FINITELY SPECTRAL OPERATORS

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1. Introduction. In the theory of spectral (and prespectral) operators in a Banach space or in a locally convex topological vector space the countable additivity (in some topology) of a resolution of the identity of the operator is a standing assumption. One might wonder why. Even if one cannot completely agree with the opinion of Diestel and Uhl ([**6**, p. 32]) stating that "countable additivity [of a set function] is often more of a hindrance than a help", it might be interesting to study which portions of the theory of (pre)spectral operators and in which form extend to the more general situation described below.

In this paper we study bounded linear operators in a complex Banach space which possess a finitely additive, uniformly bounded resolution of the identity defined on all Borel subsets of the complex plane. In the first part of the discussion (cf. Theorems 1 and 2) we summarize the most important similarities and differences between the theories of prespectral and of finitely spectral operators, and naturally omit the similar proofs. In Theorem 3 we characterize the finitely spectral operators as those having certain operational calculi. Example 3 answers a question of Gillespie [**13**, p. 44] in the negative: it shows that there is a nonspectral prespectral operator having a unique resolution of the identity (of any class). Then we pose the following apparently open problem: does there exist a finitely spectral operator of scalar type which is not prespectral of any class?

The following parts of Section 3 may be regarded as contributions toward the solution of this problem, though they can be of interest in themselves. A finitely spectral operator of scalar type has a continuous $C(K)$-operational calculus. Theorem 4 gives some necessary and sufficient conditions for operators of the latter type to be scalar prespectral of some class G. Theorem 5 presents a sufficient condition for this, whereas Example 4 shows that this condition is not necessary in general (in the case $G = X^*$, the dual of the space X, it is, cf. Spain [**24**]). Example 5, which is a version of a remarkable example of Fixman [**11**] and of Berkson and Dowson [**4**], shows that there is a finitely spectral operator in l^∞ that is not prespectral of any class. Finally, Theorem 6 demonstrates that in a space without a copy of l^∞ each finitely spectral operator is spectral, whereas in a space containing a copy of l^∞ there are proper finitely spectral operators.

2. Preliminaries and notations. Let $\mathbb{C}$ denote the complex field, and let $a, b \subset \mathbb{C}$. We shall write ab for $a \cap b$, b^c for $\mathbb{C}\backslash b$, and if M is a class of subsets of $\mathbb{C}$, we write $b \in M^c$ for $b \notin M$. If H is a set, A is a σ-field of subsets of H, then $B(H, A)$ denotes the Banach algebra of bounded $\mathbb{C}$-valued A-measurable functions on H. The Banach spaces of set functions $ca(H, A)$ and $ba(H, A)$ are defined as in [**9**; IV. 2], as well as the Banach algebra $C(K)$ for K compact in $\mathbb{C}$. The support of $f: \mathbb{C} \to \mathbb{C}$ will be denoted by $\operatorname{supp} f$.

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If X and Y are complex Banach spaces, $L(X, Y)$ denotes the space of bounded linear operators from X into Y, and $L(X) = L(X, X)$. For $T \in L(X)$, $\sigma(T)$ denotes its spectrum and $\ker T$ denotes its kernel. Let H, A be as above. A finitely (additive) spectral measure on A (with values in $L(X)$) is a Boolean algebra (B.A.) homomorphism E of the B.A. A into a B.A. of projections in $L(X)$ for which $E(H) = I$ and there is a $K \geq 1$ such that $|E(b)| \leq K$ for every $b \in A$. If we do not specify otherwise, then $H = \mathbb{C}$ and A is the σ-field of all Borel subsets of $\mathbb{C}$. Let $T \in L(X)$. A finitely spectral measure E is called a *(finite) resolution of the identity for* T, if E commutes with T and the spectrum of each restriction $T \mid E(b)X$ is contained in the closure $\bar{b}$ of b. T is called *finitely spectral*, if T has a finite resolution of the identity.

Let X^* be the adjoint space of X. Then $\langle x, x^* \rangle$ or x^*x will denote their natural pairing. If Y and Z are paired linear spaces, then $w(Y, Z)$ will denote the weak topology of Y induced by this pairing. If V is a topological (vector) space with a (vector) topology U, then it will be denoted sometimes by (V, U). Let G be a total subset of the dual X^* of the Banach space X. For the definitions and properties of a spectral measure of class G or of a prespectral operator of class G we refer to the monograph of Dowson [**7**].

In what follows let X be a complex Banach space and let $T \in L(X)$. A (closed) T-invariant subspace Y of X is said to be *spectral maximal for* T if, for any other such subspace Z of X, $\sigma(T \mid Z) \subset \sigma(T \mid Y)$ implies $Z \subset Y$. T is said to have the *n-spectral decomposition property* ($n \geq 2$ an integer) if for every open covering $(G_1, \ldots, G_n)$ of $\sigma(T)$ there are T-invariant subspaces $X_1, \ldots, X_n$ of X such that $X = X_1 + \ldots + X_n$ and $\sigma(T \mid X_i) \subset \bar{G}_i$ $(i = 1, \ldots, n)$. T is *decomposable* in the sense of Foias ([**5**]) if it has the n-spectral decomposition property for every $n \geq 2$ so that the subspaces $X_1, \ldots, X_n$ can be chosen to be spectral maximal for T. The single-valued extension property and the (local) spectrum $\sigma(x) = \sigma_T(x)$ for x in X are defined as in [**7**, p. 141]. We fix also the following terminology.

