Can. J. Math. Vol. **50** (4), 1998 pp. 719–738.

INDECOMPOSABLE ALMOST FREE MODULES—THE LOCAL CASE

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ABSTRACT. Let R be a countable, principal ideal domain which is not a field and A be a countable R-algebra which is free as an R-module. Then we will construct an $\aleph_1$-free R-module G of rank $\aleph_1$ with endomorphism algebra $\mathrm{End}_R\, G = A$. Clearly the result does not hold for fields. Recall that an R-module is $\aleph_1$-free if all its countable submodules are free, a condition closely related to Pontryagin's theorem. This result has many consequences, depending on the algebra A in use. For instance, if we choose $A = R$, then clearly G is an indecomposable 'almost free' module. The existence of such modules was unknown for rings with only finitely many primes like $R = \mathbb{Z}_{(p)}$, the integers localized at some prime p. The result complements a classical realization theorem of Corner's showing that any such algebra is an endomorphism algebra of some torsion-free, reduced R-module G of countable rank. Its proof is based on new combinatorial-algebraic techniques related with what we call *rigid tree-elements* coming from a module generated over a forest of trees.

1. **Introduction.** Let R be a fixed countable, principal ideal domain which is not a field. An R-module A is reduced for if $\bigcap_{s\in S} sA = 0$ where $S = R \setminus \{0\}$ and A is torsions-free if $sa = 0$ $(s \in S, a \in A)$ implies $a = 0$. Note that R is reduced because R is not a field. We will consider R-algebras A which are torsion-free and reduced as R-modules A_R. In particular this is the case when A_R is free.

Let $\kappa \leq \lambda$ be infinite cardinals. We are interested in R-modules of size λ which are κ-free, which is the case when all its submodules of cardinality $< \kappa$ are free R-modules.

Can we find indecomposable κ-free R-modules of cardinality λ?

We are mainly interested in the case when $\kappa = \lambda$ and in particular when this cardinal is $\aleph_1$.

Such modules—by freeness—most likely want to decompose into non-trivial direct sums and in fact, if λ is a singular cardinal, then by Shelah's [31] singular compactness theorem it follows that such R-modules are free (hence very decomposable), this holds in particular for cardinals of cofinality ω, *e.g.* for $\aleph_\omega$, a result due to Hill [26], see Eklof, Mekler [14].

On the other hand, the existence of non-free, $\aleph_1$-free R-modules of cardinality $\aleph_1$ follows from Griffith [23], Hill [26], Eklof [11], Mekler [29] and a result of Shelah's in

Received by the editors December 20, 1996; revised November 7, 1997 and January 22, 1998.

This work is supported by the project No. G-0294-081.06/93 of the German-Israeli Foundation for Scientific Research & Development.

No. 591 from Shelah's list of publications.

AMS subject classification: 20K20, 20K26, 20K30, 13C10.

Key words and phrases: indecomposable modules of local rings, $\aleph_1$-free modules of rank $\aleph_1$, realizing rings as endomorphism rings.

Eklof [12, p. 82, Theorem 8.8]. By an induction it can be shown, that there are non-free $\aleph_n$-free modules of cardinality $\aleph_n$. A similar result for non-commutative groups is due to Higman [24, 25]. The freeness-result at $\aleph_\omega$ illustrates that induction breaks down at $\aleph_{\omega+1}$ and new techniques are needed to show that for *certain cardinals* λ *only* the existence of non-free λ-free R-modules of cardinality λ follows, see Shelah, Magidor [28].

However only very little is known in ZFC about algebraic properties of the non-free λ-free R-modules of cardinality λ and this is also the case when $\lambda = \kappa = \aleph_1$, see Eklof [11] and Eklof, Mekler [14]. The following problem is immediate.

Investigate the **algebraic properties** *of λ-free modules of cardinality λ.*

The only earlier result known to us uses a construction from Shelah [33] of non-separable groups [12, 14] and is due to Eda [10]. He shows the existence of an $\aleph_1$-free group G of cardinality $\aleph_1$ with $\mathrm{Hom}(G, \mathbb{Z}) = 0$. In this paper we want to present new techniques which allow us to shed some more light on this problem. In order to work exclusively in ZFC we restrict ourself to $\kappa = \aleph_1$ and $\lambda \leq 2^{\aleph_0}$. Recall that under negation of CH the cardinal λ can be quite arbitrary, see [27]. We will state the next corollary which will follow immediately from our Main Theorem in Section 3.

MAIN COROLLARY 3.1. *Let $A \neq 0$ be a R-free R-algebra over a countable, principal ideal domain R which is not a field and let $|A| < \lambda \leq 2^{\aleph_0}$, then there exists an $\aleph_1$-free R-module G of cardinality λ with* $\mathrm{End}\, G = A$.

We will construct G as an A-module and A is identified with endomorphisms acting by scalar multiplication. If $A = R$, we derive the *existence of $\aleph_1$-free R-modules of cardinality $\aleph_1$ with* $\mathrm{End}\, G = R$, *a result about indecomposable R-modules* known only in the case $R = \mathbb{Z}$ from our recent paper [21]. The main difficulty in passing from $\mathbb{Z}$ to R can be seen in the local case when R is a local ring with just one prime p, *e.g.* if $R = \mathbb{Z}_{(p)}$ is the ring of integers localized at $p \neq 0$. Infinitely many primes—by arithmetic—provide a rigid system (= modules with no homomorphisms $\neq 0$ between them). Hence homomorphisms can be restricted in their activity on G by building into G a rigid system in a suitable way [21]. Finally they 'calm down' to scalar multiplication on G. This is no longer possible in the local case. The only chance we have is to utilize the existence of sufficiently many algebraically independent elements in the p-adic completion of $\mathbb{Z}_{(p)}$ and this is hidden in our construction.

It may be interesting to see this result in the light of its predecessors. The first example of an $\aleph_1$-free module which is not free is the Baer-Specker module R^ω, which is the cartesian product of countably many copies of the ring R, known for sixty years; *cf.* Baer [1] or [16, p. 94]. Assuming CH, this module is an example of an R-module of cardinality $\aleph_1 = 2^{\aleph_0}$. However, it is surely (by slenderness of R) a finite but not an infinite direct sum of summands $\neq 0$. Under the same set-theoretic assumption of the continuum hypothesis it can be shown that A above can be realized as the endomorphism ring of an $\aleph_1$-free R-module G of cardinality $\aleph_1$. The chronologically earlier realization theorem of this kind uses the weak diamond prediction principle which follows from $2^{\aleph_0} < 2^{\aleph_1}$. See Devlin and Shelah [5] for the weak diamond, Shelah [35] for the case $\mathrm{End}\, G = \mathbb{Z}$

and Dugas, Göbel [7] for the case $A = \operatorname{End} G$ and extensions to larger cardinals. Using, what is called *Shelah's Black Box*, the existence of $\aleph_1$-free modules G with $|G| = \lambda^{\aleph_0}$ also follows from Corner, Göbel [4] using Dugas, Göbel [8] and combinatorial fine tuning from Shelah [36, 37], see also Shelah [41, Chapter VII] and [40]. Many of the older results however do not concentrate on the additional demand that the constructed modules with prescribed endomorphism algebra are $\aleph_1$-free, see [2, 18, 19, 20]. This of course was due to other difficulties that had to be settled first.

Assuming Martin's axiom (MA) together with ZFC and $\aleph_2 < 2^{\aleph_0}$ any $\aleph_2$-free group G of cardinality $< 2^{\aleph_0}$ is separable and hence has endomorphism ring $\mathbb{Z}$ only in the trivial case when $G = \mathbb{Z}$, see [21].

Hence $\aleph_1$ in the Main Corollary can not be replaced by $\aleph_2$. This is in contrast to the result [7] which holds in Gödel's universe: All algebras A as above are of the form $A \cong \operatorname{End} G$ for all uncountable regular, not weakly compact cardinals $\lambda = |G| > |A|$ such that G is a λ-free R-module. A similar result was shown recently [22] using the generalized continuum hypothesis G.C.H. only. In view of the theorem under Martin's axiom,

the existence of indecomposable $\aleph_2$-free R-modules of cardinality $\aleph_2$ and the existence of such modules with endomorphism ring R, respectively, is undecidable.

