Philosophy of Science (2025), **92**, 511–527
doi:10.1017/psa.2024.49

PSA PHILOSOPHY OF SCIENCE ASSOCIATION

PHILOSOPHY OF SCIENCE

ARTICLE

Gauge Theory Without Principal Fiber Bundles

Henrique Gomes

Oriel College, University of Oxford, UK
Email: gomes.ha@gmail.com

(Received 24 June 2024; revised 10 September 2024; accepted 16 September 2024; first published online 28 October 2024)

Abstract

In general relativity, the strong equivalence principle is underpinned by a geometrical account of fields on spacetime, by which all fields and bodies probe the same geometry. This geometric account implies that the parallel transport of all spacetime tensors and spinors is dictated by a single affine connection. No similar account of gauge theory is put forward by standard textbooks, which use principal bundles to coordinate the parallel transport of different, interacting particles. Nonetheless, here I argue that gauge theory *does* afford such a geometric account, obviating the need for principal bundles.

1. Introduction

All quarks and gluons interact via one and the same strong nuclear force. And although these particles all carry charges for this same force, they are described by fields that take values in a variety of internal vector spaces coexisting over each spacetime point. In spite of this variety, these internal spaces don't just coexist over each spacetime point: they are intimately connected, which is what allows the different particles to interact. For instance, were we to take particular values for quarks and gluons at spacetime point p and carry them to point q along a certain spacetime curve γ, these values would perform a "synchronized rotation" in their respective internal spaces: their evolution is coordinated or "marches in step."

The usual mathematical explanation for the variety is that the fields exist as sections of distinct vector bundles over spacetime, and they march in step because those vector bundles are *associated vector bundles*, by fiat associated to a single principal G-bundle, P, where G is the symmetry group regimenting a particular interaction, e.g., $SU(3)$ in the case of the strong force. In turn, each principal bundle is endowed with a single Ehresmann affine connection ω, which defines parallel transport for fields that are charged under the corresponding force.

Thus, in the standard model (SM) of particle physics, all fields charged under the same gauge group get their parallel transport from the same mathematical object, ω. Since ω exists on the more abstract principal fiber bundle P, it does not cohabit the space of the physical matter fields. Nonetheless, from "afar," it coordinates the parallel transport of the different physical fields in their respective internal vector spaces. In the words of Weatherall (2016, p. 2401):

> Principal bundles are auxiliary [in the sense that only] vector bundles represent possible local states of matter; principal bundles coordinate between these vector bundles ... [they are auxiliary] in the sense in which a coach is auxiliary to the players on the field.

This is a beguiling metaphor, but is it explanatory? It certainly falls short of the familiar geometric explanation for parallel transport that we get in general relativity.

In the case of general relativity, in order to describe parallel transport of tensor fields on the spacetime manifold M we need not invoke principal bundles at all. Recall how general relativity textbooks expound tensor analysis, by having the metric determine the notion of parallel transport (i.e., the Levi-Civita connection), with never a mention of a principal fibre bundle. There, sections of different tensor and spinor bundles march in step under parallel transport *because* they are all constructed from the same geometric structures: namely, the tangent bundle TM, with each tangent space T_pM being endowed with a Lorentzian inner product and an orientation (necessary in the case of spinors). The parallel transport of all tensor fields is coordinated because they are sections of vector bundles built from the same tangent bundle. It is the tangent bundle that underpins a unified account of parallel transport for tensor fields.[1]

In the gauge case, the textbook tradition—indeed, so far as I know, the extant literature[2]—reveals no similarly powerful explanation for why the fields that couple through non-gravitational forces march in step under parallel transport. This I will call the *coordination problem* of gauge theory: it constitutes the main disanalogy between spacetime and gauge theories that this paper is focused on dispeling.

One might at first think that the obstacle to a resolution of the coordination problem in gauge theory, that similarly to the gravitational case does not mention principal fiber bundles, is that we can't define a covariant derivative directly for vector bundles. This is not the case: such a definition of covariant derivative is straightforward, and it is given, for an arbitrary vector bundle, in equation (A.7).[3] The

---

[1] It is true: "you won't go to jail" for describing parallel transport of tensor fields using principal bundles also in the spacetime case. That description employs an orthonormal basis of vectors at each spacetime point and a connection form that describes their parallel transport. The different orthonormal bases are related by elements of the Lie group $O(3,1)$ (or $SO(3,1)$, if spacetime orientation is important), and so the space of orthonormal bases over spacetime forms a principal fiber bundle with $O(3,1)$ as its structure group (cf. Appendix A). This is the group that leaves the Minkowski metric on a $3+1$ space invariant (and its subgroup of orientation-preserving transformations). Thus, the structure group of the principal bundle is tied to the preservation of the structure of a "typical fiber", which is a vector space over each spacetime point—viz., the tangent space, which is isomorphic to $\mathbb{R}^4$—endowed with a semidefinite inner product. Indeed, this description is often involved in the treatment of spinors. But the point is that we don't *need* to invoke this bundle to describe parallel transport of tensor fields or even spinors.

[2] I thank Lathan Boyle and David Tong for helpful discussions of this curious lacuna in the literature.

[3] That definition specializes, when the vector bundle is the tangent bundle, to the usual definition of covariant spacetime derivatives.

obstacle to resolving the coordination problem with such a definition is that it offers us no relation between the derivatives of two vector bundles (if one bundle is not obtained (e.g., tensorially) from the other). In other words, given two different but interacting particles, seen as sections of two different vector bundles, there may be no natural way for an intrinsic derivative in the first bundle to induce one on the second. Thus, using this intrinsic covariant derivative leaves the "marching-in-step" of sections of any two different vector bundles under parallel transport *completely* mysterious.