Definition 1. Let A be a complex algebra of complex-valued functions on a set $N \subset \mathbb{C}$ (or of equivalence classes $\tilde{f}$ of such functions f) with the operations of (induced by) pointwise addition and multiplication, and containing the elements 1, z($\tilde{1}$, $\tilde{z}$). $T \in L(X)$ is said to be *A-spectral* if there is an algebra homomorphism $H: A \to L(X)$ such that $TH(f) = H(f)T$ for all $f \in A$, $H(1) = I$ (the identity in $L(X)$), and the operator $Q = T - H(z)$ is quasinilpotent (replace 1, z by $\tilde{1}$, $\tilde{z}$). T is said to be *A-scalar* (or to have an *A-operational calculus* H) if H above can be chosen so that $T = H(z)$. (Here and in similar contexts we write 1 and z for the functions $z \mapsto 1$ and $z \mapsto z$, respectively.)

3. Results and examples.

Lemma 1. *If $T \in L(X)$ is finitely spectral, then T is decomposable in the sense of Foias.*

Proof. Let G_1, G_2 be open sets in $\mathbb{C}$ such that $\sigma(T) \subset G_1 \cup G_2$. Let E denote a finite resolution of the identity for T, and let $x \in X$. Then

$$x = E(G_1)x + E(G_1^c)x \in E(G_1)X + E(G_1^c)X.$$

By [**7**, Prop. 1.37] and by assumption,

$$\sigma(T \mid E(G_1^c)X) \subset \sigma(T) \cap G_1^c \subset G_2,$$
$$\sigma(T \mid E(G_1)X) \subset \bar{G}_1.$$

So T has the 2-spectral decomposition property. Albrecht [**2**], Lange [**17**] and the author [**20**] have proved independently that T is then decomposable in the sense of Foias.

Corollary 1. *If T is finitely spectral, then T has the single-valued extension property; for any closed set F in $\mathbb{C}$ the linear manifold*

$$X_T(F) = \{x \in X : \sigma_T(x) \subset F\}$$

is a spectral maximal subspace for T, and for any open set N such that $F \subset N \subset \mathbb{C}$ and for any finite resolution of the identity E for T

$$E(F)X \subset X_T(F) \subset E(N)X.$$

In particular, with the choice $F = \sigma(T)$,

$$E(\sigma(T))X \subset X = E(N)X.$$

Proof. The first two statements are immediate consequences of the fact that T is decomposable. To prove the third assertion, use as much of the proof of [**7**, Theorem 5.33] as permitted under our conditions. Finally, note that $\sigma_T(x) \subset \sigma(T)$ for any x in X.

Corollary 2. *If T is finitely spectral with a resolution of the identity E, then*

$$\sigma(T) = \bigcap \{F \subset \mathbb{C} : F \text{ closed and } E(F) = I\}.$$

Now let T be a finitely spectral operator with a finite resolution of the identity E and let N be a Borel neighborhood of $\sigma(T)$. Then $E(N) = I$ and

$$B_N = \{E(b);\ b \text{ Borel subset of } N\} \subset B_{\mathbb{C}} \equiv B$$

is a bounded Boolean algebra of projections. By [**7**, Prop. 5.3], if M denotes a bound for B, the projections $E_1, \ldots, E_n$ in B are disjoint and if $c_1, \ldots, c_n \in \mathbb{C}$, then

$$\left|\sum_{i=1}^{n} c_i E_i\right| \leq 4M \sup |c_i| .$$

For any Borel set H in $\mathbb{C}$ let $B(H)$ denote the Banach algebra of bounded Borel measurable functions on H under the supremum norm. For any f in $B(N)$ the integral

$$E(f, N) = \int_N f(z) E(dz)$$

is defined as usual (see, for example, [**7**, pp. 119–120]), and satisfies

$$|E(f, N)| \leq 4M \sup_{z \in N} |f(z)| .$$

Further, the map $f \to E(f, N)$ is a continuous algebra homomorphism of $B(N)$ into $L(X)$.

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The system of Borel neighborhoods $\{N_t\}$ is partially ordered by $N_t \leq N_u$ if and only if $N_t \supset N_u$. In this case the restriction mapping from $B(N_t)$ into $B(N_u)$ is continuous and linear, and so we can construct the inductive limit

$$B^+ = B^+(\sigma(T)) = \operatorname{ind}_{N_t} B(N_t).$$

This inductive limit can be obtained also by choosing a strictly decreasing cofinal sequence $\{N_k\}$ of Borel neighborhoods with intersection $\sigma(T)$: thus

$$B^+ = \operatorname{ind}_k B(N_k).$$

B^+ can be thought of as the topological algebra of equivalence classes of Borel measurable functions bounded on some neighborhood of $\sigma(T)$; two functions are equivalent if they coincide on a neighborhood of $\sigma(T)$.

Since $E(N_k \backslash N_m) = 0$ for $m \geq k$, the continuous linear mappings

$$E_k : B(N_k) \to L(X), \qquad E_k f = E(f, N_k)$$

are compatible (cf., for example, [**12**, pp. 117–118]). Hence the mapping $E : B^+ \to L(X)$ defined by $E\tilde{f} = E(f, N_k)$, where $f \in B(N_k)$ and $\tilde{f}$ is the equivalence class containing f, is well-defined and is a continuous algebra homomorphism. The notation E seems to be natural and will not cause any confusion. We shall also use the notation

$$E\tilde{f} = \int \tilde{f}\, dE,$$

and we clearly have for any $f \in \tilde{f}$, $f \in B(N_k)$

$$E\tilde{f} = \int_{N_k} f(z) E(dz).$$

Independently of T, we can start with a finitely additive spectral measure, and then have the following result.

LEMMA 2. *Let H be a set and A a σ-field of subsets of H. Let E be a finitely additive spectral measure on A with values in $L(X)$, and let $f \in B(H, A)$. Define for each Borel set b*

$$E(f, H) = \int_H f(z) E(dz), \qquad F(b) = E(f^{-1}(b)).$$

Then $E(f, H)$ is a finitely spectral operator with a resolution of the identity F. Further,

$$E(f, H) = \int_{\mathbb{C}} z F(dz).$$

Proof. This is as the corresponding parts of the proof of [**7**, Proposition 5.8].