Endomorphism ring results as discussed have well-known applications using the appropriate also well-known R-algebras A.

If Γ is any abelian semigroup, then we use Corner's R-algebra A_Γ, implicitly discussed in Corner, Göbel [4], and constructed for particular $\Gamma's$ in [3] with special idempotents (expressed below), with free R-module structure and $|A_\Gamma| = \max\{|\Gamma|, \aleph_0\}$. If $|\Gamma| < 2^{\aleph_0}$, we may apply the main theorem and find a family of $\aleph_1$-free R-modules $G_\alpha (\alpha \in \Gamma)$ of cardinality $\aleph_1$ such that for $\alpha, \beta \in \Gamma$,

$$G_\alpha \oplus G_\beta \cong G_{\alpha+\beta} \quad \text{and} \quad (G_\alpha \cong G_\beta \text{ if and only if } \alpha = \beta).$$

Observe that this induces all kinds of counterexamples to Kaplansky's test problems for suitable $\Gamma's$. If we consider Corner's algebra in [3], see Fuchs [17, p. 145], then it is easy to see that A_R is free and $|A| = \aleph_0$. The particular idempotents in A and our main theorem provide the existence of an $\aleph_1$-free superdecomposable R-module of cardinality $\aleph_1$, which seems to be new as well. Recall that a group is superdecomposable if any non-trivial summand decomposes into a proper direct sum.

Finally, we remark that, as the reader may suspect, it is easy to replace G in Theorem 3.1 by a rigid family of 2^λ such groups with only the trivial homomorphism between distinct members.

2. The construction of $\aleph_1$-free modules.

a. THE TOPOLOGY. Let R be a countable, principal ideal domain which is not a field, hence R is reduced. We consider any free R-algebra A of cardinality $|A| < 2^{\aleph_0}$. In particular A is torsion-free and reduced as well. Enumerating $S = R \setminus \{0\} = \{s_n : n \in \omega\}$ we obtain a descending chain of principal ideals q_nA for

(1) $$q_0 = 1 \text{ and } q_{n+1} = s_n q_n^2 \quad \text{for all } n \in \omega$$

with $\bigcap_{n\in\omega} q_nA = 0$. The system q_nA $(n \in \omega)$ generates a Hausdorff topology, the R-topology on A.

b. THE GEOMETRY OF A TREE AND A FOREST. Let $T = {}^{\omega>}2$ denote the tree of all finite branches $v\colon n \longrightarrow 2$, $n < \omega$, where $\ell(v) = n$ denotes the length of the branch v. The branch of length 0 is denoted by $\perp = \emptyset \in T$ and we also write $v = (v \restriction n-1)^\wedge v(n-1)$. Moreover, ${}^{\omega}2 = \mathrm{Br}(T)$ denotes all infinite branches $v\colon \omega \longrightarrow 2$ and clearly $v \restriction n \in T$ for all $v \in \mathrm{Br}(T)$, $n \in \omega$. We often identify infinite branches v with their nodes $v = \{v \restriction n : n \in \omega\}$ which is a countable, maximal, linearly ordered subset of T. Following convention we will call a node $v \restriction n$ *finite branch of length n* of the tree $\mathrm{Br}(T)$. If $v \neq w \in \mathrm{Br}(T)$, then

$$\mathrm{br}(v, w) = \inf\{n \in \omega : v(n) \neq w(n)\}$$

denotes the *branch point* of v and w. Hence $m = \mathrm{br}(v, w)$ is the largest ordinal with $v \restriction m = w \restriction m$.

If $C \subset \omega$, then we collect the subtree

$$T_C = \{v \in T : \text{ if } e \in \ell(v) \setminus C \text{ then } v(e) = 0\}.$$

Similarly

$$\mathrm{Br}(T_C) = \{v \in \mathrm{Br}(T) : \text{ if } e \in \omega \setminus C \text{ then } v(e) = 0\},$$

hence $v \restriction n \in T_C$ for all $v \in \mathrm{Br}(T_C)$, $n \in \omega$ and as before we omit C if $C = \omega$. Many of our arguments use a finite trunk of these trees. If $m < \omega$, then we define

(2) $$T_C^m = \{\tau \in T_C : \ell(\tau) < m\}.$$

Finally let $\overline{T_C} = T_C \cup \mathrm{Br}(T_C)$.

Next we use trees to build a forest.

Let $\kappa \leq \lambda \leq 2^{\aleph_0}$ be two fixed infinite cardinals and let κ be regular and uncountable. Then we choose a family $\mathfrak{C} = \{C_\alpha \subset \omega : \alpha < \lambda\}$ of pairwise almost disjoint, infinite subsets of ω. Let $T \times \alpha = \{v \times \alpha : v \in T\}$ be a disjoint copy of the tree T and let $T_\alpha = T_{C_\alpha} \times \alpha$ for $\alpha < \lambda$ be the forest of trees (with finite branches), say

$$T_{\mathfrak{C}} = \dot{\bigcup_{\alpha<\lambda}} T_\alpha$$

and choose disjoint sets of infinite branches from T. We have $T_\alpha^* = \mathrm{Br}(T_{C_\alpha}) \subseteq \mathrm{Br}(T)$ $(\alpha < \lambda)$ and take a family of pairwise disjoints subsets, *i.e.*

$$\mathfrak{V} = \{V_\alpha \subseteq T_\alpha^* : \alpha < \lambda\} \text{ with } |V_\alpha| = \kappa.$$

Moreover, for any $m \in \omega$ at least κ pairs of branches in V_α branch at m or above. It will be very convenient, however not necessary to restrict to perfect trees. A tree is *perfect* if it has no isolated points (in the order topology), *i.e.* every branch has an unbounded set of branch points. It is easy to see that we may assume for the forest that all trees $T_\alpha \subseteq T_{\mathfrak{C}}$ are perfect trees. This additional assumption about the trees is only used in the proof of Proposition 3.7 in form of the following:

OBSERVATION 2.1. Any perfect subtree of T has a subtree order isomorphic to T such that for any ordinal $n \in \omega$ there is at most one finite branch $v \restriction n$ of the subtree such that $v \restriction (n+1) \neq w \restriction (n+1)$ for some branch w of this subtree.

PROOF. Let T be a perfect tree. We will define a tree embedding $p\colon T \longrightarrow T$ such that $p(T)$ has the desired branching property. The map p is defined as the union of a chain of partial maps $p_i\colon T^i \longrightarrow T^{n_i}$. Let

$$B(T) = \{v \restriction \mathrm{br}(v, w) \in T : v \neq w \in T\}$$

be the set of all branch points in T. Using that $\mathrm{Br}(T)$ has no isolated infinite branches and König's Lemma, inductively we can choose a sequence of natural numbers n_i ($i \in \omega$) such that $n_0 = 0$ and if n_i is given, then n_{i+1} is the least number $x > n_i$ such that for any $v \in T^x$ the set $B(v, x) = \{e : n_i < e < x, v \restriction e \in B(T)\}$ has cardinality $|B(v, x)| > 2^{n_i}$. Now we have enough room to extend a partial embedding $p_i\colon T^i \longrightarrow T^{n_i}$ to p_{i+1} in such a way that the branching condition of the Observation 2.1 holds when restricted to $p_{i+1}(T^{i+1})$. Hence $p = \bigcup_{i\in\omega} p_i$ is the desired tree embedding of T into that perfect tree.