Principal and associated bundles are uniquely important for gauge theories because they allow us to partially solve this first coordination problem. We do so by defining the covariant derivative not intrinsically on the vector bundle, but as I described above: via the action of the Ehresmann connection ω on sections of associated vector bundles—see definition 4 and equation (A.5). Nonetheless, this is only a halfway house towards a satisfactory resolution of the coordination problem because we still need to stipulate that particles whose parallel transport should march in step are all associated to the same principal fiber bundle, with the same structure group and Ehresmann connection. This (partial) resolution places the symmetry group first, and then goes on to define all the objects and structures that are well-behaved under (some action of) this group. In the terminology of Jacobs (2021, chapter 4.1), it is a *symmetry-first* approach to gauge theory. He describes the situation as follows (Jacobs, 2023, p. 40):

> But it is a problem for this approach that [. . .] two fields survey the same connection as a matter of brute fact. There really are two connections: one defined over the first associated bundle, and one defined over the second. These connections are the same only in the sense that we can represent both with the same connection on a single principal bundle. But [. . .] there is no independent Yang–Mills field that the associated bundle connections supervene on. This makes it seem somewhat mysterious that these connections are equivalent. On [this] approach, it is a brute fact that all matter fields have the same symmetries.

In sum, in general relativity there is no coordination problem for parallel transport: Since tensors are constructed from the tangent bundle their marching-in-step under parallel transport is guaranteed, and there is no need to introduce principal and associated bundles. In contrast, gauge theory needs principal and associated bundles because, in the textbook tradition, it lacks a structure analogous to the tangent bundle, which can underpin a unified account of parallel transport for a variety of fields that interact, not gravitationally, but via the other forces of nature.

Here I will show that, contrary to the textbook tradition, gauge theory *has* a structure analogous to the tangent bundle, which can be used to the same effect. This structure suffices, for example, to account for the content of the standard model of particle physics. Thus, while there can be a practical reason for introducing principal and associated bundles—as there often is in general relativity (cf. footnote 1)—there is no mathematical necessity for doing so. For we *can* see all the constituents of both the chromodynamics and the electroweak sector of the standard model as sections of vector bundles that are tensorially constructed from a single, underpinning vector bundle. This underpinning vector bundle has fibers isomorphic to $\mathbb{C}^3 \times \mathbb{C}^2 \times \mathbb{C}$, endowed with the canonical inner product of the complex planes and an orientation, and so has an automorphism group which at each point is isomorphic to $SU(3) \times SU(2) \times U(1)$.

In this picture, the invariant structure comes first, and the symmetry group that preserves this structure comes second. That is, the gauge group is no longer postulated as fundamental: it acquires meaning as the invariance group of the typical fiber of E. So we can think of parallel transport in a frame-independent way as being a structure-preserving map, carrying the fiber's structure from one point of spacetime to another along a spacetime path. The different particle types are represented by different types of tensors over the same vector bundles.

Here is this paper's content, in slogan form: Gauge transformations can be understood naturally as automorphisms of an internal geometric structure, to which the theory is ontologically committed; and an affine connection defines parallel transport in these spaces, with never a principal bundle in sight.

2. How to dispel the disanalogy: The internal spaces

This section will dispel the putative disanalogy between parallel transport of spacetime and internal quantities expounded in section 1. But before diving in, I want to clear the ground.

The label "geometrical" might be taken to connote properties related to distance relations, and to geodesics extremizing such distances. That is not how I mean it. Although there is one interpretation of gauge theories and gauge transformations that is geometric in this sense—called Kaluza–Klein theory (see Kaluza (1921) and O'Raifeartaigh (1997) for the history)—that is not the sense I will focus on here. Here I want to assess whether gauge transformations can be understood naturally as automorphisms of an internal geometric structure; and whether parallel transport of internal quantities can be understood similarly to that of tensor fields over spacetime.

Next, let us set aside all questions about the "external" spacetime geometry. A matter field can be described as the tensor product of some interior space on which gauge fields take values—e.g., a complex scalar or Yang–Mills field, ϕ—and a spinor field ψ; or tensor fields in the case of gauge bosons. In the standard account, gauge fields use a connection and gauge frame which are independent of the spacetime manifold frame, while spinor and tensor fields mirror the connection and changes in the frame of the spacetime manifold. So gauge fields are acted on by representations of the gauge group and its Lie algebra, while spinor fields are acted on by representations of the Spin group and its Lie algebra ($\mathfrak{so}(3,1)$). The gauge component then responds to gauge transformations, while the spinor component responds to changes of frame. Here, I will focus only on the gauge, or, as I will argue, the internal part.

We will see that interacting fields can be seen as sections of bundles built up from the same internal spaces, or typical fibers. For instance, in the same way that a symmetric, covariant tensor of rank two is built from two copies of TM, (the internal part of) quarks will have components in a typical fiber isomorphic to $\mathbb{C}^3$, and gluons will be certain (traceless) tensor bundles, involving $\mathbb{C}^3$ and $\mathbb{C}^{*3}$. Thus we will have a geometric reason for the parallel transport of the different quarks and leptons marching in step.

Table 1. The representation of the SM groups on fermions

	$SU(3)$	$SU(2)$	$U(1)$
q_L	3	2	$\frac{1}{6}$
u_R	3	1	$\frac{2}{3}$
d_R	3	1	$-\frac{1}{3}$
ℓ_L	1	2	$-\frac{1}{2}$
e_R	1	1	-1

This will of course require a brief description of the particle content of the SM, which I provide in section 2.1. In section 2.2 I interpret the results in terms of sections of vector bundles for certain typical fibers. In section 2.3, I present five possible objections to my intepretation.

2.1. A closer look at the SM

In particle physics jargon, connections are the "force-carriers," and are represented by gauge bosons. But I will first set the bosons aside and focus on the fermionic content of the SM; we will get back to bosons later.

The SM is represented in terms of Weyl fermions, which are two-component spacetime spinors. But I am only interested in the structure of the internal spaces; the spaces where the gauge connections act. So here I am basically ignoring the spacetime spinor structure of the SM (though they are somewhat implicit in the notation of left- or right-handed particles used below). When representing the full fermionic content of the SM, this spinor part would be included as factors in a tensor product with the internal part that I am interested in and aim to describe in this section. I will get back to this point below.

