ON THE COMPLEMENTARY FUNCTIONS OF THE FRESNEL INTEGRALS

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1. Introduction. As is well known, the functions

$$(1.1)\qquad c(u) = \int_u^\infty \cos(p^2)dp, \quad s(u) = \int_u^\infty \sin(p^2)dp$$

have various applications in theoretical physics and engineering. It is thus worthwhile to study their behaviour for real and complex values of the argument. Since

$$c(u) = \tfrac{1}{2}\int_{u^2}^\infty t^{-\frac{1}{2}}\cos t\,dt, \quad s(u) = \tfrac{1}{2}\int_{u^2}^\infty t^{-\frac{1}{2}}\sin t\,dt$$

we may consider the functions

$$(1.2)\qquad c(z) = \int_z^\infty t^{-\frac{1}{2}}\cos t\,dt, \qquad s(z) = \int_z^\infty t^{-\frac{1}{2}}\sin t\,dt, \quad z = x + iy,$$

instead of (1.1).

We have

$$(1.3)\qquad c(z) = C - C(z), \quad s(z) = S - S(z)$$

where

$$(1.4)\qquad C(z) = \int_0^z t^{-\frac{1}{2}}\cos t\,dt, \qquad S(z) = \int_0^z t^{-\frac{1}{2}}\sin t\,dt$$

are the Fresnel integrals and

$$(1.5)\qquad C = \lim_{z\to\infty} C(z) = \sqrt{\frac{\pi}{2}}, \qquad S = \lim_{z\to\infty} S(z) = \sqrt{\frac{\pi}{2}}.$$

In order that the relation (1.3) be valid, one has to choose a path of integration which goes asymptotically parallel to the x-axis to infinity. By means of (1.3) results about $c(z)$ and $s(z)$ can be obtained from recent considerations (1) of the Fresnel integrals; since these consequences are immediate we shall not consider them in detail.

2. Relations to other known functions. The functions $c(z)$ and $s(z)$ can be represented by certain $W_{k,m}$-functions. We have

$$(2.1)\qquad W_{-1/4,1/4}(\pm iz) = e^{\mp iz/2}(\pm iz)^{-1/4}\int_0^\infty\left(1 \pm \frac{t}{iz}\right)^{-\frac{1}{2}} e^{-t}dt.$$

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Setting $1 + t/iz = w/z$ and $1 - t/iz = w/z$, respectively, we obtain

$$W_{-1/4,1/4}(\pm iz) = z^{1/4} e^{\pm i(z/2+3\pi/8)} \int_z^\infty w^{-\frac{1}{2}} e^{\mp iw}\, dw \qquad \left(\arg z \neq \frac{\pi}{2}\right),$$

where one has to integrate along the real axis. Hence we find

$$\begin{aligned} c(z) &= 2^{-1} z^{-1/4} \{ e^{-i\alpha(z)} W_{-1/4,1/4}(iz) + e^{i\alpha(z)} W_{-1/4,1/4}(-iz) \}, \\ s(z) &= 2^{-1} z^{-1/4} \{ e^{-i\beta(z)} W_{-1/4,1/4}(iz) + e^{i\beta(z)} W_{-1/4,1/4}(-iz) \}, \end{aligned} \left(\arg z \neq \frac{\pi}{2}\right) \tag{2.2}$$

where

$$\alpha(z) = \tfrac{1}{2}\left(z + \frac{3\pi}{4}\right), \qquad \beta(z) = \tfrac{1}{2}\left(z - \frac{\pi}{4}\right).$$

For both integrals on the right hand side of (2.2) one has to choose a common path of integration which coincides asymptotically with the real axis.

Using the error function

$$\operatorname{Erf}(z) = \int_z^\infty e^{-t^2} dt$$

we find

$$\begin{aligned} c(z) &= i^{\frac{1}{2}} \operatorname{Erf}(\sqrt{-iz}) + i^{-\frac{1}{2}} \operatorname{Erf}(\sqrt{iz}), \\ s(z) &= i^{-\frac{1}{2}} \operatorname{Erf}(\sqrt{-iz}) + i^{\frac{1}{2}} \operatorname{Erf}(\sqrt{iz}), \end{aligned} \qquad \left(\arg z \neq \frac{\pi}{2}\right). \tag{2.3}$$

Relations between $c(z)$, $s(z)$ and the incomplete Gamma function $Q(z, \beta)$ have been mentioned in **(1)**. There exist also relations to Lommel's functions which can be obtained from the representations given in **(4)**.