The forest $T_{\mathfrak{C}}$ of pairwise disjoint perfect trees T_α and the sequence of sets of infinite branches $\mathfrak{V}$ from T which branch at 'almost disjoint sets' will form our basic geometrical objects for building modules. The geometry will help to distinguish elements and to carry out calculation in the corresponding module. In view of Observation 2.1 we will assume

(3) If $T_\alpha \subseteq T_{\mathfrak{C}}$ and $n \in \omega$ there is at most one finite branch $v \restriction n$ such that $v \restriction (n+1) \neq w \restriction (n+1)$ for some $w \in T_\alpha$.

c. THE BASE MODULE AND ITS COMPLETION. We consider the free A-module

$$B_{\mathfrak{C}} = \bigoplus_{\tau \in T_{\mathfrak{C}}} \tau A$$

which is a pure and dense submodule of its R-adic completion $\widehat{B_{\mathfrak{C}}}$ taken in the R-topology on $B_{\mathfrak{C}}$. The A-module $B_{\mathfrak{C}}$ will be our *base module* and we will often omit $\mathfrak{C}$ for convenience. The sequence $\mathfrak{V}$ of infinite branches is used to identify certain elements in the completion $\widehat{B_{\mathfrak{C}}}$. Any infinite branch $v \in V_\alpha$, $n < \omega$ and any $g \in \widehat{B_{\mathfrak{C}}}$ give rise to an element y_{vng} in the completion $\widehat{B_{\mathfrak{C}}}$. Note that

$$y_{vng} = \sum_{i \geq n} \frac{q_i}{q_n}(v \restriction i \times \alpha) + g \sum_{i \geq n} \frac{q_i}{q_n} v(i) \tag{4}$$

https://doi.org/10.4153/CJM-1998-039-7 Published online by Cambridge University Press

is a well-defined element of the $\hat{A}$-module $\widehat{B_{\mathfrak{C}}}$. The reader should keep in mind that the *branch element* y_{vng} connects an infinite branch $v \in \mathrm{Br}(T_{C_\alpha}) \subseteq \mathrm{Br}(T)$ with finite branches from the disjoint tree T_α. We will write $y_{v0g} = y_{vg}$ and often omit the suffix g if this is clear from the context. Moreover

$$y_{vng} - s_n q_n y_{vn+1g} = v \restriction n \times \alpha + gv(n) \tag{5}$$

follows from (1) and (4). We immediate obtain an equation concerning branching branches. For $v \neq w \in V_\alpha$ and the branch point $m = \mathrm{br}(v, w)$ we have $v(i) = w(i) \in \{0, 1\}$ for $i < m$ and $v(m) - w(m) = \pm 1$. Hence

$$y_v = \sum_{i=0}^{m} q_i(v \restriction i \times \alpha) + g \sum_{i=0}^{m-1} q_i v(i) + q_{m+1} y_{vm+1} + g q_m v(m) \quad \text{and} \tag{6}$$

$$\sum_{i=0}^{m} q_i(v \restriction i \times \alpha) + g \sum_{i=0}^{m-1} q_i v(i) = \sum_{i=0}^{m} q_i(w \restriction i \times \alpha) + g \sum_{i=0}^{m-1} q_i w(i) \quad \text{and}$$

$$y_v - y_w \pm q_m g = q_{m+1}(y_{vm+1} - y_{wm+1}) \quad \text{for } \mathrm{br}(v, w) = m. \tag{7}$$

The special form of branch elements allows us to recognize the geometry of the trees through the

d. SUPPORT AND NORM OF ELEMENTS AND SUBSETS OF $\widehat{B_{\mathfrak{C}}}$. Elements g in $\widehat{B_{\mathfrak{C}}}$ have a natural *support* $[g] \subseteq T_{\mathfrak{C}}$, the at most countable set of finite branches used in the sum-representation with respect to the R-adic completion of $B_{\mathfrak{C}}$ defined by

$$g = \sum_{\tau \in [g]} g_\tau,$$

where $g_\tau \neq 0$ is a unique element in $\tau \hat{A}$. If $\tau \in T_{\mathfrak{C}}$ and $1 \in A$ we identify τ with the element $\tau 1$ in B, hence $T_{\mathfrak{C}} \subseteq B_{\mathfrak{C}}$ and support is defined on $T_{\mathfrak{C}}$ as well.

Let $v \in V_\alpha$ and note that in particular $[y_{vn0}] = [v_n \times \alpha]$, where

$$[v_n \times \alpha] = \{v \restriction j \times \alpha : j \in \omega, j \geq n\} \subseteq T_\alpha.$$

This infinite part of the branch v we also denote by $[v_n \times \alpha] = [v_n]$ because it is clear that like v it comes from α. The notion of support trivially extends to subsets X of $\widehat{B_{\mathfrak{C}}}$ by taking unions $[X] = \bigcup_{x \in X} [x]$, see also [4].

Each element $g = \sum_{\tau \in [g]} g_\tau$ of $\hat{B}$ also has a special—possibly empty—subset in $[g]$, the

$$R\text{-support} \quad [g]_R = \{\tau \in [g] : 0 \neq g_\tau \in \tau R\}.$$

Branch elements $g = y_{vn0}$ as in (4) have R-support $[g]_R = [g] = [v_n]$. The support of an element immediately gives rise to its *norm*.

If $X \subseteq T_{\mathfrak{C}}$ then $\|X\| = \inf\{\beta < \lambda : X \subseteq \bigcup_{\alpha < \beta} T_\alpha\}$ denotes the *norm* of X. If $\|X\|$ does not exist we write $\|X\| = \infty$. Moreover $\|g\| = \|[g]\|$ denotes the norm of an element $g \in B_{\mathfrak{C}}$, for example $\|y_{vn0}\| = \alpha + 1$ whenever $v \in V_\alpha$. The following lemma is used several times.

LEMMA 2.2. *Let G be an A-submodule of $\widehat{B_{\mathfrak{C}}}$ such that $\|G\| \leq \alpha$ and $g \in G$. If*

$$F_\alpha = \langle T_\alpha, y_{vng} : v \in V_\alpha, n < \omega \rangle_A$$

is a submodule of $\widehat{B_{\mathfrak{C}}}$, then $G \cap F_\alpha = \mathrm{Ag}$.

PROOF. Recall that $v(n) \in \{0, 1\}$. Take any $n < \omega$ with $v(n) = 1$. From (5) we have

$$g = y_{vng} - s_n q_n y_{vn+1g} - (v \restriction n) \times \alpha$$

and $\mathrm{Ag} \subseteq F_\alpha \cap G$. If $h \in F_\alpha$, then h can be expressed as a linear combination of elements from a finite subset of T_α and a finite set of elements of the form y_{vng}. Using (5) we can choose $m \in \omega$ subject to the following conditions.

$$h \in \langle \{g\} \cup T_\alpha^m \cup \{y_{vmg} : v \in E\} \rangle_A \subseteq F_\alpha$$

where E is a finite subset of V_α and T_α^m as defined in (2) such that

$$[g] \cap [v_m] = \emptyset \quad \text{for all } v \in E \tag{8}$$

$$[v_m] \cap [w_m] = \emptyset \quad \text{for all } v \neq w \in E \tag{9}$$

We can write $h = \sum_{v \in E} a_v y_{vmg} + a_g g + t$ where $a_v, a_g \in A$ and $t \in \langle T_\alpha^m \rangle_A$. If also $h \in G$ we take any $\tau \in [v_m]$ to see that $a_v = 0$ and similarly $t = 0$, hence $h = a_g g \in \mathrm{Ag}$ which shows the lemma.

e. THE DESIRED $\aleph_1$-FREE MODULE. We use these basic tools to construct the desired R-module.

THE R-MODULE-CONSTRUCTION 2.3. Let A be the free R-algebra over the countable principal ideal domain R which is not a field with $|A| < \lambda \leq 2^{\aleph_0}$ and $\kappa = |A|^+ + \aleph_1$. If $\mathfrak{C}, \mathfrak{B}$ and $B_{\mathfrak{C}}, \widehat{B_{\mathfrak{C}}}$ are as given, then choose a transfinite sequence b_α ($\alpha < \lambda$) which runs λ times through the non-zero elements b in $B_{\mathfrak{C}}$ with $B_{\mathfrak{C}}/Ab$ A-free. We define inductively A-submodules $G_\alpha \subseteq \widehat{B_{\mathfrak{C}}}$ subject to the following condition for any $\alpha < \lambda$. The sequence G_α is increasing, continuous with

$$G_0 = 0 \text{ and } G = \bigcup_{\alpha < \lambda} G_\alpha.$$

$$G_{\alpha+1} = \langle G_\alpha \cup T_\alpha \cup \{y_{vng_\alpha} : v \in V_\alpha, n \in \omega\} \rangle_A. \tag{10}$$

We also let $g_\alpha = b_\alpha$ if $b_\alpha \in G_\alpha$ with $\|g_\alpha\| \leq \alpha$ and $g_\alpha = 0$ otherwise.

