alpha \wedge [\widehat{Y}] = \sum (\xi_I \wedge [\widehat{Y}])\varphi_I. \tag{2}$$

Since the functions φ_I are linearly independent, (2) shows that $\xi_I \wedge [\widehat{Y}] = 0$ for each I i.e. $V(\mathfrak{q})^* \wedge [\widehat{Y}] = 0$. This proves the first part of Theorem 6.

Since $e(\mathfrak{q})^*$ generates $V(\mathfrak{q})^*$ as a K_∞ module, the condition $V(\mathfrak{q})^* \wedge [\widehat{Y}] = 0$ is equivalent to $e(\mathfrak{q})^* \wedge [\widehat{Y}] = 0$. This implies in particular, that

$$e(\mathfrak{q})^* \wedge \overline{e(\mathfrak{q})^*} \wedge [\widehat{Y}] = 0. \tag{3}$$

Translate both sides of (3) by elements $k \in K_\infty$ and integrate over K_∞. We obtain from Equation (7) of (6.2) and the K_∞ invariance of $[\widehat{Y}]$ that $(\sum b_i \lambda_{E_i}) \wedge [\widehat{Y}] = 0$. Note that here the sum is over all the E_i which are irreducible subspaces of $V^+(\mathfrak{q})$ and that $b_i > 0$. Therefore, by the definition of the cycle class $[\widehat{Y}]$, the restriction of $\sum b_i \lambda_{E_i} \wedge L^{d-k}$ to the cycle $\widehat{Y}$ is zero. Then, Lemma (6.9) implies that $(\oplus E_i) \cap E(G, H, k) = 0$. But $\oplus E_i = V^+(\mathfrak{q})$. Therefore, $V^+(\mathfrak{q}) \cap E(G, H, k) = 0$. This completes the proof of Theorem 6.

8. Cup Products

(8.1) Now let G imbed diagonally in $G \times G$. By the Kunneth isomorphism $H^\bullet(\mathrm{Sh}^0(G \times G)) = H^\bullet(\mathrm{Sh}^0 G) \otimes H^\bullet(\mathrm{Sh}^0 G)$. Moreover, $\mathrm{Res}\ (\alpha \otimes \beta) = \alpha \wedge \beta$, if Res is the restriction from $\mathrm{Sh}^0(G \times G)$ to $\mathrm{Sh}^0 G$.

Let α, α' be strongly primitive classes of degrees k and k' with $k + k' \leqslant \dim(X) = D$, of type $\mathfrak{q}, \mathfrak{q}'$. By Theorem 2, if $g(\alpha) \wedge \alpha' = 0$ for all $g \in G_f$, then $(\alpha \otimes \alpha') \wedge [\widehat{\Delta}] = 0$. This is the first part of Theorem 7.

We will now prove that the criterion (1) of Theorem 7 holds. By the general criterion of Theorem 6, the product $g(\alpha) \wedge \alpha' = 0$ for all $g \in G_f$ *only if* the intersection $E(G \times G, G, k + k') \cap V^+(\mathfrak{q} \oplus \mathfrak{q}') = 0$ (it is easily checked that $\alpha \otimes \alpha'$ is strongly primitive of type $\pi = A_{\mathfrak{q} \oplus \mathfrak{q}'}$ where π corresponds to the parabolic subalgebra $\mathfrak{q} \oplus \mathfrak{q}'$ of $\mathfrak{g} \oplus \mathfrak{g}$). It is also immediate from the definition of V^+ that $V^+(\mathfrak{q} \oplus \mathfrak{q}') = V^+(\mathfrak{q}) \otimes V^+(\mathfrak{q}')$. Therefore it is immediate from Theorem 6 that if $g(\alpha) \wedge \alpha' = 0$ for all $g \in G_f$ then

$$E(G \times G, G, k + k') \cap (V^+(\mathfrak{q}) \otimes V^+(\mathfrak{q}')) = 0.$$

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We now prove that the conditions (1) and (2) of Theorem 7 are equivalent. Consider the natural wedging map

$$\varphi : \overset{k}{\wedge}\, \mathfrak{p}^+ \otimes \overset{k'}{\wedge}\, \mathfrak{p}^+ \to \overset{k+k'}{\wedge}\, \mathfrak{p}^+.$$

Now $\mathfrak{p}^+ \oplus \mathfrak{p}^+$ is a $(K_\infty \times K_\infty)$-module and the above wedging map is equivariant for K_∞, with K_∞ acting diagonally on the left.