THEOREM 1. *Let T be a finitely spectral operator with a resolution of the identity E and let N be any bounded Borel neighborhood of $\sigma(T)$. Define*

$$S(E) = E\tilde{z} = \int_N z E(dz), \qquad Q(E) = T - S(E).$$

Then $S(E)$ is a finitely spectral operator with a resolution of the identity E, and $Q(E)$ is a quasinilpotent commuting with E. Conversely, if E is a finitely additive spectral measure on the Borel sets vanishing outside a bounded Borel set N, $S(E)$ is defined as above, and $Q(E)$ is a quasinilpotent commuting with E, then the operator

$$T = S(E) + Q(E)$$

is finitely spectral with a resolution of the identity E.

Proof. See, for example, the proofs of [**7**, Theorem 5.10, 5.15].

Also, the commutativity theorem for prespectral operators and its consequences generalize for finitely spectral operators.

THEOREM 2. *Let T be a finitely spectral operator with a resolution of the identity E. Let $V \in L(X)$ and let $VT = TV$. Assume that $\tilde{f} \in B^+$ has a representative f which is continuous in some neighborhood N of $\sigma(T)$. Then*

$$VE\tilde{f} = E\tilde{f}V.$$

Further, if F is any resolution of the identity for T, then

$$E\tilde{f} = F\tilde{f}.$$

In particular, with the notation of the preceding theorem

$$S(E) = S(F).$$

Proof. See the proofs of [**7**, Theorems 5.12, 5.13], or [**1**, Theorem 3].

If E is a finitely additive spectral measure vanishing outside a bounded Borel set, we shall call the operator

$$S(E) = \int_C zE(dz)$$

a scalar operator. If T is finitely spectral, then the operators $S = S(E)$ and $Q = Q(E)$ figuring in Theorems 1 and 2 will be called the scalar and the quasinilpotent parts of T, respectively.

Now we shall show by examples that the "irregular" behavior of a resolution of the identity of a finitely spectral operator, possible by Corollary 1 to Lemma 1, can indeed occur.

EXAMPLE 1. Let T be a prespectral operator with a resolution of the identity E_0 (of some class), and let Z be a closed set in $\sigma(T)$ such that $E_0(Z) \neq I$. Let $P = E_0(Z^c)$ and let B denote the Boolean algebra of all Borel subsets of Z^c. Let F denote the proper filter of B consisting of all sets of the form $A \backslash Z$, where A is a Borel neighborhood in $\mathbb{C}$ of Z. Let M be a maximal filter of B containing F (cf. [**23**, 6.1(ii)]). Any Borel set in the plane is of the form $b \cup m$, where b is a Borel subset of Z and m is a Borel subset of Z^c which belongs to M or to M^c. Define

$$E(b \cup m) = \begin{cases} E_0(b) + P & (m \in M), \\ E_0(b) & (m \in M^c). \end{cases}$$

It is easy to see that E is a finitely additive spectral measure vanishing outside each neighborhood of Z. The operator

$$S = S(E) = \int z E(dz)$$

is finitely spectral with a resolution of the identity E (Theorem 1) and with spectrum $\sigma(S)$ contained in Z (Corollary 2 to Lemma 1). However,

$$E(A \backslash Z) = P \neq 0$$

for any Borel neighborhood A (in $\mathbb{C}$) of Z.

EXAMPLE 2. Let Z be the set $\sigma(I) = \{1\}$ (I the identity in $L(X)$), let B, F, M be as above, and define for b and m as above

$$E(b \cup m) = \begin{cases} I & (m \in M), \\ 0 & (m \in M^c). \end{cases}$$

We have then

$$S = S(E) = E\bar{z} = I.$$

On the other hand, it is clear that

$$\int_{\sigma(I)} z E(dz) = 0 \neq S(E).$$

Referring now to Definition 1 and to the considerations and notations preceding Lemma 2, we have the following characterization of finitely spectral operators.

THEOREM 3. *For $T \in L(X)$ the following conditions are equivalent.*

(i) *T is finitely spectral,*

(ii) *T is $B^+(\sigma(T))$-spectral with respect to a continuous homomorphism H of the topological algebra B^+ into $L(X)$,*

(iii) *T is $B(N)$-spectral, where N is a compact neighborhood of $\sigma(T)$.*

Proof. If T is finitely spectral with a finite resolution of the identity E then, by Theorem 1

$$T = E\bar{z} + Q,$$

where the quasinilpotent Q commutes with E. According to the considerations preceding Lemma 2, T is $B^+(\sigma(T))$-spectral with respect to the continuous homomorphism $H(\bar{f}) = E\bar{f}(\bar{f} \in B^+)$. Thus (i) implies (ii).

Any homomorphism $H: B^+ \to L(X)$ induces a homomorphism $H_N: B(N) \to L(X)$

defined by

$$H_N(f) = H(\tilde{f}) \quad (f \in B(N))$$

if N is any compact neighborhood of $\sigma(T)$. Hence (ii) implies (iii).

Assume now that (iii) holds with respect to a homomorphism H. The set of idempotents in the commutative Banach algebra $B(N)$ is bounded in norm by 1; therefore, by a result of Bade and Curtis [**3**, Theorem 2.3], its image under H is bounded. Let $k(h)$ denote the characteristic function of a set h and define

$$E(b) = H(k(b \cap N)) \quad (b \text{ Borel set in } \mathbb{C}).$$

Then there exists a number $M > 0$ such that for all b

$$|E(b)| \le M,$$

and E is a finitely additive spectral measure commuting with each $H(f)$.