Note that $\kappa \leq \lambda \leq 2^{\aleph_0}$ is a regular cardinal. The constructed A-module G has visibly cardinality λ, and we want to show that it is $\aleph_1$-free as A-module. We reserve G to denote this module for the rest of this paper.

In view of Pontryagin's theorem we say that an R-module is $\aleph_1$-free if any submodule of finite rank is contained in a free R-submodule. If R is a principal ideal domain, Pontryagin's theorem [16, p. 93, Theorem 19.1] ensures that any countably generated submodule is free. This gives us the following:

OBSERVATION 2.4. Let A be a free R-algebra over a principal ideal domain R and M an A-module such that any finite subset is contained in an A-free and R-pure submodule, then M is an $\aleph_1$-free R-module, *i.e.* all its countably generated R-submodules are free.

Next we will show that G is $\aleph_1-$free. This will be the case when $\beta = 0$ in the next proposition.

PROPOSITION 2.5. *Let A be a free R-algebra and $G = \bigcup_{\alpha\in\lambda} G_\alpha$ be the constructed R-module. Then $|G| = \lambda$ and G/G_β is an $\aleph_1-$free R-module for any $\beta < \lambda$.*

PROOF. In view of Observation 2.4 we consider any non-empty finite set $E \subseteq G/G_\beta$. Choose $\alpha < \lambda$ minimal with $E \subseteq G_\alpha/G_\beta$. First note that $\alpha > \beta$ must be a successor because E is a proper finite set, hence $\gamma = \alpha - 1 \geq \beta$ exists. Also note that G_α/G_β is a quotient of A-modules, hence an A-module. By induction it is enough to show that

(11)

$$E \subseteq (U + G_\beta)/G_\beta \oplus G_\gamma/G_\beta \subseteq_* G_{\gamma+1}/G_\beta \quad \text{for some free } A\text{-module } (U + G_\beta)/G_\beta.$$

First we want to find inductively an A-submodule $U \subseteq G_\alpha$. We note by (10) that $G_{\gamma+1} = G_\gamma + F_\gamma$ where

$$F_\gamma = \langle T_\gamma \cup \{y_{vng_\gamma} : v \in V_\gamma, n < \omega\}\rangle_A \tag{12}$$

If $E' \subseteq G_{\gamma+1}$ is a set of representatives of the elements in E, then by (12) and (5) there is a finite set $F \subseteq V_\gamma$, and a number $m < \omega$ such that

$$E' \subseteq U + G_\gamma \quad \text{where } U = \langle T_\gamma^m \cup \{y_{vmg_\gamma} : v \in F\}\rangle_A$$

Moreover we may assume that $[v_m] \cap [w_m] = \emptyset$ for all $v \neq w \in F$. A support argument shows that the defining generators of U are A-independent *modulo* G_γ, hence $U + G_\beta/G_\beta$ must be A-free and $G_\gamma/G_\beta \cap (U + G_\beta)/G_\beta = 0$.

Now it is easy to show that U is R-pure in G which also implies the purity in (11). If $h \in G \setminus G_{\gamma+1}$, then an easy support argument shows that $G_{\gamma+1}$ is pure in G that is to say that $dh \notin G_{\gamma+1}$ for any $0 \neq d \in R$ and in particular $dh \notin U$. We may suppose that $h \in G_{\gamma+1}$, and by the last considerations we find a finitely generated A-submodule

$$U' = \langle T_\gamma^{m'} \cup \{y_{vm'g_\gamma} : v \in F'\}\rangle_A$$

for some number $m' \geq m$ and finite set $F \subseteq F' \subseteq V_\gamma$ with $h \in U'$. We may assume that m' is chosen such that also

$$[v_{m'}] \cap [w_{m'}] = \emptyset \quad \text{for all } v \neq w \in F'.$$

One more support argument now shows that U is a summand of U', we leave it to the reader to write down a complement of U in U'. If $dh \in U$ for some $0 \neq d \in R$, then $h \in U$ follows from $h \in U'$, which shows that U is pure in G.

3**. The constructed modules and their endomorphism algebras.** The following Definition 3.2 rigid tree-elements is the critical tool of this paper. The short proof of our Main Theorem 3.1, following immediately below, is based on a Main-Lemma 3.3 which indicates our strategy. Moreover, the Definition 3.4 explains how to convert rigid tree-elements into algebraic content. We think that it may help the reader if we start at the end:

The main result of this paper is the following:

THEOREM 3.1. *If A is a free R-algebra over a countable, principal ideal domain R which is not a field and $|A| < \lambda \leq 2^{\aleph_0}$, then there exists an $\aleph_1$-free R-module G of cardinality λ with* $\operatorname{End} G = A$*.*

REMARK. G will be the A-module constructed in (2.3) and we have identified $a \in A$ with $a \cdot \mathrm{id}_G$.

PROOF. Let G be the A-module from the Construction 2.3. Clearly $A \subseteq \operatorname{End} G$ by scalar-multiplication because A acts faithfully on G, and G is an $\aleph_1-$free R-module of cardinality λ by Proposition 2.5. It remains to show that $\operatorname{End} G \subseteq A$.

Suppose $\varphi \in \operatorname{End} G \setminus A$. Recall from (2.3) that $T_\alpha \subseteq G$ for all $\alpha < \lambda$, hence $B_{\mathfrak{C}} \subseteq G$. Inspection of (10) shows that $G/B_{\mathfrak{C}}$ is torsion-free divisible. This is needed to prove that there exists

$$g \in B_{\mathfrak{C}} \text{ with } B_{\mathfrak{C}}/gA\ A\text{-free and } g\varphi \notin \mathrm{Ag}\,. \tag{13}$$

Note that $B_{\mathfrak{C}}$ is a free A-module freely generated by some set $J \subseteq B_{\mathfrak{C}}$. If (13) does not hold, then $e\varphi \in Ae$ for all $e \in J$, say $e\varphi = a_e e$. If also $f \in J \setminus \{e\}$ then $f\varphi = a_f f$ and similarly $e + f$ is another basic element and the negation of (13) would also give $(e + f)\varphi = a_{e+f}(e + f)$ for some $a_{e+f} \in A$, hence $a_e = a_{e+f} = a_f$ by independence. The element $a = a_e$ does not depend on $e \in J$, and $e\varphi = ae$ for all $e \in J$, hence $\varphi \restriction B_{\mathfrak{C}} = a \cdot \mathrm{id}_{B_{\mathfrak{C}}}$. The endomorphism extends uniquely to the A-module G by density, and $\varphi = a \cdot \mathrm{id}_G \in A$, which was excluded. Condition (13) is shown.

By Construction 2.3 we can find $\alpha' < \lambda$ such that the element g from (13) belongs to $G_{\alpha'}$, moreover we find $\alpha' < \alpha < \lambda$ with $g = g_\alpha$, hence $G_{\alpha'} \subseteq G_\alpha$ and $g = g_\alpha \in G_\alpha$. In particular $g\varphi \neq 0$ by (13) and, since G is reduced, we find $m_0 \in \omega$ such that

$$g = g_\alpha, \text{ and } g\varphi \notin q_m G \quad \text{for all } m \geq m_0.$$

We now apply (10)

$$G_{\alpha+1} = \langle G_\alpha \cup T_\alpha \cup \{y_{vn} : v \in V_\alpha, n \in \omega\}\rangle_A \text{ where } y_{vn} = y_{vng} \tag{14}$$

and (7) implies for $v \neq w \in V_\alpha$ that

$$y_v - y_w \pm q_m g = q_{m+1}(y_{vm+1} - y_{wm+1}) \quad \text{if } \operatorname{br}(v, w) = m.$$

We may assume $m \geq m_0$ by assumption on $\mathfrak{V}$. Let $t_v = y_v\varphi$ ($v \in V_\alpha \in \mathfrak{V}$) and apply φ to the last equation. We derive the existence of a *family of rigid tree-elements* as defined on the next page:

The set of elements $t_v \in G$ ($v \in T$), T a subset of V_α satisfying the hypothesis of (3.3), constitute a family of rigid tree-elements for $g\varphi$, where

$$t_v - t_w \pm q_m g\varphi \in q_{m+1}G.$$

We now apply our Main-Lemma 3.3 and obtain

$$g\varphi \in \mathrm{Ag}_\alpha = \mathrm{Ag}$$

which contradicts (13). Hence φ does not exist and the Main Theorem 3.1 follows.