The part of the (minimal) SM that I am interested in consists of 45 complex numbers, organized into three generations, which means it has the same structure repeated three times. We can understand this repetition in terms of direct sums:

$$\mathbb{C}^{45} = \mathbb{C}^{15} \oplus \mathbb{C}^{15} \oplus \mathbb{C}^{15}. \tag{2.1}$$

Table 1 tells us how these components transform, and it is organized into blocks whose elements can transform into each other (elements from different generations, or blocks, cannot). So each $\mathbb{C}^{15}$ breaks down into the five rows of the table (I will here only focus on the first generation).[4] Now let us unpack table 1. First, the columns are labeled with the groups that are associated to the types of interaction: strong ($SU(3)$), weak ($SU(2)$), and hyperweak ($U(1)$).

[4] The three generations differ mostly with respect to their Yukawa couplings to the Higgs, which I am ignoring here. These are non-gauge interactions that lead to different masses of the three generations. Also note that here I am describing the *minimal* SM, and so I am not including the right-handed neutrinos, which have not yet been directly observed, but, after the discovery of neutrino oscillations, are generally assumed to exist.

The **quarks** are represented by the first three rows of the table. As to the first column, quarks clearly feel the strong forces, and they transform under the standard, or fundamental, representation of $SU(3)$, labeled "3," which just means $SU(3)$ acts on elements of $\mathbb{C}^3$ via matrices which preserve the volume element and complex inner product of $\mathbb{C}^3$. So the components of quarks corresponding to the first row can be seen as vectors in internal spaces isomorphic to (a structured) $\mathbb{C}^3$. Now, q_L is a left-handed quark doublet, which is a doublet of the form $q_L = (u_L, d_L)$. In the first generation this would be called up-left and down-left, respectively; in the second generation it would be charm-left and strange-left; and in the third generation it would be top-left and bottom left. The reason q_L is called a doublet—unlike the two rows beneath it, representing the up-right and the down-right quarks u_R and d_R, which are singlets—is that the components of q_L, namely u_L and d_L, are charged under the weak nuclear force, and transform into each other under the action of $SU(2)$. In the entry corresponding to $q_L \times SU(2)$ this transformation property is represented by the number "2," which means that q_L transforms as an element of $\mathbb{C}^2$ under the fundamental representation of $SU(2)$. The number "1" for the entries $u_R \times SU(2)$ and $d_R \times SU(2)$ means that u_R and d_R are neutral under the weak forces, so cannot transform into each other (because, being singlets, they don't transform at all under $SU(2)$). Finally, the left-handed quark has a "weak hypercharge" of $-1/6$ under $U(1)$, which means that it is a complex number (an element of $\mathbb{C}$) which, under the action of a given $U(1)$ phase-shift generator ξ, has its phase rotate at the rate of $-\xi/6$ (or $e^{i\xi/6}$); *mutatis mutandis* for the down-right and up-right quarks.[5]

The **leptons** are represented by the remaining two rows in the table and have a kind of parallel structure to the quarks, but of course they are all neutral under $SU(3)$ (they are not charged under strong interactions). ℓ_L is the left-handed lepton doublet, which is of the form $\ell_L = (e_L, \nu_L)$. In the first generation these are the left-handed electron and neutrino (in the second and third they get "muon" and "tau" prefixes). Again, we put e_L and ν_L in the same row because they are charged under $SU(2)$ (they are charged under the weak forces), and transform into each other, unlike the particle of the remaining row—the right-handed electron e_R, which is neutral under $SU(2)$. The hypercharge of ℓ_L is $-1/2$ (which does not coincide with its electric charge; see footnote 5). The electric charge of the right-handed electron is, as expected, 1.

With the basic ingredients in place, I will now defend my intepretation of table 1, arguing that it dispels the (putative) disanalogy to gravity described in section 1.

2.2 *Interpretation*

The first two columns of table 1 contain only one kind of non-trivial representation: the fundamental. So, in these columns, elements of $SU(3)$ and $SU(2)$ are 3×3 and 2×2 matrices, respectively, acting on elements of $\mathbb{C}^3$ and $\mathbb{C}^2$, preserving their canonical inner product and oriented volume.[6] The third column, under $U(1)$, is, in

[5] Note that for $U(1)$ it is a "0" entry—and not a "1," as it is for $SU(3)$ and $SU(2)$—that tells us a particle does not transform, or is neutral with respect to this interaction.

[6] A special unitary matrix is a unitary transformation with determinant 1. We can interpret the restriction to determinant 1 as preserving the oriented volume because the signed n-dimensional volume of an n-dimensional parallelepiped is expressed by a determinant, and the determinant of a linear

one sense, the most familiar from classical electromagnetism: it represents an overall phase, where different charges transform with different rotation speeds under $U(1)$.[7]

So we clearly have $\mathbb{C}^3, \mathbb{C}^2, \mathbb{C}^1$ over each spacetime point, where particles take their values. These are the typical fibers of three different fundamental vector bundles, call them $(E^3, M, \mathbb{C}^3)$, $(E^2, M, \mathbb{C}^2)$, $(E^1, M, \mathbb{C}^1)$, or E^3, E^2, E^1 for short, where, for each, a fiber at a point is isomorphic to a complex vector space with inner product and orientation: for $\pi_n : E^n \to M$, $\pi_n^{-1}(x) \simeq \mathbb{C}^n$ (but recall that there is no canonical isomorphism). Each of these vector bundles is analogous to TM in the spacetime case, and we also naturally have the dual bundles (of linear functionals), E^{3*}, E^{2*}, E^{1*}, that are necessary in order to represent the corresponding anti-particles. The group of automorphisms of these fibers are, again, (non-canonically) isomorphic to $SU(3)$, $SU(2)$, and $U(1)$. These are structures that should be preserved by an affine connection on the corresponding vector bundles.