Let $U(D)$ be the unitary group of the Hermitian metric preserved by K_∞ on its action on $\mathfrak{p}^+$. Then $\overset{k+k'}{\wedge}\, \mathfrak{p}^+$ is an irreducible representation of $U(D)$ and occurs with multiplicity one in the $(k+k')$–fold tensor product $\underbrace{(\mathfrak{p})^+ \otimes \cdots \otimes (\mathfrak{p})^+}_{(k+k')-\text{times}}$ (this is well known by the Theory of Young Diagrams: see [F-H], Theorem (6.3), (2), applied to $\lambda = (1, 1, \cdots, 1)$ and $d = k + k'$). Hence $\overset{k+k'}{\wedge}\, \mathfrak{p}^+$ occurs with multiplicity one in the representation $\overset{k}{\wedge}\, \mathfrak{p}^+ \otimes \overset{k'}{\wedge}\, \mathfrak{p}^+$.

But the inclusion of $\overset{k+k'}{\wedge}\, \Delta(\mathfrak{p}^+)$ (where $\Delta(\mathfrak{p}^+) =$ the diagonal in $\mathfrak{p}^+ \oplus \mathfrak{p}^+$) in $\overset{k+k'}{\wedge}\, (\mathfrak{p}^+ \oplus \mathfrak{p}^+)$ induces an inclusion $\overset{k+k'}{\wedge}\, \mathfrak{p}^+ \hookrightarrow \overset{k}{\wedge}\, \mathfrak{p}^+ \otimes \overset{k'}{\wedge}\, \mathfrak{p}^+$. On the other hand there is a natural (wedging) projection $\overset{k}{\wedge}\, \mathfrak{p}^+ \otimes \overset{k'}{\wedge}\, \mathfrak{p}^+ \overset{\varphi}{\to} \overset{k+k'}{\wedge}\, \mathfrak{p}^+$; both these maps are $U(D)-$ equivariant, and $U(D) \supset \Delta(K_\infty)$ where K_∞ acts diagonally as $\overset{k}{\wedge}\, \mathfrak{p}^+ \otimes \overset{k'}{\wedge}\, \mathfrak{p}^+$. Since $\overset{k+k'}{\wedge}\, \mathfrak{p}^+$ occurs with multiplicity one, $\mathrm{Ker}\varphi =$ Orthogonal complement of $\overset{k+k'}{\wedge}\, \Delta(\mathfrak{p}^+)$ in $\overset{k}{\wedge}\, \mathfrak{p}^+ \otimes \overset{k}{\wedge}\, \mathfrak{p}^+$ under a metric $\langle , \rangle$ on the latter space invariant under $\underline{U(D) \times U(D)}$ action.

Suppose now $V^+(\mathfrak{q}) \wedge V(\mathfrak{q}') = 0$. Then

$$V^+(\mathfrak{q}) \otimes V(\mathfrak{q}') \subset \mathrm{Ker}\varphi = \left(\overset{k+k'}{\wedge}\, \Delta\mathfrak{p}^+ \right)^\perp.$$

Let $\xi \in \overset{k+k'}{\wedge}\, \Delta(\mathfrak{p}^+)$, and $v \in V(\mathfrak{q}) \otimes V(\mathfrak{q}')$, and $k \in K_\infty \times K_\infty$. Since $V^+(\mathfrak{q}) \otimes V^+(\mathfrak{q}')$ is $K_\infty \times K_\infty$-invariant, we get

$$0 = \langle k(v), \xi \rangle = \langle v, k(\xi) \rangle$$

for all $k \in K_\infty \times K_\infty$, and all $\xi \in \overset{k+k'}{\wedge}\, \Delta(\mathfrak{p}^+)$. By taking the $(K_\infty \times K_\infty)$-span of ξ, we obtain

$$\langle v, E(G \times G, G, k + k') \rangle = 0 \ (\forall v \in V^+(\mathfrak{q}) \otimes V^+(\mathfrak{q}')).$$

Therefore: $V^+(\mathfrak{q}) \otimes V^+(\mathfrak{q}')$ is in the orthogonal complement of $E = E(G \times G, G, k + k')$, i.e.,

$$[V^+(\mathfrak{q}) \otimes V(\mathfrak{q}')] \cap E = 0.$$

Conversely, suppose $[V^+(\mathfrak{q}) \otimes V^+(\mathfrak{q}')] \cap E = 0$. Then $V^+(\mathfrak{q}) \otimes V^+(\mathfrak{q}') \subset E^\perp$. In particular, $V^+(\mathfrak{q}) \otimes V^+(\mathfrak{q}') \subset (\overset{k+k'}{\wedge}\, \Delta(\mathfrak{p}^+))^\perp = \mathrm{Ker}\varphi$. Therefore $\varphi(V^+(\mathfrak{q}) \otimes V^+(\mathfrak{q}')) = 0$

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i.e.

$$V^{+}(\mathfrak{q}) \wedge V^{+}(\mathfrak{q}') = 0.$$

We have shown that (1) $\Leftrightarrow$ (2). This completes the proof of Theorem 7.