The spectrum of the operator $V = H(z)$ is, by assumption, equal to $\sigma(T)$. Further, V is $B(N)$-scalar in the sense of [**25**, IV. 7.2], therefore by [**25**, IV. 7.3] it is decomposable in the sense of Foias, and its spectral maximal space for any closed set F in $\mathbb{C}$ is

$$X_V(F) = \bigcap \{\ker H(f) : f \in B(N), \operatorname{supp} f \subset F^c\}.$$

We show that V is finitely spectral with a finite resolution of the identity E. It will suffice to prove that for any Borel set b with closure $F \subset N$, we have $\sigma(V \mid E(b)X) \subset F$.

For any Borel set $h \subset F$ we have

$$E(h)X = H(k(h))X \subset X_V(F),$$

since $H(f)H(k(h))X = \{0\}$ for $f \in B(N)$ with $\operatorname{supp} f \subset F^c \subset h^c$. Let

$$P(b) = E(b) \mid X_V(F), \qquad V(F) = V \mid X_V(F).$$

Since V and $E(b)$ commute, the idempotent $P(b)$ commutes with $V(F)$, so that

$$\sigma(V \mid E(b)X) = \sigma(V(F) \mid P(b)X_V(F)) \subset \sigma(V(F)) \subset F$$

(cf. [**7**, Proposition 1.37] and [**5**, Theorem 2.1.5]).

As N is a neighborhood of $\sigma(V) = \sigma(T)$, by Theorem 1, there is a quasinilpotent Q_1 commuting with E such that

$$V = \int_N zE(dz) + Q_1.$$

Since T commutes with H, T commutes with E and V, hence also with Q_1. The operator $Q_0 = T - V$ is quasinilpotent, and commutes with E and V. Hence Q_0 and Q_1 commute, and

$$T = \int_N zE(dz) + Q_1 + Q_0.$$

The operator $Q = Q_0 + Q_1$ is quasinilpotent and commutes with E hence, by Theorem 1, T is finitely spectral. Thus (iii) implies (i), and the proof is complete.

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Let X be a Banach space. We shall say that two finitely additive spectral measures E and F vanishing outside some bounded set and with values in $L(X)$ are equivalent if they are resolutions of the identity of the same scalar operator T. It is well-known that if E is a spectral measure which is countably additive in the strong operator topology, then E is the unique spectral measure of any class in its equivalence class $\tilde{E}$ (cf. [**7**, Theorem 6.7]). It is also clear (cf. Example 2) that an equivalence class $\tilde{E}$ may simultaneously contain a spectral measure E_1 of some class and a spectral measure E_2 of no class. Fixman [**11**] constructed an example showing that an equivalence class $\tilde{E}$ can contain spectral measures of different classes (cf. [**7**, pp. 147–148]).

T. A. Gillespie has asked in [**13**, p. 44] whether a prespectral operator having a unique resolution of the identity (of any class) is necessarily spectral. The negative answer is contained in the following example.

Example 3. There is a Banach space X and an equivalence class $\tilde{E}$ such that $\tilde{E}$ contains a spectral measure of class $G_1 \neq X^*$ but no other spectral measures of any class.

Let $X = l^\infty$ and for $x = \{x_k\}$ in X define $Tx = \{(1/k)x_k\}$. Then $\sigma(T) = \bigcup_{k=1}^{\infty} \{1/k\} \cup \{0\}$ and T is a scalar operator with a resolution of the identity E_1 of class $G_1 = l^1$ satisfying (with δ the Kronecker symbol)

$$E_1\left(\left\{\frac{1}{n}\right\}\right)\{x_k\} = \{\delta_{nk}x_k\}.$$

Since the sets $\{1/k\}$ and their complements in $\sigma(T)$ are closed, if E is a resolution of the identity of class G for T then, by [**7**, Proposition 5. 27],

$$E\left(\left\{\frac{1}{n}\right\}\right) = E_1\left(\left\{\frac{1}{n}\right\}\right).$$

Further, by [**7**, Theorem 11. 12], $E(\{0\})X = \ker T = \{0\}$, and hence $E(\{0\}^c) = I$. G is a total linear manifold in $X^* = l^1 \oplus c_0^\perp$ (cf. [**16**, p. 426]. If $G \subset l^1$, then E and E_1 are both resolutions of the identity of class G for T. By [**7**, Theorem 5.13], we have $E = E_1$.

If G is not contained in l^1, then there is $g \in G$ such that

$$g = g_1 + g_0, \qquad g_1 \in G_1, \qquad 0 \neq g_0 \in c_0^\perp.$$

Let b_n denote the set $\{1, \frac{1}{2}, \ldots, 1/n\}$, and let $e \in X$ be such that $g_0 e \neq 0$. If $n \to \infty$, then

$$g_1E(b_n)e = g_1E_1(b_n)e \to g_1e,$$
$$0 = g_0E(b_n)e \not\to g_0e;$$

therefore

$$gE(b_n)e \not\to gE(\{0\}^c)e = ge.$$

This contradiction shows that T has the unique resolution of the identity E_1 of any class.

Problem. Does each equivalence class $\tilde{E}$ on each Banach space X contain a spectral measure of some class?

In this direction we have the following results.

Let E be a spectral measure of some class Γ on the Banach space X, and let G denote the set of all x^* in X^* such that the set function $\langle E(\cdot)x, x^*\rangle$ is countably additive for every x in X. Then E is also of class G and, as Gillespie has noted ([**14**, Lemma 3]), G is a w^*-dense, norm closed subspace of X^* which is invariant under the adjoint of $\int f(z)E(dz)$ for every Borel measurable complex valued function f which is bounded on the support of E. Further, if F is the resolution of the identity of class X of the adjoint of $\int zE(dz)$, then G is also invariant under F (cf. [**14**, p. 20] and [**21**, Lemma 3]). According to these remarks there is no restriction of the generality in the assumptions of the following result, which makes use of some general ideas of Schaefer (cf. [**22**, pp. 469–470]).