We proceed with the definition of *rigid tree-elements*.

DEFINITION 3.2. Let G be the A-module and $\kappa \leq \lambda$ be the regular cardinal from (2.3) and $T \subseteq V_\alpha$ for some $\alpha < \lambda$ be a set of cardinality κ. If $\beta \leq \lambda$, a family $\{t_v \in G_\beta : v \in T\}$ is called a *family of rigid tree-elements* for $z \in G$ at a tree T_α, if

(t) $\qquad t_v - t_w \pm q_m z \in q_{m+1}G \quad$ for all $v \neq w \in T$ with $\mathrm{br}(v, w) = m$.

MAIN-LEMMA 3.3. *Let G be the A-module constructed in (2.3) and $\alpha < \lambda$, $\beta \leq \lambda$. If T is a set of branches from V_α branching above some $m \in \omega$ with $|T| = \kappa$ and $\{t_v \in G_\beta : v \in T\}$ is a family of rigid tree-elements for $z \in G \setminus q_m G$, then $z \in \mathrm{Ag}_\alpha$.*

The proof of (3.3) follows after a number of steps where we replace T above by equipotent subsets with 'stronger' families of rigid tree-elements. Our final goal is a family as in the following definition.

DEFINITION 3.4. Let m_0 be a natural number. We will say that the family $\{t_v \in G_\beta : v \in T\}$ as in (3.2) is an *independent* family of rigid tree-elements for z at some tree T_α over m_0 if there are a sequence of ordinals $\alpha_1 < \cdots < \alpha_s < \lambda$, $m, j^* > m_0$, a finite set F with elements $a_x \in A$ for $x \in F$ and an injective map

$$\delta\colon T \times F \longrightarrow \dot{\bigcup_{i \leq s}} V_{\alpha_i}, \tag{i}$$

such that any t_v ($v \in T$) can be expressed as

$$t_v = \sum_{x \in F} a_x y_{\delta(x,v)m}. \tag{ii}$$

Moreover,

$$\Big(\bigcup_{i \leq s} [g_{\alpha_i}]\Big) \cap [y_{\delta(x,v)m}] = \emptyset, \tag{iii}$$

the branches $\delta(x, v) \restriction j^*$ ($x \in F$) are all distinct and independent of v,

$$F = \dot{\bigcup_{i \leq s}} F_i \text{ and } \delta(T \times F_i) \subseteq V_{\alpha_i} \quad (i \leq s). \tag{iv}$$

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For convenience we will some times omit α above, writing V_i for V_{α_i}. Often we only deal with partial maps of δ say

$$\delta_v = \delta \restriction \{v\} \times F \text{ and write } \delta_v \colon F \longrightarrow F_v = \mathrm{Im}(\delta_v).$$

In order to find an independent family of rigid tree-elements we first concentrate on finding a weaker family which satisfies (w) in (3.5) and comes from a given family of rigid tree-elements at some fixed tree.

PIGEON-HOLE-LEMMA 3.5. *Let G be the A-module, κ be the cardinal given by (2.3), and let $\alpha < \lambda$ and $\beta \leq \lambda$. Assume that we also have a family of elements*

$$t_v \in G_\beta \quad (v \in T) \text{ for some subset } T \subseteq V_\alpha \text{ of cardinality } \kappa$$

and elements $h_1, \ldots, h_k \in G_\beta$ with the property that

$$\text{if } v, w \in T \text{ and } \mathrm{br}(v, w) = m, \text{ then } t_v - t_w \in \langle h_1, \ldots, h_k \rangle_A + q_m G_\beta. \qquad (\beta)$$

Then we find an equipotent subset T' of T and ordinals $\beta_1 < \cdots < \beta_s < \beta$ with

$$t_w \in \langle T_{\beta_i}, y_{vng_{\beta_i}} : v \in V_{\beta_i}, n < \omega \text{ and } i \leq s \rangle_A \quad \text{for all } w \in T'. \qquad \text{(w)}$$

Note that (β) is a weak form of the definition of a family of rigid tree-elements. If for the above family $\beta = \lambda$, then by (w) we also find $\beta < \lambda$ such that $t_v \in G_\beta$ $(v \in T)$ for an equipotent subfamily.

PROOF. We must collect a small 'pigeon-hole'—the right-hand side of (w)—and enough 'pigeons' t_w to land in (w). There are plenty of pigeons and we just discard all trouble makers.

The proof is by induction on $\beta \leq \lambda$.

If $\beta = 0$, then $G_0 = 0$ and it is nothing to show.

Next we assume that $\beta \leq \lambda$ is a limit ordinal. We will distinguish three cases (a),(b) and (c) depending on the cofinality $\mathrm{cf}(\beta)$ of β.

(a) Suppose $\mathrm{cf}(\beta) > \kappa$.

Hence $|T| < \mathrm{cf}(\beta)$ and note that $\mathrm{cf}(\beta)$ is a regular cardinal. Then we can find $\gamma < \beta$ with $h_1, \ldots, h_k \in G_\gamma$ and $t_v \in G_\gamma$ for all $v \in T$. Condition (γ) in (3.5) holds for $\gamma < \beta$ and the induction hypothesis applies to finish this case.

(b) Suppose $\mathrm{cf}(\beta) < \kappa$.

Note that $G_\beta = \bigcup_{i<\mathrm{cf}(\beta)} G_{\sigma(i)}$ for some strictly increasing, continuous sequence $\sigma(i)$ converging to β. In this case we immediately find some $i < \mathrm{cf}(\beta)$ and $\gamma = \sigma(i) < \beta$ with $h_1, \ldots, h_k \in G_\gamma$ and $T' = \{v \in T : t_v \in G_\gamma\}$ has cardinality κ. The claim follows by induction like (a).

(c) Suppose $\mathrm{cf}(\beta) = \kappa$.

Let $\langle \beta_\xi \in \beta : \xi < \kappa \rangle$ be a strictly increasing, continuous chain of ordinals converging to β, and enumerate $T = \{v_\xi : \xi < \kappa\}$ without repetition. We also may assume that $h_1, \ldots, h_k \in G_{\beta_0}$. Consider the set

$$Y = \{\xi \in \kappa : t_{v_\xi} \in G_{\beta_\xi}\} \subseteq \kappa.$$

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Suppose first that

(i) Y is a stationary subset of κ.

If $\xi \in Y$, then we find a smallest G_{β_δ} ($\delta \in \kappa$) which contains t_{v_ξ}. Since $\langle \beta_\xi : \xi \in \kappa \rangle$ and $\langle G_\nu : \nu \in \kappa \rangle$ are continuous sequences, δ must be a successor ordinal, say $\delta = \gamma + 1$. We get a function $g\colon Y \longrightarrow \kappa$ $\big(\xi \longrightarrow g(\xi) = \gamma\big)$ and note that $g(\xi) < \xi$ from $t_{v_\xi} \in G_{\beta_\xi}$. Hence g is regressive and Fodor's lemma applies, see [27, Theorem 22, p. 59]. There is a stationary subset X of Y—which must be stationary in κ by hypothesis (i)—on which g is constant, taking some fixed value $\delta \in \kappa$. Clearly $|X| = \kappa$ and $T' = \{t_{v_\xi} : \xi \in X\}$ and G_{β_δ} ($\beta_\delta < \beta$) satisfy the induction hypothesis (β_δ) in (3.5). Again, the claim follows in case (i) by induction.

Finally we assume that

(ii) Y is not a stationary subset of κ.