3. Improvement of the accuracy of the asymptotic expansion. From **(1)** we obtain the following asymptotic expansions valid for all complex arguments with the exception of purely imaginary ones:

$$\begin{aligned} c(z) &\sim -z^{-\frac{1}{2}} (a(z) \cos z + b(z) \sin z) \\ s(z) &\sim z^{-\frac{1}{2}} (b(z) \cos z - a(z) \sin z) \end{aligned} \tag{3.1}$$

where

$$a(z) = \sum_{m=1}^{\infty} (-1)^m \frac{1.3\ldots(4m-3)}{(2z)^{2m-1}}, \quad b(z) = 1 + \sum_{m=1}^{\infty} (-1)^m \frac{1.3\ldots(4m-1)}{(2z)^{2m}}.$$

For real values $z = x$ the optimal accuracy of (3.1) can be considerably improved by summing the divergent part of (3.1) by Euler's method; we have to multiply the smallest term of the series by the function obtained through that Euler summation process. In detail: The term of $a(z)$ which has the smallest absolute value corresponds to the largest value of m for which

$$4m^2 - \tfrac{1}{4} \leqslant |z|^2. \tag{3.2}$$

Let us denote this term by $a_{m_0}(z)$. We first consider values of x which are integers. Equation (3.2) is valid for $x = 2m$. Then we have

$$a(x) = a_1(x) + \ldots + a_{m_0}(x)K_I(x)$$

where

$$(3.3) \qquad K_{\rm I}(x) = 1 - \frac{(2x-1)(2x+1)}{(2x)^2} + \frac{(2x-1)(2x+1)\ldots(2x+5)}{(2x)^4} - + \ldots$$

or

$$K_{\rm I}(x) = 1 - \left(1 - \frac{1}{(2x)^2}\right) + \left(1 + \frac{8}{2x} + \frac{14}{(2x)^2} - \frac{8}{(2x)^3} - \frac{15}{(2x)^4}\right) - + \ldots$$

$$= (1 - 1 + 1 - + \ldots) + (8 - 24 + 48 - + \ldots)\frac{1}{2x}$$

$$+ (1 + 14 - 205 + 924 - + \ldots)\frac{1}{(2x)^2} + \ldots .$$

The sequence of terms occurring in the coefficient of $(2x)^{-p}$, $p = 0, 1, \ldots,$ is such that Euler's summation always yields a finite expression for each of these coefficients. In this way we find

$$(3.4) \qquad K_{\rm I}(x) = 0.5 + \frac{1}{2x} - \frac{1.75}{(2x)^2} + \frac{4.5}{(2x)^3} - \frac{3.875}{(2x)^4} - \frac{146}{(2x)^5} + \ldots .$$

Setting

$$b(x) = b_1(x) + \ldots + b_{m_0-1}(x) + b_{m_0}(x) + \ldots$$

or

$$b(x) = b_1(x) + \ldots + b_{m_0-1}(x) + a_{m_0}(x)K_{\rm II}(x)$$

and $x = 2m$, in consequence of

$$b_m(x) = a_m(x)\,\frac{4m-1}{2x}, \qquad m = 1,2, \ldots,$$

we obtain

$$K_{\rm II}(x) = \frac{2x-1}{2x} - \frac{(2x-1)(2x+1)(2x+3)}{(2x)^3} + \ldots .$$

If we sum this expression by a method analogous to that described above we find

$$(3.5) \qquad K_{\rm II}(x) = 0.5 - \frac{0.5}{2x} - \frac{0.75}{(2x)^2} + \frac{6.25}{(2x)^3} - \frac{44.375}{(2x)^4} + \ldots .$$