THEOREM 4. *Let X be a Banach space, let $S \in L(X)$, and let $K \supset \sigma(S)$ be a compact subset of $\mathbb{C}$. Assume that there is a continuous algebra homomorphism $H: C(K) \to L(X)$ such that $H(1) = I$ and $H(z) = S$, and that G is a w^*-dense, norm closed subspace of X^* which is invariant for $H(f)^*$ for every $f \in C(K)$. Let $L_G(X)$ denote the completion of $L(X)$ equipped with the weak topology $w(L(X), X \otimes G)$, and let F denote the resolution of the identity of class X of the adjoint S^** (cf. [**7**, Theorem 5.21]). *Let*

$$m_{x,g}(\cdot) = \langle x, F(\cdot)g\rangle, \qquad N = \operatorname{span}\{m_{x,g} : x \in X,\ g \in G\}.$$

Then there is a unique linear continuous extension

$$\tilde{H} : (B(K), w(B(K), N)) \to L_G(X)$$

of H. S is scalar prespectral of class G if and only if $\tilde{H}$ maps $B(K)$ into $L(X)$.

Proof. Let T denote the topology $w(B(K), N)$ (which may not be Hausdorff). It is easily seen that $C(K)$ is dense in $(B(K), T)$, and H, regarded as a map of $(C(K), T)$ into $(L(X), w(L(X), X \otimes G))$, is continuous. To obtain the last assertion we note that

$$\int_K f_\alpha \, dm_{x,g} \to \int_K f \, dm_{x,g} \quad (f_\alpha, f \in C(K), (x, g) \in X \times G)$$

clearly implies that

$$gH(f_\alpha)x = \langle x, H(f_\alpha)^*g\rangle \to \langle x, H(f)^*g\rangle = gH(f)x,$$

since $H(f)^*g = \int_K f(z)F(dz)g$ (cf. [**7**, Theorem 5.21]). Therefore there is a unique linear continuous mapping $\tilde{H}$ from $(B(K), T)$ into the completion $L_G(X)$ which extends H.

Assume now that S is scalar prespectral with resolution of the identity E of class G. Let

$$J(f) = \int_K f(z)E(dz) \quad (f \in B(K)).$$

Then the restriction of J is a continuous algebra homomorphism of the Banach algebra $C(K)$ into $L(X)$ and therefore, by [**1**, Theorem 3], $J(f) = H(f)$ $(f \in C(K))$. If $f_\alpha \to f$ in the topology T of $B(K)$, then

$$gJ(f_\alpha)x = \int_K f_\alpha(z)\langle E(dz)x, g\rangle = \int_K f_\alpha(z)\langle x, F(dz)g\rangle \to \int_K f\,dm_{x,g} = gJ(f)x,$$

since $F(\cdot)g = E(\cdot)^*g$ for every g in G (see [**21**, Lemma 3]). Hence $\tilde{H}(f) = J(f) \in L(X)$ for every f in $B(K)$.

Assume now that $\tilde{H}$ maps $B(K)$ into $L(X)$, and let M denote the class of all Borel subsets of K. Define

$$E(b) = \tilde{H}(k(b)) \quad (b \in M),$$

where $k(b)$ denotes the characteristic function of the set b. Let A denote the class of such Borel sets $a \in M$ that

$$\langle E(a)x, g\rangle = \langle x, F(a)g\rangle \quad (x \in X, g \in G).$$

The class A is clearly closed with respect to taking complements in K. Further, if $a_n \in A$ $(n = 1, 2, \ldots)$, and for $n \to \infty$ we have for some $a \in M$

$$\langle x, F(a_n)g\rangle \to \langle x, F(a)g\rangle \quad (x \in X, g \in G),$$

then the continuity of $\tilde{H}$ and $k(a_n) \to k(a)$ in the topology T imply that

$$\langle E(a_n)x, g\rangle = \langle \tilde{H}(k(a_n))x, g\rangle \to \langle \tilde{H}(k(a))x, g\rangle = \langle E(a)x, g\rangle.$$

Since F is a spectral measure of class X, the class A contains the union of any increasing (the intersection of any decreasing) sequence of its elements. We shall show that A contains the closed subsets of K, which will imply that A is the class of all Borel subsets of K.

Let c be a compact subset of K. There is a sequence $\{f_n\} \subset C(K)$ such that $|f_n| \leq 1$ and $\lim_n f_n = k(c)$ pointwise. The Lebesgue dominated convergence theorem yields that $f_n \to k(c)$ in the topology T; therefore

$$\langle E(c)x, g\rangle = \lim_n \langle H(f_n)x, g\rangle = \lim_n \int_K f_n\,dm_{x,g} = \langle x, F(c)g\rangle,$$

for each $x \in X, g \in G$.

Now we show that E is a spectral measure (then clearly of class G). It will suffice to prove the multiplicative property. Define the set functions n by

$$n(b) = n(b, x, g, h) = \int_b h(z)m(dz, x, g) \quad (b \in M, x \in X, g \in G, h \in C(K)),$$

where $m(\cdot, x, g)$ is a more convenient notation for $m_{x,g}(\cdot)$. For any f in $C(K)$ we have then

$$\int_K f(z)n(dz) = \int_K f(z)h(z)m(dz, x, g) = \langle x, H(fh)^*g\rangle$$
$$= \langle H(h)x, H(f)^*g\rangle = \int_K f(z)m(dz, H(h)x, g).$$

Since $n(\cdot)$ and $m(\cdot, H(h)x, g)$ are regular countably additive measures (cf. [**9**, III. 9. 22]), they are identical. Therefore, using the H^*-invariance of G, for every b in M

$$n(b) = m(b, H(h)x, g) = \langle H(h)x, F(b)g\rangle = \langle x, F(b)H(h)^*g\rangle$$
$$= \langle E(b)x, H(h)^*g\rangle = \int_K h(z)m(dz, E(b)x, g).$$

Hence we obtain for $b \in M$, $x \in X$, $g \in G$, $h \in C(K)$

$$\int_K h(z)k(b, z)m(dz, x, g) = n(b, x, g, h) = \int_K h(z)m(dz, E(b)x, g),$$

and so for every $a \in M$ we have $m(ab, x, g) = m(a, E(b)x, g)$. Thus

$$\langle E(ab)x, g\rangle = m(ab, x, g) = \langle E(b)x, F(a)g\rangle = \langle E(a)E(b)x, g\rangle.$$

Hence E is a spectral measure of class G and, by duality,

$$S = \int_K zE(dz).$$

The theorem is proved.