In this case we have to work showing that (ii) can not occur. There is a cub C in κ with $Y \cap C = \emptyset$. Inductively we may replace C by an equipotent subset, called C again, and replace the β_ξ's by new ones such that

$$t_{v_\xi} \in G_{\beta_{\xi+1}} \setminus G_{\beta_\xi} \quad \text{for all } \xi \in C.$$

Note that $T' = \{t_{v_\xi} : \xi \in C\}$ still has cardinality κ, and apply Proposition 2.5 to note that $G_{\beta_{\xi+1}}/G_{\beta_\xi}$ is $\aleph_1$-free for all $\xi \in C$. Moreover,

$$0 \neq t_{v_\xi} + G_{\beta_\xi} \in G_{\beta_{\xi+1}}/G_{\beta_\xi}$$

and there are elements $0 \neq d_\xi \in R$ such that

$$t_{v_\xi} + G_{\beta_\xi} \notin d_\xi(G_{\beta_{\xi+1}}/G_{\beta_\xi}). \tag{d}$$

From $|R| < |C|$ we can find $0 \neq d \in R$ with $C' = \{\xi \in C : d_\xi = d\}$ of cardinality κ. Pick any $j \in \omega$ with $d|q_j$.

The set $X' = \{t_{v_\xi} : \xi \in C'\}$ of cardinality κ must have elements $t_{v_\epsilon}, t_{v_\eta}$ for $\epsilon < \eta \in C'$ with branch point $\mathrm{br}(v_\epsilon, v_\eta) = m > j$. In particular $d|q_m$. We derive from the hypothesis (β) that

$$t_{v_\epsilon} - t_{v_\eta} \in \langle h_1, \ldots, h_k \rangle_A + q_m G_{\beta_{\eta+1}}.$$

However $h_1, \ldots, h_k \in G_{\beta_0} \subseteq G_{\beta_\eta}$ and also $t_{v_\epsilon} \in G_{\beta_\eta}$ from $\epsilon < \eta$. Hence

$$t_{v_\eta} + G_{\beta_\eta} \in q_m(G_{\beta_{\eta+1}}/G_{\beta_\eta}) \subseteq d(G_{\beta_{\eta+1}}/G_{\beta_\eta}) = d_{v_\eta}(G_{\beta_{\eta+1}}/G_{\beta_\eta})$$

which contradicts (d) and case (ii) cannot come up. This finishes the case of limit ordinals β.

We may assume that $\beta = \gamma + 1$ is a successor ordinal, and the lemma holds for $\gamma < \beta$. We also have

$$G_\beta = G_{\gamma+1} = \langle T_\gamma, G_\gamma, y_{vng_\gamma} : v \in V_\gamma, n \in \omega \rangle_A.$$

As in (10) and (12) we can write $G_{\gamma+1} = G_\gamma + F_\gamma$ with

$$F_\gamma = \langle T_\gamma \cup \{y_{vng_\gamma} : v \in V_\gamma, n < \omega\} \rangle_A.$$

Obviously $\|G_\gamma\| \leq \gamma$ by construction (2.3) of G and Lemma 2.2 applies. We derive

$$\mathrm{Ag}_\gamma = G_\gamma \cap F_\gamma$$

and $G_\gamma / \mathrm{Ag}_\gamma \oplus F_\gamma / \mathrm{Ag}_\gamma$ is a direct sum.

If $h_1, \ldots, h_k \in G_\beta$ and $t_v \in G_\beta$ ($v \in T$) are given by hypothesis, then we can write $t_v = t_v^0 + t_v^1$ for all $v \in T$ and similarly $h_i = h_i^0 + h_i^1$ ($i \leq k$) with $t_v^0, h_i^0 \in G_\gamma$ and $t_v^1, h_i^1 \in F_\gamma$. Moreover, if $v, w \in T$ branch at $\mathrm{br}(v, w) = m$, then by hypothesis $t_v - t_w \in \langle h_1, \ldots, h_k \rangle_A + q_m G_\beta$ and

$$(t_v^0 - t_w^0) + (t_v^1 - t_w^1) = \sum_{i=1}^{k} a_i h_i^0 + \sum_{i=1}^{k} a_i h_i^1 + q_m g'_\gamma + q_m f_\gamma$$

follows for some $g'_\gamma \in G_\gamma$ and $f_\gamma \in F_\gamma$. Hence

$$\Big(t_v^0 - t_w^0 - \sum_{i=1}^{k} a_i h_i^0\Big) - q_m g'_\gamma = \Big(t_w^1 - t_v^1 + \sum_{i=1}^{k} a_i h_i^1\Big) + q_m f_\gamma.$$

The left-hand side of the displayed equation is in G_γ while the right-hand side is in F_γ. The sum must be 0 *modulo* Ag_γ by the direct sum above. In particular

$$t_v^0 - t_w^0 \in \langle h_1^0, \ldots, h_k^0, g_\gamma \rangle_A + q_m G_\gamma.$$

The induction applies for $\gamma < \beta$. We find $T' \subseteq T$ with $|T'| = \kappa$ and $\beta_1 < \cdots < \beta_k < \gamma$ such that

$$t_w^0 \in \langle T_{\beta_i}, y_{vng_{\beta_i}} : v \in V_{\beta_i}, i \leq k, n \in \omega \rangle_A$$

for all $w \in T'$. Finally let $\beta_{k+1} = \gamma$ and note that

$$t_w = t_w^0 + t_w^1 \in \langle T_{\beta_i}, y_{vng_{\beta_i}} : v \in V_{\beta_i}, i \leq k+1, n \in \omega \rangle_A$$

for all $w \in T'$. This completes the induction.

Next we will use the Pigeon-Hole-Lemma 3.5 to find an independent family of rigid tree-elements from an ordinary family of rigid tree-elements. Then we are ready to prove the Main-Lemma 3.3 which established already the Main-Theorem 3.1.

PROPOSITION 3.6. *Let m_0 be a natural number and G be the A-module constructed in (2.3). Suppose there is a family of rigid tree-elements for $0 \neq z \in G$ at some tree. Then we can find an independent family of rigid tree-elements over m_0 for z at the same tree.*

REMARK. The new family need not be a subfamily of the old one.

PROOF. Let $\{t_v \in G : v \in T\}$ be the given family of rigid tree-elements for $z \in G$ where $T \subseteq V_\alpha$ for some $\alpha < \lambda$ has cardinality κ.

Hence

$$t_v - t_w \in Az + q_{m+1} G \quad \text{for all } v \neq w \in T \text{ with } \mathrm{br}(v, w) = m$$

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follows from (t) in Definition 3.2. Shrinking T, we may assume $m > m_0$. Then we apply the Pigeon-Hole-Lemma 3.5 for $k = 1$. There is an equipotent subset T replacing T and there are ordinals $\beta_1 < \cdots < \beta_s < \lambda$ with

$$t_v \in \sum_{i=1}^{s} F_i \quad \text{for all } v \in T$$

where we write [as before in (12)]

$$F_i = \langle T_{\beta_i} \cup \{y_{wng_{\beta_i}} : w \in V_{\beta_i}, n < \omega\}\rangle_A.$$

We replace β_i by i and put

$$g_{\beta_i} = g_i, T_{\beta_i} = T_i, V_{\beta_i} = V_i \text{ and } y_{wng_{\beta_i}} = y_{wn} \quad \text{for } w \in V_i.$$

Moreover let $\mathbb{B} = \{\beta_1, \ldots, \beta_s\}$ and $B_{\mathbb{B}} = \langle T_i : i \leq s\rangle_A$. For each $v \in T$ we now can write

$$\text{(rp)} \qquad t_v = b_v + \sum_{i=1}^{s} \sum_{l=1}^{t} \sum_{n=0}^{m(v_{li})} a_{v_{li}n} y_{v_{li}n}$$

where $b_v \in B_{\mathbb{B}}$, $a_{v_{li}n} \in A$ and $y_{v_{li}n}$ are branch-elements with $v_{li} \in V_i$ depending on $v \in T$, $l \in \omega$ and $i \leq s$.

Next we want to improve the representation (rp) by using relations in G and discarding some of the elements from T.