The term of $b(z)$ which has the smallest absolute value corresponds to the largest value of m for which

$$(3.6) \qquad (4m+1)(4m+3) \leqslant 4|z|^2 .$$

Let us denote this term of $b(z)$ by

$$b_{m_0^*}(z).$$

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Again we consider real values $z = x$. Equation (3.6) is valid for $x = 2m + 1$. Then we have

$$b(x) = b_1(x) + \ldots + b_{m_0^*}(x) K_{\text{I}}(x)$$

and, since

$$a_{m+1}(x) = -b_m(x) \frac{4m+1}{2x}, \qquad m = 0,1, \ldots,$$

for that value of x we also have

$$a(x) = a_1(x) + \ldots + a_{m_0^*}(x) + a_{m_0^*+1}(x) + \ldots$$

$$= a_1(x) + \ldots + a_{m_0^*}(x) - b_{m_0^*} K_{\text{II}}(x),$$

where $K_{\text{I}}(x)$ and $K_{\text{II}}(x)$ are given by (3.4) and (3.5), respectively. For non-integer values of x one has to proceed similarly. Using this method the error in the values of the functions for arguments between 3 and 4, say, is about 2–3 per cent of the smallest error obtained by applying (3.1) in the usual way; the relative improvement increases with increasing argument.

4. Properties of the Zeros. In contrast to the Fresnel integrals, the complementary functions $c(z)$ and $s(z)$ have real zeros but no complex ones.

LEMMA 1. *The function $c(z)$ possesses exactly one zero in each of the intervals $J_n : n\pi \leqslant x \leqslant (2n+1)\pi/2$ $(n = 0, 1, \ldots)$ of the real axis; the other intervals, $K_n : (2n+1)\pi/2 < x < (n+1)\pi$ $(n = 0, 1, \ldots)$, do not contain zeros. The function $s(z)$ has exactly one zero in every interval K_n but no zeros in the intervals J_n.*

Proof. We consider $c(z)$. For large values of x the statement is a consequence of (3.1); a certain zero lies at a distance α_n from $n\pi$,

$$x_n = n\pi + \alpha_n, \qquad \alpha_n > 0,$$

where α_n tends to zero if n tends to infinity. Furthermore, since $\sqrt{x}$ is monotone we obtain from the form of the integrand of $c(z)$ that $\alpha_{n-1} > \alpha_n$ where

$$x_{n-1} = (n-1)\pi + \alpha_{n-1}$$

is the preceding zero of $c(z)$. Moreover, from this property the existence and above indicated position of the smaller zeros of $c(z)$ follows. In consequence of Rolle's theorem we have

$$\alpha_n < \frac{\pi}{2}, \qquad n = 0,1, \ldots,$$

and from these inequalities we conclude the existence of intervals which cannot contain zeros. The statement for $s(z)$ can be proved by a similar argument. All zeros are simple.

Using Lemma 1 we obtain

THEOREM 2. *The functions $c(z)$ and $s(z)$ do not have complex zeros.*

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Proof. We consider $c(z)$. Since this function is real for real values of the argument we may restrict ourselves to positive values of y. Furthermore we consider only positive values of x; for negative values of x the proof is similar. From (1.2) we have

$$c(x+iy) = c(x) + L(z) \tag{4.1}$$

where

$$L(z) = -\,i\int_0^y (x+iw)^{-\frac{1}{2}} \cos\,(x+iw)dw.$$

Let us denote by x_1 and x_2 ($> x_1$) the real zeros of $c(z)$ contained in the strip S_n: $2n\pi \leqslant x \leqslant (2n+2)\pi$ of the z-plane. For

$$\tfrac{1}{2}(4n+1)\pi \leqslant x \leqslant (2n+1)\pi, \quad \tfrac{1}{2}(4n+3)\pi \leqslant x \leqslant (2n+2)\pi$$

the imaginary part of $L(z)$ has constant sign; for $x_1 \leqslant x \leqslant \frac{1}{2}(4n+1)\pi$ and $x_2 \leqslant x \leqslant \frac{1}{2}(4n+3)\pi$ the real part of $c(z)$ has constant sign. Consequently, if there were complex zeros of $c(z)$ in S_n their real parts would lie in the intervals

$$2n\pi \leqslant x \leqslant x_1 \quad \text{or} \quad (2n+1)\pi \leqslant x \leqslant x_2. \tag{4.2}$$

Setting $(x+iw)^{-\frac{1}{2}} = a + ib$ we have

$$\mathfrak{J}\, L(z) = -\sin x \int_0^y b \sinh w\, dw - \cos x \int_0^y a \cosh w\, dw.$$