As a corollary we obtain the following generalization of a result of Dunford [**8**] (see also [**7**, Theorem 6.10]).

COROLLARY 1. *Assume the situation described in the first two sentences of Theorem* 4. *If X is sequentially complete with respect to the $w(X, G)$ weak topology, then S is scalar prespectral of class G.*

Proof. According to the proof of Theorem 4, it is enough to show that $E(b) = \tilde{H}(k(b))$ belongs to $L(X)$ for each Borel set $b \subset K$. It will suffice to prove that $\tilde{H}(k(b))$ is a linear map of X into X satisfying (with the notations of Theorem 4)

$$\langle \tilde{H}(k(b))x, g\rangle = \langle x, F(b)g\rangle \quad (x \in X, g \in G), \tag{$*$}$$

because an application of [**9**, II. 2.7] will then yield that $E(b) \in L(X)$. If b is a compact subset of K, and $\{f_n\} \subset C(K)$ is such that $|f_n| \leq 1$ and $f_n \to k(b)$ pointwise, then for $x \in X$, $g \in G$, we have

$$\lim_n \langle H(f_n)x, g\rangle = \lim_n \int f_n \, dm_{x,g} = \langle x, F(b)g\rangle.$$

Since X is $w(X, G)$-sequentially complete, there is a unique $D(b)x \in X$ such that

$$\langle D(b)x, g\rangle = \lim_n \langle H(f_n)x, g\rangle = \langle x, F(b)g\rangle.$$

On the other hand, $f_n \to k(b)$ in the topology T; thus the continuity of $\tilde{H}$ implies that $\tilde{H}(k(b))x = D(b)x$. So $(*)$ holds if b is a compact subset of K. A similar argument, using the $w(X, G)$-sequential completeness of X and the continuity of $\tilde{H}$, shows that the class A

of sets b for which $(*)$ holds is closed with respect to taking increasing unions (decreasing intersections) of sequences of its elements, and A clearly contains complements. So A contains the class of all Borel sets, and the proof is complete.

REMARK. Theorem 4 and its Corollary 1 remain valid if the condition $H(f)^*G \subset G$ $(f \in C(K))$ is replaced by $F(b)G \subset G$ (b Borel set).

Indeed, for every $f \in C(K)$

$$H(f)^* = \int_K f(z)F(dz).$$

Since G is closed, the definition of the integral implies that $H(f)^*G \subset G$.

In [**24**] Spain has given a characterization of scalar spectral operators in terms of their $C(K)$-operational calculi, which may be used to determine whether a given equivalence class $\tilde{E}$ of spectral measures contains a spectral measure of class X^*. Seeking a generalization of Spain's result, we obtain the following sufficient condition for a class $\tilde{E}$ to contain a spectral measure of class G.

THEOREM 5. *Assume that $S \in L(X)$ has a continuous $C(K)$-operational calculus (here K denotes a compact set containing $\sigma(S)$) H such that for each x in X the map*

$$H_x : C(K) \to X, \qquad H_x f = H(f)x \quad (f \in C(K))$$

is compact with respect to the weak topology $w(X, G)$ of X. (Here G denotes a total F-invariant linear manifold in X^*, and F is the resolution of the identity of class X for S^*.) Then S is a scalar prespectral operator of class G.

Proof. By a result of Lewis [**18**, Theorem 3.1], characterizing weakly compact operators from $C(K)$ into a locally convex Hausdorff linear topological space, for every x in X there is a measure m_x defined on the σ-algebra B of the Borel subsets of K and with values in X such that

(i) the set function $gm_x : B \to \mathbb{C}$ is countably additive for each g in G,

(ii) the (Pettis-type) integral ([**18**, Definition 2.1])

$$\int_K f(z)m_x(dz) \in X$$

exists for every function f in $C(K)$ and is equal to $H(f)x$.

For every $x \in X$, $g \in G$, $f \in C(K)$ we have then

$$\begin{aligned}\int_K f(z)gm_x(dz) &= g\left(\int_K f(z)m_x(dz)\right) = \langle H(f)x, g\rangle \\ &= \langle x, H(f)^*g\rangle = \int_K f(z)\langle x, F(dz)g\rangle,\end{aligned}$$

where both occurring scalar-valued set functions are regular and countably additive on B. For every $b \in B$ define the map

$$E(b) : X \to X, \qquad E(b)x = m_x(b).$$

Then

$$\langle E(b)x, g\rangle = \langle x, F(b)g\rangle,$$

so that $E(b)$ is linear, and [**9**, II. 2.7] shows that $E(b) \in L(X)$. The F-invariance of G implies that E is multiplicative, and so E is a spectral measure of class G. Since

$$\left\langle \int_K zE(dz)x, g \right\rangle = \langle x, H(z)^*g\rangle = \langle Sx, g\rangle \quad (x \in X, g \in G),$$

the theorem is proved.

However, the following example shows that the converse of this theorem does not hold in general.