Now, as usual, we can join these vector bundles in different ways, using different kinds of products; and, as for tensor fields over spacetime, here too the most important for our purposes is the tensor product.[8] Of course, a group action or representation on a vector space V induces a representation on arbitrary tensor products of V and V^*, and so it is here: The structure of the typical fiber defines a group that acts on that typical fiber, and that action naturally extends to all tensor products of the space and its dual.[9]

In the first row the left-handed quark doublet has components lying along $\mathbb{C}^3$, $\mathbb{C}^2$, and $\mathbb{C}^1$: we must locate it within a space of three colors, and of two isospin charges, and of one hypercharge. The internal part of the left-handed quark doublet is a section of the bundle

$$q_{\mathrm{L}} \in \Gamma(E^3 \otimes E^2 \otimes E^1). \tag{2.2}$$

Unlike the first row of table 1, the particles in the following two rows have no component along $\mathbb{C}^2$, which is why they are not charged under $SU(2)$. In contrast, the left-handed lepton doublet has no components along $\mathbb{C}^3$, but has components along $\mathbb{C}^2$; and the right-handed electron has no components along either $\mathbb{C}^3$ or $\mathbb{C}^2$ (that is why it is not charged under either the strong or the weak interactions), it only has components along $\mathbb{C}^1$ (cf. footnote 5).

endomorphism determines how the orientation and the n-dimensional volume are transformed under the endomorphism. Alternatively, $U(n)$ is the n-fold cover of $SU(n) \times U(1)$.

[7] I should also note that weak hypercharge, denoted Y_{W}, is not the same as electric charge, Q. The relation between the two types of charge emerges only after symmetry breaking, which requires an interaction between the Higgs and weak isospin: it is given by the equation (in our convention) $Q = 2T_3 + Y_{\mathrm{W}}$, where T is the $SU(2)$ charge, and we have assumed the Higgs potential selects the third component of isospin. It coincides with electric charge only for the rows that transform trivially under $SU(2)$, namely, for all the right-handed particles in the table. Thus, the electric charge of the down-right quark is $-1/3$, for an up-right quark it is $2/3$, etc. The way these charges combine after symmetry-breaking gives a mnemonic device for the numbers of the last column: the entry for the left-handed particles is the average of the two entries below, for right-handed particles.

[8] Given two vector bundles E, E' over the same spacetime M, the tensor bundle is a bundle over M whose fiber over $x \in M$ is $E_x \otimes E'_x$.

[9] For instance, if $\rho(g)$ is a representation of G on V, then G acts on the dual space V^* via the inverse of the transpose, $\rho(g^{-1})^T$.

As I said above, the odd man out in table 1 is the third column, corresponding to the $U(1)$ weak hypercharges, since there we have multiple non-neutral values. How should we interpret the different weak hypercharges as properties of sections of vector bundles? One immediate answer comes from a rather trivial technical point. Since $\mathbb{C}^1$ has complex dimension 1, arbitrary tensor products of $\mathbb{C}^1$ will also have complex dimension 1.[10] But if a particle is, formally, a section of a vector bundle $E^1 \otimes E^1 := E^1_2$, under a rotation of E^1's typical fiber $\mathbb{C}$ by θ, because of the multilinearity of the tensor product, that section of E^1_2 picks up a phase of 2θ. Thus, formally, taking the lowest charge as the unit, we can think of a weak hypercharge of $N/6$ as being due to the Nth tensor product of E^1, which we call E^1_N, and negative charges are sections of tensor products of $(E^1)^*$. But, precisely because these tensor products are still one-dimensional, not much changes in terms of the representation of these sections: there are no added degrees of freedom.[11]

In the first two columns, the representations 3 and 2 describe the number of degrees of freedom of the particle in these spaces: vectors in $\mathbb{C}^3$ have three and in $\mathbb{C}^2$ have two. Indeed, for the same reason, we label with an "8" the representation of the gluon, whose internal component, as described in equation (A.4), would, in our geometric treatment, be a section of $\Gamma(E^3 \otimes_{\mathrm{T}} E^{*3})$, where T stands for traceless (which is necessary for parallel transport to be not only linear, but compatible with the inner product). So "8" is the number of internal degrees of freedom that such a field would have, and its tensor structure implies it is acted on by the adjoint representation of the group action on E^3.

So, even though the gluon distinguishes itself for determining parallel transport for itself and for the other particles, here, unlike the standard picture, the gluon is of a piece with other matter fields in that they are all sections of (different) tensor bundles over *the same* underpinning vector bundle. But the gluon does not fit table 1 because it is not a fermion, and does not decompose into a tensor product with Weyl spinors as the rest of the table does; it is a boson, and its spacetime part is a 1-form. Indeed, this is the case for all the affine connections, which, in particle physics terminology, are called the gluon, the W, and the Z bosons. These are the degrees of freedom dictating the parallel transport of color, isospin, and (hyper)charge, which, along a given spacetime curve $\gamma : [0,1] \to M$ take, respectively, the fibers of E^3, E^2, and E^1 over $\gamma(0) \in M$ to the fibers of E^3, E^2, and E^1 over $\gamma(1) \in M$ as a linear, structure-preserving transformation (cf. equation (A.12)).[12]

[10] Here, it is important to distinguish the dimensions of a vector space qua complex space, i.e., in which addition is linear under complex scalar multiplication, from dimensions of a vector space qua real vector space. For V and W complex vector spaces of dimension p and q respectively, $\dim_{\mathbb{C}}(V \otimes_{\mathbb{C}} W) = pq$, while $\dim_{\mathbb{R}}(V \otimes_{\mathbb{R}} W) = 4pq$.

[11] In the standard presentation, the fact that all representations of $U(1)$ are one-dimensional is a consequence of *Schur's lemma*. Namely, an irreducible unitary complex $U(1)$ representation must be one-dimensional by Schur's lemma, since all $U(1)$ elements commute with each other and so are multiples of the identity, and each one-dimensional subspace is an invariant subspace of multiples of the identity. I find the proof in terms of tensor spaces that I mention in the main text much more transparent.

[12] As with the fermions, we can of course have different sections of vector bosons. According to (A.4), given any fixed D_o (e.g., over a trivialization $E|U \simeq U \times V$, we can set $\mathrm{D}_o = \mathrm{d} \otimes \mathrm{Id}$), any such vector boson defines an affine connection D that is compatible with the fiber structure.