Since $x > 0$ and $y > 0$ we have $-\,b < a$. Furthermore, since

$$\sin x \leqslant \cos x \qquad (2n\pi \leqslant x \leqslant (2n+\tfrac{1}{4})\pi),$$

for these values of x we have $\mathfrak{J}\, L(z) < 0$. By similar reasoning we find $\mathfrak{J}\, L(z) > 0$ for values of z which satisfy $(2n+1)\pi \leqslant x \leqslant (2n+\frac{5}{4})\pi$. Hence the corresponding strips of S_n cannot contain complex zeros of $c(z)$. We prove finally that these strips contain the strips defined by (4.2). We have to show that

$$\alpha_n \equiv x_1 - 2n\pi < \tfrac{1}{4}\pi \quad , \quad \alpha_{n+1} \equiv x_2 - (2n+1)\pi < \tfrac{1}{4}\pi.$$

Since α_n is a monotone decreasing function of n (cf. the proof of Lemma 1) it suffices to prove that $\alpha_0 < \frac{1}{4}\pi$. From (1.3)–(1.5) we have

$$c(\tfrac{1}{4}\pi) = \sqrt{(\tfrac{1}{2}\pi)} - \int_0^{\frac{1}{4}\pi} t^{-\frac{1}{2}} \cos t\, dt.$$

If $0 \leqslant t \leqslant \frac{1}{4}\pi$ then $\cos t \geqslant 2^{-\frac{1}{2}}$ and consequently

$$c(\tfrac{1}{4}\pi) < \sqrt{(\tfrac{1}{2}\pi)} - 2^{-\frac{1}{2}} \int_0^{\frac{1}{4}\pi} t^{-\frac{1}{2}}\, dt = 0.$$

Since $c(x)$ is continuous, $c(0) = \sqrt{(\frac{1}{2}\pi)} > 0$, and $c(\frac{1}{4}\pi) < 0$, the interval $(0, \frac{1}{4}\pi)$ of the real axis must contain a zero of $c(z)$; hence $\alpha_0 < \frac{1}{4}\pi$. This completes the proof of Theorem 2 for $c(z)$. The statement for $s(z)$ can be proved in a similar way.

5. Computation of the Zeros. As follows from §4 approximate values for the zeros (except for the smallest zero of $c(z)$) can be obtained from (3.1). We have

$$c(x) \sim -x^{-\frac{1}{2}}\sin x + \tfrac{1}{2}x^{-3/2}\cos x + \ldots = 0.$$

Hence the first approximation is given by

$$x_{1,n} = n\pi, \qquad n = 1, 2, \ldots .$$

Setting

$$f(x) = -x^{-\frac{1}{2}}\sin x + \tfrac{1}{2}x^{-3/2}\cos x$$

we obtain

$$f(x_{1,n}) = (-1)^n 2^{-1}(n\pi)^{-3/2}$$

and

$$f'(x_{1,n}) = (-1)^{n+1}(n\pi)^{-\frac{1}{2}} + O((n\pi)^{-5/2}).$$

Hence the second approximation is given by

$$x_{2,n} = n\pi + (2n\pi)^{-1}, \qquad n = 1, 2, \ldots . \tag{5.1}$$

Similarly, we obtain for the zeros of $s(z)$ the second approximation

$$x^*_{2,n} = \tfrac{1}{2}(2n+1)\pi + ((2n+1)\pi)^{-1}, \qquad n = 0, 1, \ldots . \tag{5.2}$$

From (5.1) we obtain the second zero ($n = 1$) of $c(z)$ within an error of 1 per cent and the higher zeros more accurately. From (5.2) the first zero ($n = 0$) of $s(z)$ can be obtained within an error of 7 per cent, the second zero ($n = 1$) within an error of 0.3 per cent, etc.

For a more exact determination of the zeros we can use either the Taylor series development of the Fresnel integrals at $z = 0$, cf. **(1)**, in connection with (1.3) or, if n is large enough, the asymptotic expansion (3.1). In the latter case the method corresponds to that described in **(1**, § 7**)**. Some of the quotients occurring in the procedure for $c(z)$ involve the function $\tan x_n$. Setting

$$\tan x_n \approx \tan x_{2,n} = \tan (2n\pi)^{-1} \approx (2n\pi)^{-1}$$

we obtain the third approximation for the zeros of $c(z)$ in the form

$$x_{3,n} = n\pi + (2n\pi)^{-1} + 3(2n\pi)^{-3} \tag{5.3}$$

and the higher approximations in the form

$$\begin{aligned} x_{2q,n} &= n\pi + (2n\pi)^{-1} + \sum_{p=2}^{q} (-1)^{p+1} 1.3\ldots(4p-7)(4p-5)(4p-4)(2n\pi)^{1-2p} \\ x_{2q+1,n} &= x_{2q,n} + (-1)^{q+1} 1.3\ldots(4q-1)(2n\pi)^{-2q-1}, \qquad q = 2,3, \ldots . \end{aligned} \tag{5.4}$$