EXAMPLE 4. Let $K = [0, 1]$, let M denote the σ-algebra of the Borel subsets of K and let

$$X = B(K) = B(K, M)$$

with the supremum norm. The operator $T \in L(X)$ defined by

$$(Tx)(z) = zx(z)$$

is a scalar prespectral operator of class $G = ca\ (K, M)$, where G is clearly total in $X^* = ba\ (K, M)$ (for these spaces see [**9**, Chapter IV]). $\sigma(T) = K$, and the resolution of the identity E of class G is

$$(E(b)x)(z) = k(b; z)x(z) \quad (b \in M).$$

The adjoint $T^* \in L(X^*)$ is given by

$$(T^*x^*)(b) = \int_b zx^*(dz) \qquad (x^* \in X^*).$$

The resolution of the identity F of class X for Y^* satisfies

$$(F(b)g)(h) = g(b \cap h) \qquad (b, h \in M, g \in G)$$

and hence G is F-invariant.

Let us assume now that the map

$$A : C(K) \to (X, w(X, G)), \qquad Af = H_1 f = \int_K f(z)E(dz)1$$

(here H is the, by [**1**, Theorem 3] unique, continuous $C(K)$ operational calculus for T) is compact (or, equivalently, is weakly compact). By Theorem 3.1 of Lewis [**18**] there is a unique set function $m : M \to X$ such that gm is countably additive and regular for each g in G (cf. [**18**, p. 159]), and

$$Af = \int_K f(z)m(dz) \quad (f \in C(K)).$$

Hence $m(\cdot) = E(\cdot)1$. Further, by the cited theorem again, the $w(X, G)$-closed absolutely convex hull Q of $m(M)$ is $w(X, G)$-compact. By Šmulian's criterion for weak compactness (see e.g. [**15**, 16.6]), this is the case if and only if the polar Q^0 (in G) of Q is radial at 0 and each linear functional on G which is bounded on Q^0 is represented by some element of X.

It is easily seen (cf. [**15**, 16.3]) that

$$\begin{aligned} Q^0 &= \{g \in G : |gm(b)| \le 1 \quad \text{for} \quad b \in M\} \\ &= \{g \in G : |g(b)| \le 1 \quad \text{for} \quad b \in M\} = \{g \in G : |g| \le 1\}, \end{aligned}$$

where $|g|$ is the supremum norm in $ba(K, M)$. So the linear functionals on G which are bounded on Q^0 are exactly the elements of $G^* = ca(K, M)^*$.

Let $z \in K$, and let m_z denote the element of G defined by

$$m_z(\{z\}) = 1, \qquad m_z(b) = 0 \quad \text{for} \quad b \subset K\backslash\{z\}, \qquad b \in M.$$

Let H denote the closed linear span of $\{m_z : z \in K\}$ in G. Since the restriction n of the Lebesgue measure to M cannot be approximated by linear combinations of m_z's in G, we have $n \in G\backslash H$, and so there is a g^* in G^* such that $g^*H = \{0\}$ and $g^*n = 1$.

If g^* is represented by $f \in X$, then

$$\int_K f(t)m_z(dt) = 0 \quad (z \in K)$$

implies that $f(t) = 0$ for $t \in K$. Hence $g^*n = 0$, a contradiction. Thus we have an example of a scalar prespectral operator T of class G, for which the map H_1 from $C(K)$ into $(X, w(X, G))$ is not compact.

The operators S and A of the following example were (essentially) considered in [**7**, Example 10.11] for other purposes. We shall use them for the construction of a finitely spectral operator T which is not prespectral of any class. This type of examples goes essentially back to Fixman [**11**], cf. also Berkson and Dowson [**4**]. Note that with a slight modification the underlying space X could be l^∞ rather than $l^\infty(Z)$ below.

Example 5. Let $z_1 = 1$, $z_n = 1 - 1/n$ $(n = 2, 3, \ldots)$ and let Z denote the set consisting of the numbers z_k $(k = 1, 2, \ldots)$. Let F_0 denote the family of all cofinite subsets of Z, i.e. of all subsets whose complements in Z are finite sets. Then F_0 is a filter, and is contained in a maximal filter M on Z. The corresponding two-valued finitely additive measure m is defined by

$$m(h) = 1 \quad \text{for} \quad h \in M, \qquad m(h) = 0 \quad \text{for} \quad h \notin M.$$

Since $m \in ba(Z)$, the integral

$$Lf = \int_Z f\,dm$$

exists for all $f \in l^\infty(Z)$, and L is a linear multiplicative continuous functional on $X = l^\infty(Z)$ (see, e.g., [**10**, 1.20, 1.25–1.27]).

For any $f \in X$ write $f_k = f(z_k)$ $(k = 1, 2, \ldots)$. Define

$$Sf = \{z_k f_k\}, \qquad Af = \{Lf, 0, 0, \ldots\} \quad (f \in X).$$

Then S and A belong to $B(X)$, $|A| = 1$ and $A^2 = 0$ (cf. [**7**, pp. 203–204]). Further, $LSf = L\{z_k\}Lf = Lf$ implies

$$AS = A = SA,$$

the spectrum of S is Z, and S is a scalar type prespectral operator with resolution of the identity E of class $l^1(Z)$ satisfying

$$E(b)f = k(b)f \quad (b \subset Z, f \in X),$$

where $k(b)$ denotes the characteristic function of the set b. Define $T = S + A$; then T is not prespectral of any class (cf. [**7**, pp. 146–147]). For completeness' and later contrast's sake we give the proof: if it were with resolution of the identity G of class K then, by [**7**, Theorem 5.24], G is also a resolution of the identity for the scalar part S, and the nilpotent part A commutes with G. By [**7**, Theorem 5.33], $G(\{1\})X = E(\{1\})X$, and therefore

$$AG(\{1\})X = AE(\{1\})X = \{0\}.$$

However, $A\{1, 1, \ldots\} = \{1, 0, 0, \ldots\} \in E(\{1\})X$ implies that

$$G(\{1\})AX \neq \{0\}.$$

Now we define the following projections:

$$F(b)f = \begin{cases} k(b \cup \{1\})f & (b \in M), \\ k(b \backslash \{1\})f & (b \notin M), \end{cases} \quad (f \in X).$$

It is easy to show that F is a finitely additive spectral measure. For example, let $b_1 \in M$, $b_2 \notin M$. Since the complement M^c of M is a (maximal) ideal, $b_1 b_2 \in M^c$. Hence

$$F(b_1)F(b_2) = E(b_1 \cup \{1\})E(b_2 \backslash \{1\}) = E(b_1 b_2 \backslash \{1\}) = F(b_1 b_2).$$

Clearly, F is uniformly bounded and commutes with S. We show that it also commutes with A.