Summing up, apart from (2.2), we get

$$u_{\mathrm{R}} \in \Gamma\left(E^3 \otimes E^1_4\right), \quad d_{\mathrm{R}} \in \Gamma(E^3 \otimes E^1_{-2}), \quad \ell_{\mathrm{L}} \in \Gamma\left(E^2 \otimes E^1_{-3}\right), \quad e_{\mathrm{R}} \in \Gamma\left(E^1_{-6}\right), \tag{2.3}$$

and adding the vector bosons (one for each $SU(n)$), for which we include its 1-form component in spacetime,

$$\omega_n \in \Gamma(T^*M \otimes E^n \otimes_{\mathrm{T}} E^{n*}). \tag{2.4}$$

We can conceive of each generation as having the following decomposition into five factors:

$$\mathbb{C}^{15} = \left(\mathbb{C}^3 \otimes \mathbb{C}^2 \otimes \mathbb{C}^1_1\right) \oplus \left(\mathbb{C}^3 \otimes \mathbb{C}^1_4\right) \oplus (\mathbb{C}^3 \otimes \mathbb{C}^1_{-2}) \oplus \left(\mathbb{C}^2 \otimes \mathbb{C}^1_{-3}\right) \oplus \mathbb{C}^1_{-6}. \tag{2.5}$$

And we can finally answer the main question of this section: Why do the parallel transports of different, mutually interacting particles, as sections of different vector bundles, march in step?

Recall that in the textbook tradition (see, e.g., Nakahara, 2003, chapter 9), the answer is postulated. The gauge symmetry group is not derived as preserving some physical structure, it is postulated in the definition of the principal bundle. But here I've argued that, just as tensor bundles are constructed from the underpinning geometry of TM and tensors have components in the spaces thus constructed, particle fields have components in internal spaces corresponding to color, isospin, and (hyper) charge that are constructed from the underpinning geometry isomorphic to that of $\mathbb{C}^3$, $\mathbb{C}^2$, and $\mathbb{C}^1$, endowed with an inner product and, except in the case of $\mathbb{C}^1$, an orientation. Parallel transport marches in step because it concerns the underpinning internal geometry. The structure groups $SU(3) \times SU(2) \times U(1)$ are the symmetries that preserve the internal geometry. In this picture, the structure groups are *not* postulated. They are isomorphism-invariant automorphism groups that moreover can emerge explicitly upon comparisons of parallel transported tensors, as the (isomorphism-invariant) holonomy group Hol(D), described in Appendix A.2 (cf. equation (A.12)).

In this tensorial representation of the fields of gauge theory, there is no need for indices, except to denote the type of tensor under consideration: in the analogous spacetime case, this is called *the abstract index notation* for spacetime tensors.[13] Just as in the case of spacetime tensors, these gauge tensors are invariant under passive transformations. It is only upon introducing a trivialization of the vector bundle—i.e., a local isomorphism between E and $U \times F$, where $U \subset M$ is some patch of spacetime—that we can talk about a tensor's components transforming under a change of trivialization. But, again, just as spacetime tensors are *not* invariant under active diffeomorphisms, here the gauge tensors are *not* invariant under a fibre-wise linear isomorphism of tangent bundles. The transformation between these tensors corresponds to the active view of gauge transformations (cf. equation (A.17) for the transformation of the connection).

[13] This notation uses indices to indicate the types of tensors or spinors, rather than their components in a particular basis. The indices are mere placeholders, not related to any basis, and, in particular, are non-numerical.

2.3 Possible objections

Here I will address five possible objections about the geometric viewpoint. The first is more technical, the second is conceptual, the third is metaphysical, the fourth is about completeness, and the fifth is about applications beyond the SM. All but the first two lead to concessions about my framework.

First, the technical possible objection. I said above that the spinor structure of the fields comes in as a factor in a tensor product with the internal tensorial structure. But that is not exactly right for the table as I presented it: it would require me to represent the SM solely in terms of one chirality, which is certainly possible. Instead of having both right- and left-handed spinors, one can include in the table only left-handed ones; I preferred not to mix particles and anti-particles in the table, which is why I instead used both chiralities. Using a single chirality would have the advantage of being rigorous about the tensor product between internal spaces and spinors, but would have the disadvantage of having to introduce complex conjugates of the representations, e.g., using $\bar{3}$ instead of 3 for the first and fourth rows of the table, and also having to introduce q_{L}^{c}, the anti-left-handed quark doublet, and ℓ_{L}^{c}, the anti-left-handed lepton doublet. But of course doing this would not offend my main thesis, since complex conjugation of $\mathbb{C}^3$ is an operation that requires no more structure than I have posited; it is analogous to taking T^*M to be defined by TM (as linear functionals thereof).

Now I'll address the second, conceptual objection. Given the Lagrangian of the SM written in a local coordinate system, I could extract all of the invariances and symmetry transformations directly. Invariance of the Lagrangian would constrain the internal values of the different particle fields to appropriately march in step. This is a true statement, but I don't think it is explanatory. For the same could of course be said about general covariance in general relativity. There, it is the geometric interpretation that underpins the universal coupling of all of the fields to a single spacetime geometry. But this universality could fail; for instance, if "bi-metric" Lagrangians for gravity were adopted, we could have more than one Levi-Civita connection, which could dictate parallel transport differently for different fields. Reversing the explanatory arrow, the fact that such bi-metric theories have little empirical support can be explained by the more parsimonious, familiar geometric interpretation of general relativity. Similarly, my argument here shows that the most parsimonious explanation for the current form of the standard model (without the analogous "bi-metrics") is that it concerns an internal structured space, isomorphic to $\mathbb{C}^3 \times \mathbb{C}^2 \times \mathbb{C}^1$.