Similarly, for the zeros x_n^* of $s(z)$ we obtain the third approximation

$$x^*_{3,n} = \tfrac{1}{2}(2n+1)\pi + ((2n+1)\pi)^{-1} + 3((2n+1)\pi)^{-3} \tag{5.5}$$

and the higher approximations

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$$(5.6)\quad \begin{aligned} x^*_{2q,n} &= \tfrac{1}{2}(2n+1)\pi + ((2n+1)\pi)^{-1} + \\ &\sum_{p=2}^{q} (-1)^{p+1} 1.3 \ldots (4p-7)(4p-5)(4p-4)((2n+1)\pi)^{1-2p} \\ x^*_{2q+1,n} &= x_{2q,n} + (-1)^{q+1} 1.3 \ldots (4q-1)((2n+1)\pi)^{-2q-1},\ q = 2,3,\ldots . \end{aligned}$$

6. Modulus surfaces of $c(z)$ and $s(z)$, tables of zeros. Figure 1 represents the surface $W = |c(z)|$ in three-dimensional xyW-space. Figure 2 shows the surface $W = |s(z)|$ in this space. These graphs yield a clear picture of the distribution of the function values of $c(z)$ and $s(z)$ for complex values of the argument. The surfaces can be investigated by means of differential geometry in a way similar to that used in **(1)** in the study of the Fresnel integrals $C(z)$ and $S(z)$.

The tables contain the first 25 zeros of $c(z)$ and $s(z)$. The calculation was done by means of the methods developed in the preceding sections. Tables of values of the functions for complex values of the argument are obtained from **(2; 3)** by means of the relation (1.3) of this paper.