(8.2) *Proof of Theorem 8.* Let α, α' be non-zero classes of type $\mathfrak{q}, \mathfrak{q}'$ on $\mathrm{Sh}^0 G$, $G = U(n, 1)$ of degrees k and k' with $k + k' \leqslant n$. *Note that $D = n$ in this section* . The compact dual of X is $\mathbb{P}^n(\mathbb{C})$, whose cohomology is generated by a non-zero element L of $H^2(\mathbb{P}^n(\mathbb{C}))$. Let $\widehat{\Delta}$ be the diagonal in the product of $\mathbb{P}^n(\mathbb{C})$ with itself, and $[\widehat{\Delta}]$ the associated cycle class in the cohomology of the product. Then the Kunneth isomorphism implies that

$$[\widehat{\Delta}] = \sum \varepsilon_i (L^i \otimes L^{n-i}) \tag{1}$$

where the sum is over i from 0 to n and $\varepsilon_i = \pm 1$.

Suppose to the contrary, that $g(\alpha) \wedge \alpha' = 0$ for all $g \in G_f$. Now, Theorem 7 says that if $g(\alpha) \wedge \alpha' = 0$ for all $g \in G_f$, then

$$(\alpha \otimes \alpha') \wedge [\widehat{\Delta}] = 0. \tag{2}$$

Compare the Kunneth components of both sides of (2). In particular, we get from (1) that

$$(\alpha \wedge L^{n-k}) \otimes (\alpha' \wedge L^{k}) = 0. \tag{3}$$

Note that by assumption, $k \leqslant n - k'$. The Lefschetz hyperplane section Theorem then implies that $\alpha \wedge L^{n-k} \neq 0$ and that $\alpha' \wedge L^k \neq 0$. This contradicts (3). Hence Theorem 8 follows.

(8.3) *Proof of Theorem 9.* The proof is similar to that of Theorem 8. If $\widehat{X}$ is the compact dual, then $H^2(\widehat{X}) = \mathbb{C}L$. If $k, k' < [n/2]$, then $\overset{k}{\wedge}\, \mathfrak{p}^{+} = \overset{k}{\wedge}\, \mathbb{C}^n$ and $\overset{n-k'}{\wedge}\, \mathfrak{p}^{+} = (\overset{n-k'}{\wedge}\, \mathbb{C}^n)$ are *irreducible* representations of $(SO(2) \times SO(n)) = K_\infty$ (see [F-H], Theorem (19.2) and Theorem (19.14)). Therefore by Schur's Lemma and (7) of (1.3),

$$H^{2k}(\widehat{X}) = \mathbb{C}L^k, \quad H^{n-k'}(\widehat{X}) = \mathbb{C}L^{n-k'}.$$

If α and α' are as in Theorem 9, and if $g(\alpha) \wedge \alpha' = 0$ for all $g \in G_f$, then we get, as in the proof of Theorem 8, that

$$(\alpha \wedge L^{n-k}) \otimes (\alpha' \wedge L^{k}) = 0. \tag{4}$$

By assumption $k < [n/2] < n - k'$, and the degrees of α and α' are k and k'. Then (cf. proof of Theorem 8) Lefschetz's Theorem on hyperplane sections contradicts (4).

Therefore Theorem 9 follows.

https://doi.org/10.1023/A:1002600432171 Published online by Cambridge University Press

Acknowledgements

Most of this work was done when I was visiting the department of Mathematics, University of Orsay, France under the Indo-French Exchange Programme CEFIPRA. I gratefully acknowledge the financial and travel support provided by CEFIPRA. I also thank the Department of Mathematics, Orsay and L. Clozel for their warm hospitality.

I thank R. Parthasarathy for very helpful conversations.

I was introduced to these questions by L. Clozel (in an earlier joint work where we studied restriction in the holomorphic case), and it is a pleasure to acknowledge his help.

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