Since the operators $F(b)$ and A belong to $L(X)$, it is sufficient to prove that they commute on a fundamental set in X, e.g. that

$$AF(b)k(h) = F(b)Ak(h) \quad (b, h \subset Z).$$

The left-hand side is the vector

$$\{L[k(b \cup \{1\})k(h)], 0, 0, \ldots\} \quad \text{for} \quad b, h \in M.$$

Since M is a filter, by the definition of m and L, the vector above is

$$\{1, 0, 0, \ldots\}.$$

Since M^c is an ideal, if b or h belongs to M^c, the left-hand side is the vector 0. For $h \in M^c$

the right-hand side is also 0; for $h \in M$ it is

$$F(b)\{1, 0, 0, \ldots\}.$$

The definition of $F(b)$ yields then that F and A commute. Hence F and T commute.

Now we can in two ways finish the proof that T is finitely spectral: either by noting that

$$S = \int zF(dz),$$

and then invoking Theorem 1, or by showing that

$$\sigma(T \mid F(b)X) \subset \bar{b} \quad \text{for any} \quad b \subset Z.$$

This containment relation can be proved as follows. Since $A \mid F(b)X$ is a nilpotent commuting with $S \mid F(b)X$,

$$\sigma(T \mid F(b)X) = \sigma(S \mid F(b)X) = \begin{cases} \sigma(S \mid E(b \cup \{1\})X) & (b \in M), \\ \sigma(S \mid E(b \backslash \{1\})X) & (b \in M^c). \end{cases}$$

Since S is prespectral with a resolution of the identity E, for $b \in M^c$ the containment relation clearly holds. For $b \in M$ we note that then 1 is an accumulation point of b.

Gillespie [**13**, Theorem 1] has proved that the Banach space X does not contain a subspace isomorphic to l^∞ if and only if every prespectral operator on X is spectral. The following result strengthens in one direction Gillespie's result and shows, loosely speaking, that if on a space there are proper prespectral operators, then there are proper finitely spectral operators, too.

THEOREM 6. *If X does not contain a subspace isomorphic to l^∞, then every finitely spectral operator on X is spectral. If X contains a subspace isomorphic to l^∞, then there is a finitely spectral operator on X which is not prespectral of any class.*

Proof. Assume first that X does not contain a subspace isomorphic to l^∞ and that T is a finitely spectral operator on X with a finitely additive resolution of the identity E. Let N be a compact neighborhood of $\sigma(T)$, let $x \in X$ and define

$$H_x(f) = \int_N f(z)E(dz)x \quad (f \in B(N)).$$

Then $H_x : B(N) \rightarrow X$ is a bounded linear operator and, by [**6**, Corollary VI. 1.3], H_x is weakly compact. Hence, its restriction H_x^c to $C(N)$ is also weakly compact. The scalar part S of T has the $C(N)$-operational calculus H defined by

$$H(f) = \int_N f(z)E(dz) \quad (f \in C(N)),$$

and so an evident modification of Spain's result [**24**] implies that S is a scalar spectral operator. If Q denotes the quasinilpotent part of T, then $T = S + Q$ and Q commutes with S; hence T is spectral.

Assume now that X contains the closed subspace X_0 and that there is an invertible bounded linear operator J from X_0 onto l^∞. The space l^∞ is injective, i.e. for every Banach space containing l^∞ as a subspace, there is a bounded linear projection from the whole space onto l^∞ (see [**19**, p. 105]). Hence there is $\bar{J} \in L(X, l^\infty)$ which extends J, by [**19**, Proposition 2.f.2]. For any x in X define

$$Px = J^{-1}\bar{J}x.$$

Then P is a bounded linear projection of X onto X_0, and hence there is a closed subspace Y of X such that

$$X = X_0 \oplus Y.$$

Let T be the operator of Example 5; then $J^{-1}TJ \in L(X_0)$. Define

$$V = J^{-1}TJ \oplus 0$$

with respect to the decomposition of X above. Then V is clearly finitely spectral with a resolution of the identity

$$E_1(b) = J^{-1}F(b)J \oplus p(b)I_Y,$$

where F is the (merely) finitely additive resolution of the identity for T figuring in Example 5, p is the 0–1 valued measure on the Borel sets assuming 1 if and only if $0 \in b$, and I_Y is the identity operator on Y. Further, with the notations of Example 5,

$$V = (J^{-1}SJ \oplus 0) + (J^{-1}AJ \oplus 0) = S_1 + N,$$

where S_1 is a prespectral operator of scalar type, and N is nilpotent. If we assume that V is prespectral with resolution of the identity D of some class G, then we obtain a contradiction, as in Example 5. Indeed, with the notations there

$$D(\{1\})X = J^{-1}E(\{1\})JX_0 \oplus \{0\};$$

hence $ND(\{1\})X = J^{-1}AJJ^{-1}E(\{1\})l^\infty \oplus \{0\} = \{0\}$.

On the other hand

$$D(\{1\})NX = D(\{1\})(J^{-1}Al^\infty \oplus \{0\}) \ni D(\{1\})(J^{-1}\{1, 0, 0, \ldots\} \oplus \{0\}).$$

The last element belongs to $D(\{1\})X$ and is not 0, so $D(\{1\})$ and N do not commute. Therefore V is not prespectral of any class.

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