The third objection is very similar in spirit to the second one, but it plays out one level lower in the hierarchy of mathematical structures. Whereas the second was about the basic geometric objects describing parallel transport, the third concerns the underpinning spaces in which the fields in question live. For the interior complex spaces I have presented are not analogous to tangent spaces with Lorentzian inner product in all relevant senses: there is a privilege afforded to the tangent space which isn't similarly afforded to complex internal spaces, since each element of the tangent space is identified with an infinitesimal path through the base manifold; the tangent space is "soldered" onto spacetime. Thus, the particular vector bundle E has to be postulated and, we must assume, shared by interacting fields.[14] Nonetheless, I maintain that the explanation afforded here distinguishes itself by putting structure, rather than symmetry, first. In contrast, as alluded to in the passage from Jacobs (2023) quoted in section 1, the standard principal fiber bundle formalism posits both the symmetry group G and the vector bundles, and demands their compatibility, which goes unexplained.[15]

Fourth, my description of the SM here was not complete. The attentive reader will have noticed a glaring omission: the Higgs particle is nowhere to be found in table 1. There are, at bottom, two reasons for this omission. The first is that the Higgs would not fit in table 1: it is a scalar field on M, not a spin-$1/2$ fermion, and so does not fit the required (but implicit) tensor product structure. The second, more relevant, reason is that the Higgs and spontaneous symmetry breaking (SSB) make things rather more complicated, with added *non-gauge* interactions between the Higgs and other particles through Yukawa couplings. It is mostly differences in these couplings that distinguish the three generations of the SM. The up, charm, and top quarks have the same electric charge, along with the same weak and strong interactions—they primarily differ in their mass, which comes from the Higgs field. The same thing holds for the down, strange, and bottom quarks, along with the electron, muon, and tau leptons. And yet there is a single generation of bosons, meaning that they are all parallel transported by the same connections. The striking similarity and apparent redundancy of the three generations is one of the great mysteries of the SM, even within the standard approach. In order to address this issue in this formalism, one would need to better understand gauge-invariant construals of the Higgs mechanism and Yukawa couplings (see, e.g., Struyve (2011) and Berghofer et al. (2023, chapter 5)) in terms of invariant geometric structures along the lines that I have proposed here. I leave a full treatment of Yukawa couplings, the Higgs, and SSB for further work.

Here is the fifth possible objection, about applications beyond the SM: The interpretation of the SM that I have proposed here was very straightforward because different non-neutral charges appear only in the $\mathbb{C}^1$ sector.[16] In that one-dimensional sector, the different charges arise from tensor products (by multi-linearity) at no additional ontological price, since these products imply no additional degrees of freedom for the particles in question. So a worry might emerge that we could not account for arbitrarily different charges for the other forces, and that the scope of the

[14] There is a second distinction due to soldering. We could still act on E with a fibre-wise linear isomorphism, with a corresponding action on the matter fields and connection forms. This is the global, or active, view of gauge transformations, on a par with the active view of smooth diffeomorphisms on a spacetime manifold. Thus, in the same way that tensors over spacetime are not invariant under active diffeomorphisms, here the gauge fields are not invariant under active linear isomorphisms. The difference between the spacetime and the gauge case is again solely due to soldering: we cannot act with a linear isomorphism over the tangent spaces without moving the spacetime points as well.

[15] See Jacobs (2021, chapter 4.1), and references therein, for a defense of the advantages of structure-first explanations of symmetry.

[16] In the higher-dimensional $\mathbb{C}^2$ and $\mathbb{C}^3$, corresponding to $SU(2)$ and $SU(3)$, non-neutral charged matter fields of the SM appear only in the (anti-)fundamental representation, which allowed my straightforward interpretation as vectors in the internal (dual) vector space.

geometric interpretation is narrower than the scope of the standard interpretation in terms of principal fiber bundles and their associated bundles.

However, at least for $SU(n)$, the geometric interpretation pursued here *can* recover all the different representations (representing different kinds of particles) by using tensor products and the internal geometric structures of the fibers $\mathbb{C}^n$—see, e.g., Coleman (1965) and Zee (2016, chapter IV.4). Indeed, we saw one such construction for the gauge boson, that lives in the adjoint representation, in equation (2.4). That representation corresponds to a traceless tensor product between an internal space and its adjoint. And although for $n > 1$ the number of degrees of freedom of such internal tensor fields is different for different valences, this is as it should be: the number of degrees of freedom of sections of tensor fields of valence (j, k) depends on j and k even for spacetime, after all. However, I believe that my interpretation might fail, or would at least become less natural, for some of the exceptional Lie groups, whose geometric interpretation is much more involved (cf. Adams, 1996). I also leave this for further study.

3 Conclusions and outlook

In particle physics, *fundamental forces* are uniquely associated to structure groups. I have argued here that those structure groups merely reflect the geometric structure of vector spaces that are internal along spacetime. Gauge invariance is then described as an ontological commitment to this structure.

The standard model cleanly illustrates this idea. In just the same way that a Lorentzian inner product on the tangent bundle TM leads directly to the local symmetry group $SO(3,1)$, the geometric structure of the internal spaces in which the fundamental particles take their values—$\mathbb{C}^3 \times \mathbb{C}^2 \times \mathbb{C}^1$ endowed with an inner product and orientation—leads directly to the familiar local symmetry group $SU(3) \times SU(2) \times U(1)$ representing the fundamental forces. Any particle field that interacts with a fundamental force has components in the corresponding internal space; as we move from one point of spacetime to another, the standard of constancy for that internal space will dictate the parallel transport of those components.

Thus, I conclude that gauge theory as applied to physics is geometrical in a very strong sense, but its geometry is that of internal vector spaces, not of principal fiber bundles. In light of this conclusion, one could replace Weatherall's (2016, p. 2401) metaphor, that "[p]rincipal bundles are auxiliary [. . .] in the sense in which a coach is auxiliary to the players on the field"[17] with another metaphor, drawn not from sports but from music. Just as all agree that, in their public performances, after arduous preparation, a top-quality orchestra such as the Vienna or Berlin Philharmonic hardly needs the conductor, who is by then almost an epiphenomenon, so also in gauge theories, the vector bundles play all the music and the principal fibre bundle is almost an epiphenomenon.[18]

[17] Because, as he says, "vector bundles represent possible local states of matter; principal bundles coordinate between these vector bundles."

[18] There is a close analog here to the debate between the dynamical and the geometric views on Lorentz invariance; cf. H. Brown (2006) for an extended defense of the dynamical approach, and H. R. Brown and Read (2022) for a recent survey. Roughly, that debate focuses on an order of explanation: Are dynamical laws (locally) Lorentz invariant *because* they at most survey a geometric

Indeed, the new viewpoint achieved in this paper opens up a novel interpretative project for gauge theory as a whole.