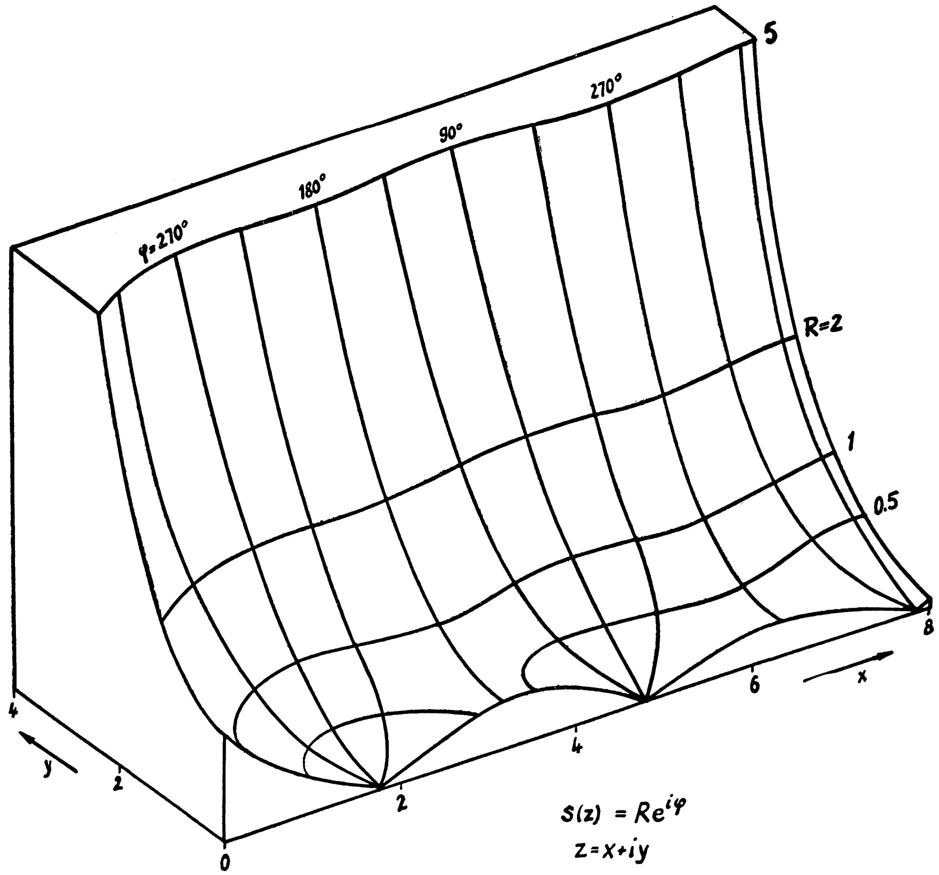

Figure 1

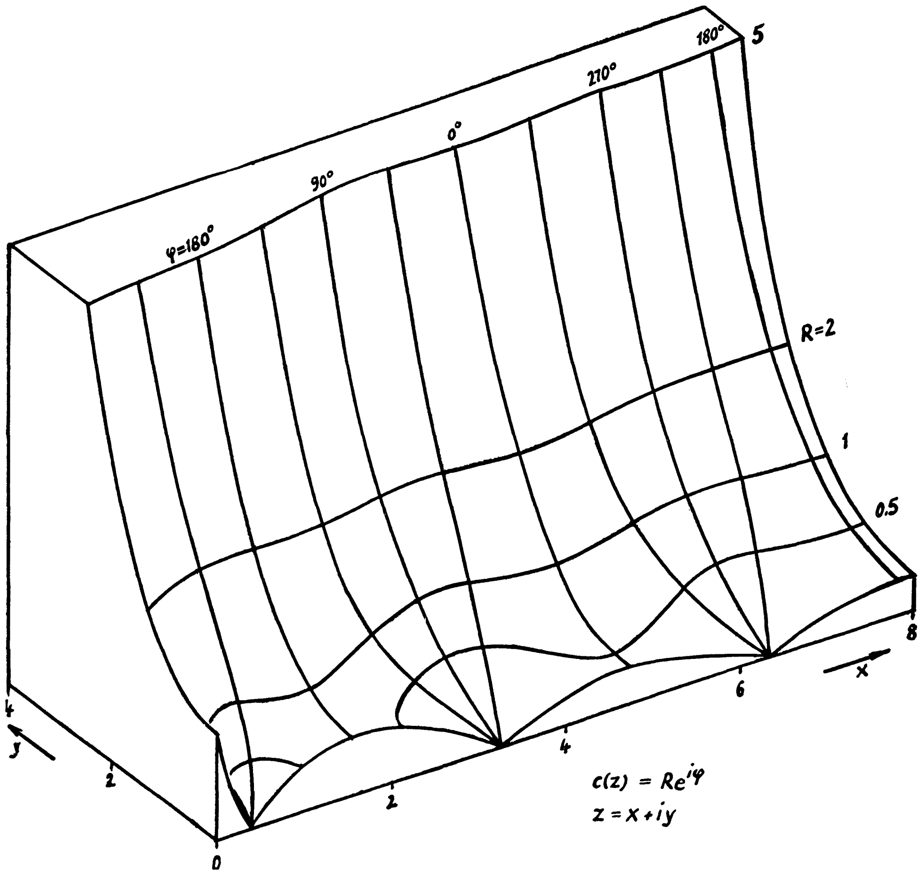

Figure 2

Zeros x_n of $c(z)$ and x_n^* of $s(z)$

n	x_n	x_n^*	n	x_n	x_n^*	n	x_n	x_n^*
0	0.41	1.77	10	31.43	33.00	20	62.84	64.41
1	3.27	4.81	11	34.57	36.14	21	65.98	67.55
2	6.36	7.92	12	37.71	39.28	22	69.12	70.69
3	9.48	11.04	13	40.85	42.42	23	72.26	73.83
4	12.61	14.17	14	43.99	45.56	24	75.41	76.98
5	15.74	17.31	15	47.14	48.71	25	78.55	80.12
6	18.88	20.44	16	50.28	51.85			
7	22.01	23.58	17	53.42	54.99			
8	25.15	26.72	18	56.56	58.13			
9	28.29	29.86	19	59.70	61.27			