Note that $w \restriction n \in T_i$ for $w \in V_i$ and (5) can be applied for $g = g_i$ to replace $\sum_{n=0}^{m(v_{li})} a_{v_{li}n} y_{v_{li}n}$ by multiples of g_i, an element in $B_{\mathbb{B}}$ and of $y_{v_{li}m(v)}$ where $m(v)$ is larger then the maximum of all the $m(v_{li})$'s and m_0, which is taken over a finite set of at most $st + 1$ numbers. We find new elements $a_{v_{li}m(v)}, a_{vi} \in A$, $b_v \in B_{\mathbb{B}}$ and $m(v) \in \omega$ and new representations for all $v \in T$ which are

$$\text{(newrp)} \qquad t_v = b_v + \sum_{i=1}^{s} \Big(\sum_{l=1}^{t} a_{v_{li}m(v)} y_{v_{li}m(v)} \Big) + a_{vi} g_i.$$

Moreover we may assume, enlarging $m(v)$ for each $v \in T$ up to the supremum of all branch points of distinct pairs $\{v_{li}, v_{ji}\}$ and the finite set $\bigcup_{i \neq j \leq s} C_i \cap C_j$ where the C_i's are from $\mathfrak{C}$, that

$$[y_{v_{li}m(v)}] \cap [y_{v_{nj}m(v)}] = \emptyset \quad \text{for all } (li) \neq (nj),\ 1 \leq l,\ n \leq t,\ 1 \leq i,\ j \leq s.$$

Also $V = \bigcup_{i \leq s} [g_i]$ is finite and $V \cap \bigcup_{li} [y_{v_{li}m(v)}] = \emptyset$ can be obtained by enlarging $m(v)$. This ensures the first part of (iii) in the Definition 3.4 of an independent family of tree-elements. Next we apply a pigeon-hole argument to simplify (*newrp*) even further. Recall that $\max\{\omega, |A|, |B_{\mathbb{B}}|\} < \kappa = |T|$. There is a subset of cardinality κ of T which we denote by T as well, with the following property.

There is a finite number of parameters for $1 \leq i \leq s$, $1 \leq l \leq t(i)$ with elements

$$t(v, i) = t(i), \quad m(v) = m > m_0,\ a_{v_{li}m(v)} = a_{li},\ a_{vi} = a_i,\ b_v = b$$

independent of $v \in T$. Equations (*newrp*) become

$$\text{(better)} \qquad t_v = b + \sum_{i=1}^{s}\Big(\sum_{l=1}^{t(i)} a_{li} y_{v_{li} m}\Big) + a_i g_i.$$

Recall that $v_{li} \in V_i \subseteq T_i^*$, $a_{li}, a_i \in A$, $b \in B_{\mathbb{B}}$. Let

$$E = \{(li) : 1 \le i \le s, 1 \le l \le t(i)\}, \quad E_v = \{v_x : x \in E\} \subseteq \dot{\bigcup_{i \le s}} V_i$$

and it is easy to verify from (*better*) that $\delta_v : E \longrightarrow E_v (x \longrightarrow v_x)$ is a bijection.

We can also choose $j_v \in \omega$ large enough and $> m_0$ such that the restriction map

$$(\delta_v \restriction j_v) : E \longrightarrow \bigcup_{i \le s} T_i : (x \longrightarrow v_x \restriction j_v)$$

is injective with image $(E_v \restriction j_v) = \{\delta_v(x) \restriction j_v : x \in E\}$. Note that $\bigcup_{i \le s} T_i$ is a countable set while T is uncountable. By a pigeon-hole argument we can shrink T such that $j_v = j^* > m_0$ and $(E_v \restriction j^*) = \Delta'$ are constant for all $v \in T$, however the finite branches in Δ' that is $\delta_v(x) \restriction j^*$ $(x \in E)$ are all distinct.

In order to show that the total map δ is injective, we replace the old family of rigid tree-elements t_v by a new family $t_v - b - \sum_{i=1}^{s} a_i g_i$ $(v \in T)$ and observe that the new family is a family of rigid tree-elements for z as well. The new family has a better representation, we can write

$$t_v = \sum_{x \in E} a_x y_{\delta_v(x) m}.$$

The set $\{E_v : v \in T\}$ of finite sets of infinite branches constitutes a Δ-system and the Δ-Lemma applies, see Jech [27, p. 225]. There is a new equipotent subset T replacing the old T such that

$$E_w \cap E_v = \Delta \quad \text{for all } v \neq w \in T.$$

If $F_v = E_v \setminus \Delta$ for $(v \in T)$ and $F = E \setminus \delta_v^{-1}\Delta$, then the F_v's are pairwise disjoint, hence δ is injective on $T \times F$ and

$$d = d_v = \sum_{x \in E \setminus F} a_x y_{\delta_v(x) m} \quad (v \in T)$$

does not depend on v any more. Replacing t_v again by $t_v - d$, we obtain a new family of rigid tree-elements for z at the tree T_α with the best representation

$$\text{(best)} \qquad t_v = \sum_{x \in F} a_x y_{\delta_v(x) m} \quad (v \in T).$$

The new family is the desired independent family of rigid tree-elements. Recall that $F_v = \dot{\bigcup}_{i \le s} (F_v \cap V_i)$ and the preimage of this decomposition is the decomposition of F in (3.4). The proposition follows.

The ultimate step in proving the Main-Lemma 3.3 is the following proposition. The Main-Lemma 3.3 is now immediate from

PROPOSITION 3.7. *Let G be the A-module constructed in (2.3). If there is an independent family of rigid tree-elements branching above some m_0 for $z \in G \setminus q_{m_0} G$ at the tree T_α, then $z \in A g_\alpha$.*

PROOF. We want to extract the arithmetical strength hidden in the given independent family of rigid tree-elements $\{t_v \in G : v \in T\}$ for z at some tree T_α with $T \subseteq V_\alpha$ of cardinality κ. By the last Proposition 3.6 and Definition 3.4 the elements can be expressed in the form

$$t_v = \sum_{x \in F} a_x y_{\delta_v(x)m} \quad \text{for } a_x \in A$$

with pairwise disjoint sets F_v of infinite branches from $\bigcup_{i=1}^{s} V_i$ where

$$F = \dot{\bigcup_{i \le s}} F_i,\ \delta_v \colon F \longrightarrow F_v,\ \delta_v(F_i) = F_v \cap V_i \subseteq V_i \quad (i \le s),$$
$$C_i = C_{\alpha_i} \in \mathfrak{C}, \quad V_i = V_{\alpha_i}, \quad T_i = T_{\alpha_i}, \quad \alpha_1 < \cdots < \alpha_s < \lambda.$$

Let $I = \{\alpha_i : i \le s\} \cup \{\alpha\}$. Recall from (1) that $\frac{q_{n+1}}{q_n} = q_n s_n$. Hence

$$z \notin \frac{q_j}{q_{j-1}} G = q_{j-1} s_{j-1} G \subseteq q_{j-1} G \subseteq q_{m_0} G \quad \text{for any } j > m_0 \tag{15}$$

by assumption on m_0. Moreover, note that F_v is a finite set of distinct branches, $\bigcup_{a \neq b \in I} C_a \cap C_b$ is a finite subset of ω, $\bigcap_{i \in \omega} q_i G = 0$ and $\bigcap_{i \in \omega} q_i A = 0$. Also note that $F_v \cap F_w = \emptyset$ for distinct $v, w \in T$ by Definition 3.4. We also have an element $j^* > m_0$ satisfying (3.4). All branches $\delta_v(x) \restriction j^*$ of length j^* for any $x \in F$ are pairwise distinct but independent of v. From these facts it is clear that the following combinatorial conditions hold.

$$\delta_v(y) \restriction j^* \neq \delta_v(x) \restriction j^* \tag{16}$$

$$\mathrm{br}\big(\delta_v(y), \delta_w(y)\big) \neq \mathrm{br}\big(\delta_v(x), \delta_w(x)\big) \ge j^* \quad \text{for } v \neq w \in T\ x \neq y \in F \tag{17}$$

$$q_{j^*-1} z \in G \setminus q_{j^*} G \tag{18}$$

$$\sup \bigcup_{a \neq b \in I} C_a \cap C_b < j^*, \tag{19}$$

and

$$a_x \neq 0 \iff a_x \notin q_{j^*-1} A \quad \text{for all } x \in F. \tag{20}$$