To close, here I showed that the geometric interpretation is available for gauge theory *as it appears in the (minimal) SM* without the Higgs (and thus applying only to one generation of particles). Can it be extended to other applications of gauge theory? Not only to encompass the Higgs and right-handed neutrinos, but also supersymmetry and Chern–Simons theories? What about exceptional Lie groups, whose geometric interpretation is much more daunting? Those are questions for another day.

References

Adams, J. Frank. 1996. *Lectures on Exceptional Lie Groups*. Chicago, IL: University of Chicago Press.

Berghofer, Philipp, Jordan François, Simon Friederich, Henrique Gomes, Guy Hetzroni, Axel Maas, and René Sondenheimer. 2023. *Gauge Symmetries, Symmetry Breaking, and Gauge-Invariant Approaches*. Cambridge: Cambridge University Press.

Brown, Harvey R. 2006. *Physical Relativity: Space-Time Structure from a Dynamical Perspective*. Oxford: Oxford University Press.

Brown, Harvey R., and James Read. 2022. "The dynamical approach to spacetime theories." In *The Routledge Companion to Philosophy of Physics*, edited by Eleanor Knox and Alastair Wilson. Abingdon: Routledge.

Coleman, Sideny. 1965. "Fun with SU(3)." In *Proc. Seminar on High-Energy Physics and Elementary Particles*, report number IAEA-STI-PUB-117, 331–52. Vienna: IAEA.

Jacobs, Caspar. 2021. "Symmetries as a guide to the structure of physical quantities." PhD diss., University of Oxford.

Jacobs, Caspar. 2023. "The metaphysics of fibre bundles." *Studies in History and Philosophy of Science* 97:34–43. doi: https://doi.org/10.1016/j.shpsa.2022.11.010

Kaluza, Theodor. 1921. "Zum Unitätsproblem der Physik." *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften (Berlin)* 1921:966–72.

Michor, Peter W. 2008. *Topics in Differential Geometry*. Providence, RI: American Mathematical Society.

Nakahara, Mikio. 2003. *Geometry, Topology and Physics*. Bristol: Institute of Physics.

O'Raifeartaigh, Lochlainn. 1997. *The Dawning of Gauge Theory*. Princeton, NJ: Princeton University Press.

Struyve, Ward. 2011. "Gauge invariant accounts of the Higgs mechanism." *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics* 42 (4):226–36. doi: https://doi.org/10.1016/j.shpsb.2011.06.003

Weatherall, James Owen. 2016. "Fiber bundles, Yang–Mills theory, and general relativity." *Synthese* 193 (8):2389–425.

Zee, Anthony. 2016. *Group Theory in a Nutshell for Physicists*. Princeton, NJ: Princeton University Press.

A. A primer on fiber bundles

A.1. Vector, principal, and associated fiber bundles

Definition 1 (Vector bundle) *A vector bundle* (E, M, V) *consists of a smooth manifold* E *that admits the action of a surjective projection* $\pi_E : E \to M$ *so that any point of* M *has a neighborhood,* $U \subset M$*, such that, for all proper subsets of* U*,* E *is locally of the form*

landscape that is Lorentz invariant, or is such a geometry just a convenient way to codify Lorentz-invariant dynamical laws? Transposing that debate to gauge theory: Since thus far it lacked a comprehensive geometric framework that was on a par with its relativistic cousin, gauge theory might have been more favorable to the dynamical view of symmetries. Now gauge theory finds a natural home within (at least a very close analog of!) the geometric view. And, on the same grain, if either the Higgs mechanism or future developments of the SM cannot be incorporated into the geometric framework, this would count in favor of the dynamical approach.

$\pi^{-1}(U) \simeq U \times V$, where V is a vector space (e.g., $\mathbb{R}^k$ or $\mathbb{C}^k$) which is linearly isomorphic to $\pi^{-1}(x)$, for any $x \in M$.

Note that the isomorphism between $\pi^{-1}(U)$ and $U \times V$ is not unique, which is why there is no canonical identification of elements of fibers over different points of spacetime. Each choice of isomorphism is called a “trivialization” of the bundle.

Definition 2 (A section of E) *A section of E is a map $\kappa : M \to E$ such that $\pi_E \circ \kappa = \mathrm{Id}_M$. We denote the space of smooth sections by $\kappa \in \Gamma(E)$.*

Definition 3 (Principal fiber bundle) *(P, M, G) consists of a smooth manifold P that admits a smooth free action of a (path-connected, semi-simple) Lie group, G, That is, there is a map $G \times P \to P$ with $(g, p) \mapsto g \cdot p$ for some left action $\cdot$ such that, for each $p \in P$, the isotropy group is the identity (i.e., $G_p := \{g \in G | g \cdot p = p\} = \{e\}$). P has a canonical, differentiable, surjective map, called a projection, under the equivalence relation $p \sim g \cdot p$, such that $\pi : P \to P/G \simeq M$, where here $\simeq$ stands for a diffeomorphism.*

It follows from the definition that $\pi^{-1}(x) = \{G \cdot p\}$ for $\pi(p) = x$. Thus there is a diffeomorphism between G and $\pi^{-1}(x)$, fixed by a choice of $p \in \pi^{-1}(x)$. It also follows (more subtly) from the definition that local sections of P exist. Similarly to a section of E, a local section of P over $U \subset M$ is a map, $\sigma : U \to P$, such that $\pi \circ \sigma = \mathrm{Id}_U$. Unlike sections of vector bundles, sections of principal bundles are generally only local.

Definition 4 (Associated vector bundle) *A vector bundle over M with typical fiber V is associated to P with structure group G when*

$$P \times_\rho V = P \times V / \sim, \quad \text{where } (p, v) \sim (gp, \rho(g^{-1})v), \tag{A.1}$$

where $\rho : G \to GL(V)$ is a representation of G on V.