Values of $c(z)$ and $s(z)$ for real argument $z = x$

x	$c(x)$	$s(x)$	x	$c(x)$	$s(x)$
0	1.253	1.253	7.5	−0.331	0.146
0.2	0.362	1.193	8.0	−0.350	−0.030
0.4	0.013	1.086	8.5	−0.283	−0.190
0.6	−0.241	0.951	9.0	−0.153	−0.294
0.8	−0.425	0.797	9.5	0.007	−0.323
1.0	−0.556	0.632	10.0	0.158	−0.272
1.2	−0.643	0.465	10.5	0.263	−0.159
1.4	−0.690	0.292	11.0	0.299	−0.012
1.6	−0.702	0.129	11.5	0.263	0.130
1.8	−0.682	−0.022	12.0	0.164	0.236
2.0	−0.635	−0.158	12.5	0.029	0.280
2.2	−0.566	−0.277	13.0	−0.107	0.255
2.4	−0.478	−0.376	13.5	−0.212	0.169
2.6	−0.377	−0.451	14.0	−0.263	0.045
2.8	−0.267	−0.503	14.5	−0.248	−0.085
3.0	−0.153	−0.531	15.0	−0.174	−0.190
3.2	−0.040	−0.536	15.5	−0.061	−0.247
3.4	0.069	−0.519	16.0	0.064	−0.241
3.6	0.168	−0.482	16.5	0.169	−0.178
3.8	0.257	−0.427	17.0	0.230	−0.074
4.0	0.330	−0.357	17.5	0.234	0.045
4.5	0.438	−0.142	18.0	0.181	0.150
			18.5	0.085	0.215
5.0	0.430	0.085	19.0	−0.029	0.227
			19.5	−0.133	0.183
5.5	0.319	0.271			
6.0	0.142	0.376	20.0	−0.202	0.096
6.5	−0.056	0.383			
7.0	−0.226	0.297			

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Values of $c(z)$ for complex argument $z = x + iy$

x	$y = 0$	$y = 1$		$y = 2$		$y = 3$		$y = 4$		$y = 5$	
	$\mathfrak{R}$[1]	$\mathfrak{R}$	$\mathfrak{I}$	$\mathfrak{R}$	$\mathfrak{I}$	$\mathfrak{R}$	$\mathfrak{I}$	$\mathfrak{R}$	$\mathfrak{I}$	$\mathfrak{R}$	$\mathfrak{I}$
0	1.25	−0.31	−1.56	−1.71	−2.96	−4.54	−5.79	−11.1	−12.3	−26.5	−27.8
1	−0.56	−1.09	−0.46	−2.75	−0.66	−6.57	−0.58	−15.5	0.10	−37.0	2.40
2	−0.64	−0.93	0.39	−1.92	1.40	−4.12	4.12	− 9.01	11.1	−20.1	28.9
3	−0.15	−0.14	0.66	0.01	1.99	0.68	5.12	3.02	12.8	9.94	31.6
4	0.33	0.56	0.36	1.49	0.97	4.13	2.19	11.1	4.74	29.4	10.1
5	0.43	0.64	−0.16	1.52	−0.61	3.81	−1.95	9.56	− 5.86	23.9	−16.9
6	0.14	0.18	−0.46	0.31	−1.43	0.50	−3.92	0.59	−10.5	− 0.13	−27.5
7	−0.23	−0.38	−0.33	−0.99	−0.95	−2.80	−2.42	− 7.81	− 5.99	−21.5	−14.7
8	−0.35	−0.54	0.07	−1.30	0.27	−3.38	0.93	− 8.85	2.96	−23.0	9.07
9	−0.15	−0.22	0.36	−0.46	1.12	−1.03	3.12	− 2.33	8.51	− 5.10	23.0
10	0.16	0.26	0.31	0.68	0.92	1.95	2.44	5.55	6.33	15.6	16.3
11	0.30	0.46	−0.01	1.12	−0.06	2.99	−0.28	7.98	− 1.10	21.4	− 3.82
12	0.16	0.24	−0.29	0.54	−0.90	1.34	−2.54	3.33	− 6.99	8.23	−19.1
13	−0.11	−0.18	−0.29	−0.47	−0.89	−1.35	−2.41	− 3.88	− 6.37	−11.0	−16.8
14	−0.26	−0.41	−0.04	−1.00	−0.10	−2.67	−0.18	− 7.20	− 0.23	−19.4	0.02
15	−0.17	−0.26	0.23	−0.61	0.73	−1.55	2.06	− 4.00	5.71	−10.2	15.8
16	0.06	0.10	0.28	0.29	0.86	0.86	2.34	2.53	6.26	7.34	16.7
17	0.23	0.36	0.08	0.88	0.21	2.37	0.53	6.44	1.25	17.6	2.89
18	0.18	0.27	−0.18	0.65	−0.58	1.68	−1.65	4.43	− 4.59	11.7	−12.7
19	−0.03	−0.05	−0.26	−0.15	−0.82	−0.47	−2.24	− 1.41	− 6.05	− 4.25	−16.3
20	−0.20	−0.32	−0.11	−0.78	−0.31	−2.09	−0.81	− 5.71	− 2.06	−15.5	− 5.19

[1] $\mathfrak{I} = 0$.