We also may assume

$$\mathrm{br}(v, w) > j^* > m_0 \quad \text{for all } v \neq w \in T. \tag{21}$$

Next we will show that the branch point of any two distinct branches $v, w \in T$ is bounded by the branch point of some 'δ-pair' of branches:

$$\mathrm{br}(v, w) \ge \mathrm{br}\big(\delta_v(x), \delta_w(x)\big) \quad \text{for some } x \in F \tag{22}$$

If

$$n = \mathrm{br}(v, w), \quad \text{then } t_v - t_w \pm q_n z \in q_{n+1} G \tag{23}$$

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by Definition 3.2 (t) of tree-elements and if also $t_v - t_w \in q_{n+1}G$, then $q_n z \in q_{n+1}G$. Hence $z \in q_n s_n G \subseteq q_{j^*-1} s_{j^*-1} G \subseteq q_{j^*-1} G$ by (21), which contradicts (15). We have

$$t_v - t_w \in q_n G \setminus q_{n+1} G \quad \text{for } n = \mathrm{br}(v, w). \tag{24}$$

On the other hand $t_v = \sum a_x y_{\delta_v(x)m}$ and $t_w = \sum a_x y_{\delta_w(x)m}$ for $a_x \in A$, hence

$$t_v - t_w = \sum_{x \in F} a_x (y_{\delta_v(x)m} - y_{\delta_w(x)m}). \tag{25}$$

If $\mathrm{br}\big(\delta_v(x), \delta_w(x)\big) > n$ for all $x \in F$, then by (7) and the last expression $t_v - t_w \in q_{n+1}G$ contradicts (24) and (22) follows.

We want to calculate (25) more accurate and define

$$\mathrm{Br}(F) = \Big\{ \mathrm{br}\big(\delta_v(x), \delta_w(x)\big) \in \omega : x \in F \text{ with } a_x \neq 0 \Big\}.$$

Then $\min\big(\mathrm{Br}(F)\big) = k = k(v, w) \leq n$ follows from (22).

Hence

$$F' = \Big\{ x \in F : \mathrm{br}\big(\delta_v(x), \delta_w(x)\big) = k \Big\}$$

is a non-empty set.

We suppose that $k < n$ for contradiction. Then $k + 1 \leq n$ and $q_n G \subseteq q_{k+1} G$ and $t_v - t_w \in q_{k+1} G$ follows from (24).

If $x \in F \setminus F'$, then $\mathrm{br}\big(\delta_v(x), \delta_w(x)\big) > k$, and $a_x(y_{\delta_v(x)m} - y_{\delta_w(x)m}) \in q_{k+1} G$ and

$$0 \equiv t_v - t_w \equiv \sum_{x \in F} a_x (y_{\delta_v(x)m} - y_{\delta_w(x)m}) \equiv \sum_{x \in F'} a_x (y_{\delta_v(x)m} - y_{\delta_w(x)m}) \bmod \frac{q_{k+1}}{q_m} G. \tag{26}$$

For any $x \in F$ we let $i(x)$ be the unique integer $i \leq s$ with $x \in F_i$. From (16) we infer

$$\mathrm{br}\big(\delta_v(x), \delta_w(x)\big) = k \geq j^* \tag{27}$$

for $x \in F'$ and using (7) and (10) we can reduce (26) further

$$\sum_{x \in F'} a_x q_k g_{i(x)} \in q_{k+1} G.$$

If $i(x) = i(y)$ for some $x \neq y \in F'$, then $x, y \in F' \cap F_i$ and all elements $\delta_v(x)$, $\delta_w(x)$, $\delta_v(y)$, $\delta_w(y)$ are branches of the same tree T_i and if $x \neq y$, then by (16) the pairs of branches $\big(\delta_v(x), \delta_w(x)\big)$ and $\big(\delta_v(y), \delta_w(y)\big)$ have two distinct branch points on T_i at the same level k. This contradicts our assumption (3), which followed from T_i being a perfect tree, hence $x = y$, see Observation 2.1.

We have seen that $|F' \cap F_i| \leq 1$ and note that $|F' \cap F_i| = 1$ for at least one of the i's, because $F' \neq \emptyset$. We discard all other i's and may assume

$$F' \cap F_i = \{x_i\} \quad \text{for all } i \leq s.$$

If $g_i = g_{i(x)} = g_{\alpha_i}$ and $a_{x_i} = a_i$, then the last displayed sum becomes

$$\sum_{i \leq s} a_i q_k g_i \in q_{k+1} G.$$

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Recall from (27) that $k = \mathrm{br}\big(\delta_v(x), \delta_w(x)\big) \geq j^*$, hence $k \notin C_i \cap C_j$ for any $i \neq j$ and k can not be the splitting level of pairs of branches from two distinct trees T_i, T_j. Hence splitting of branches $\big(\delta_v(x), \delta_w(x)\big)$ at this level k can only happen at one pair, say for the one with label $i(x) = 1$. We can reduce the last sum expression to

$$a_1 q_k g_1 \in q_{k+1} G.$$

Hence

$$a_1 g_1 \in \frac{q_{k+1}}{q_k} G = s_k q_k G \subseteq q_{j^*} G$$

by (1) and (27). However, g_1 is pure in G, hence $0 \neq a_1 \in q_{j^*} A$ contradicts (20). It follows that $k = n$, $F = F'$ and

(28) all pairs $\delta_v(x), \delta_w(x)$ $(x \in F)$ branch at level $\geq n = \mathrm{br}(v, w)$.

Note that elements $a_x(y_{\delta_v(x)m} - y_{\delta_w(x)m})$ such that v, w branches strictly above n are absorbed into $q_{n+1}G$ by (7). As before, but now for $n = k$, it follows

(29) $t_v - t_w \equiv a_x(y_{\delta_v(x)m} - y_{\delta_w(x)m}) \bmod q_{n+1} G \quad \text{and} \quad F = \{x\}.$

If $\delta_v(x) = v', \delta_w(x) = w'$ and $a_x = a$ then $v', w' \in V_{\alpha_1}$ and $v, w \in T \subseteq V_\alpha$. The independent family of t_v's for z by (29) simply turns into

$$t_v - t_w \equiv a(y_{v'm} - y_{w'm}) \bmod q_{n+1} G.$$

The pair v', w' can not branch at level $> n$ because $0 \not\equiv t_v - t_w \bmod q_{n+1} G$ by (24). Hence v, w and v', w' branch at the same level $n > j^*$ by (28). Either $\{v, w\} = \{v', w'\}$ or the pairs are different. In the second case branching of two distinct pairs of branches at such a high level n can only happen at the 'same tree'. Hence in either case we must have

$$\alpha_1 = \alpha \text{ and also } g_1 = g_\alpha$$

by (20). Using (7), (10) and (t) we have

$$0 \not\equiv q_n z \equiv a q_n g_\alpha \bmod q_{n+1} G.$$

As before, we derive $z \equiv a g_\alpha \bmod \frac{q_{n+1}}{q_n} G = q_n s_n G$ from (1). The set T of infinite branches has size κ and hence its branches split at arbitrarily large level. Choose any sequence of pairs (v, w) of branches from T with branch points converging to infinity and note that G is $\aleph_1$-free by Proposition 2.5, hence G is reduced. We derive that $z - a g_\alpha \in \bigcap_{n \in \omega} q_n s_n G = 0$, hence $z = ag_\alpha \in Ag_\alpha$ as required.

PROOF OF THE MAIN-LEMMA 3.3. The family of rigid tree-elements given by (3.3) can be traded into an independent family of rigid tree-elements over the given number m and the same tree by Proposition 3.6. Now the assumptions for Proposition 3.7 are satisfied for $m = m_0$ and that z. Hence the conclusion of (3.3) follows from (3.7).

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https://doi.org/10.4153/CJM-1998-039-7 Published online by Cambridge University Press