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Values of $s(z)$ for complex argument $z = x + iy$

x	$y = 0$	$y = 1$		$y = 2$		$y = 3$		$y = 4$		$y = 5$	
	$\mathfrak{R}$[1]	$\mathfrak{R}$	$\mathfrak{I}$	$\mathfrak{R}$	$\mathfrak{I}$	$\mathfrak{R}$	$\mathfrak{I}$	$\mathfrak{R}$	$\mathfrak{I}$	$\mathfrak{R}$	$\mathfrak{I}$
0	1.25	1.76	−0.51	3.02	−1.77	5.80	−4.55	12.3	−11.0	27.8	−26.5
1	0.63	0.68	−0.96	0.73	−2.72	0.69	−6.62	−0.10	−15.5	−2.41	−37.1
2	−0.16	−0.41	−0.71	−1.40	−1.84	−4.12	−4.10	−11.0	−9.01	−28.8	−20.2
3	−0.53	−0.85	−0.06	−2.05	0.04	−5.15	0.70	−12.7	3.01	−31.5	9.94
4	−0.36	−0.51	0.46	−1.03	1.46	−2.21	4.12	−4.75	11.1	−10.1	29.4
5	0.08	0.18	0.50	0.60	1.47	2.20	3.73	5.86	9.55	16.9	23.8
6	0.37	0.59	0.12	1.47	0.28	3.93	0.48	10.5	0.58	27.6	−0.14
7	0.29	0.44	−0.30	0.99	−0.97	2.43	−2.78	5.99	−7.80	14.9	−21.5
8	−0.03	−0.07	−0.41	−0.27	−1.24	−0.92	−3.35	−2.95	−8.81	−9.05	−23.0
9	−0.30	−0.47	−0.15	−1.16	−0.43	−3.14	−1.02	−8.53	−2.32	−23.0	−5.10
10	−0.28	−0.41	0.21	−0.96	0.67	−2.46	1.95	−6.34	5.55	−16.2	15.6
11	−0.02	−0.01	0.35	0.05	1.09	0.28	1.09	1.09	7.99	3.81	21.3
12	0.23	0.37	0.18	0.93	0.52	2.54	1.34	6.99	3.33	19.2	8.24
13	0.25	0.39	−0.14	0.92	−0.46	2.52	−1.35	6.37	−3.87	16.6	−11.0
14	0.04	0.05	−0.31	0.10	−0.96	0.20	−2.45	0.23	−7.20	−0.03	−19.4
15	−0.19	−0.30	−0.19	−0.76	−0.58	−2.07	−1.54	−5.72	−3.99	−15.7	−10.3
16	−0.24	−0.37	0.09	−0.89	0.29	−2.35	0.86	−6.27	2.53	−16.7	7.34
17	−0.08	−0.11	0.27	−0.23	0.85	−0.54	2.36	−1.25	6.44	−2.89	17.5
18	0.15	0.23	0.21	0.60	0.62	1.65	1.68	4.59	4.43	12.8	11.7
19	0.22	0.35	−0.04	0.84	−0.15	2.25	−0.46	6.06	−1.41	16.3	−4.23
20	0.09	0.14	−0.24	0.32	−0.75	0.81	−2.08	2.06	−5.70	5.19	−15.6

[1] $\mathfrak{I} = 0$.

References

1. E. Kreyszig, *On the zeros of the Fresnel integrals*, Can. J. Math., *9* (1957), 118–131.
2. ———, *Ueber den allgemeinen Integralsinus Si*(z,α), Acta math., *85* (1951), 117–181.
3. ———, *Der allgemeine Integralkosinus Ci*(z,α), Acta math., *89* (1953), 107–131.
4. E. Lommel, *Beugungserscheinungen einer kreisrunden Oeffnung und eines kreisrunden Schirmchens*, Abh. Bayr. Akad. Wiss. Muenchen *15* (1886), 231–330.

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