Given any vector bundle (E, M, V), the bundle of frames for E, called $L(E)$, is itself a principal fiber bundle $(L(E), M, GL(V))$; here, elements of $\pi^{-1}(x)$ are linear frames of E_x, and $G \simeq GL(F)$ acts via ρ on the typical fibers. By construction, $E \simeq L(E) \times_\rho V$. If V has more than just the structure of a linear vector space (e.g., if it is endowed with an inner product), then we have a *bundle of admissible frames*, e.g., orthonormal frames. This is also a principal fiber bundle, $(L'(E), M, G)$, whose structure group is a proper subgroup of the general linear group, $G \subset GL(V)$, taken to be the group that preserves the structure of V.

A.2 Connections and parallel transport

Given the tangent bundle to a principal fiber bundle TP, the vertical linear subspace at a given point $V_p \subset T_pP$ is the canonical subspace tangent to the orbits of the group, i.e., $V_p = T_p(\pi^{-1}(x))$. Thus, the group action on P gives a canonical linear isomorphism between the vertical subspace at a point p and the Lie algebra, $\#_p : \mathfrak{g} \to \mathfrak{V}_\mathfrak{p} \subset \mathfrak{T}_\mathfrak{p}\mathfrak{P}$. A vertical projection $\hat{V}$ on TP is a linear operator $\hat{V} : TP \to TP$ such that, for every $p \in P$,

$$\hat{V}_p \circ \hat{V}_p = \hat{V}_p; \quad \mathrm{Im}(\hat{V}_p) = V_p. \tag{A.2}$$

The kernel of this projection defines a *horizontal distribution* of linear subspaces, $H \subset TP$, such that, for each p,

$$H_p \oplus V_p = T_pP; \quad H_{g \cdot p} = g_* H_p \tag{A.3}$$

for $g_* : TP \to TP$ the induced (push-forward) map of the diffeomorphism given by $g : P \to P; p \mapsto g \cdot p$. An *Ehresmann connection* is sometimes taken to be simply the vertical projector, and sometimes taken to be the Lie algebra–valued map resulting from composing the vertical projection with the canonical isomorphism between vertical spaces and the Lie algebra,

$$\omega := \#^{-1} \circ \hat{V}. \tag{A.4}$$

A horizontal lift γ^h through $p \in P$ of a curve in M through $\pi(p) = x \in M$ is the unique curve through p whose tangent is everywhere horizontal and such that $\pi(\gamma^h) = \gamma$. Taking $P = L(E)$, we interpret the horizontal lift of a curve as the parallel transport of a frame at x along γ. Thus, one defines the covariant derivative of a section of E as the rate of change of the section's components in this basis. That is, for $\gamma' \in T_xM$,

$$\mathrm{D}_{\gamma'}\kappa(x) := \left[\gamma^h(0), \frac{d}{dt}|_{t=0} v_\kappa\big(\gamma^h(t)\big)\right], \tag{A.5}$$

where $v_\kappa(p)$ are the components of $\kappa(\pi(p))$ in the basis p, and $\frac{d}{dt}\big|_{t=0} v_\kappa$ acts component by component.

We can also describe covariant derivatives directly in terms of a vector bundle. Given a vector bundle (E, M, V), a covariant derivative D is an operator

$$\mathrm{D} : \Gamma(E) \to \Gamma(T^*M \otimes E) \tag{A.6}$$

such that the product rule

$$\mathrm{D}(f\kappa) = \mathrm{d}f \otimes \kappa + f\mathrm{D}\kappa \tag{A.7}$$

is satisfied for all smooth, real- (or complex-)valued functions $f \in \Gamma(M)$. Call $\mathcal{C}(E)$ the space of covariant derivatives for E. Now let the space of connections over E be defined as

$$\Delta(E) := \Gamma(T^*M \otimes \mathrm{End}(E)), \tag{A.8}$$

where $\mathrm{End}(E)$ are the linear, fiber-preserving endomorphisms of E, isomorphic to $\Gamma(E^* \otimes E)$. Given any $\mathrm{D}_o, \mathrm{D} \in \mathcal{C}(E)$, it is possible to show that there exists a $\omega_\mathrm{D} \in \Delta(E)$ such that $\mathrm{D}_o - \mathrm{D} = \omega_\mathrm{D}$ (here, I will abuse notation and use the same ω used for an Ehresmann connection on P for a connection on E). Therefore, the map

$$\begin{aligned} \Delta(E) &\to \mathcal{C}(E) \\ \omega &\mapsto \mathrm{D}_o - \omega \end{aligned} \tag{A.9}$$

is a bijection, for any choice of D_o; that is, the space of covariant derivatives is an affine space over the vector space of connections. This is why, in any trivialization of E, we can take $\mathrm{D}_o \to \mathrm{d}$, and take connections to parametrize the space of covariant derivatives; it is why the covariant derivatives are described as vector bosons: 1-forms valued on $\mathrm{End}(E)$.

If E is endowed with further structure, say, an inner product, I will require the connection to preserve that structure, so that parallel transport is well-defined within the bundle. This preservation is guaranteed if we characterize covariant derivatives via the principal fiber bundle of admissible frames as described above.

Given a covariant derivative (A.7) and a curve $\gamma \in$ such that $\gamma(0) = x$, where E is the vector bundle and E_x is the fiber over $x \in M$, we define the parallel transport along

γ as a unique linear isomorphism

$$\tau_{\gamma(t)} : E_x \to E_{\gamma(t)} \tag{A.10}$$

such that, given any $X_x \in E_x$,

$$D_{\gamma'}\left(\tau_{\gamma(t)}(X_x)\right) = 0, \tag{A.11}$$

where $\tau_{\gamma(t)}(X_x) \in \Gamma\left(E|_\gamma\right)$, or $\gamma, \gamma' : [0,1] \to M$, with $\gamma(0) = \gamma\prime(0)$ and $\gamma(1) = \gamma'(1) = y$:

$$g \cdot \tau_\gamma = \tau_{\gamma'}, \quad !g \in \mathrm{End}\left(E_y\right) \tag{A.12}$$

If the covariant derivative preserves the structure on the typical fiber (so would correspond to an Ehresmann connection on the bundle of admissible frames, as described below), then in (A.12) we have $g \in \mathrm{Aut}\left(E_y\right) \subset \mathrm{End}\left(E_y\right)$, where $\mathrm{Aut}\left(E_y\right)$ is the group of linear automorphisms that are not only linear (so not only in $\mathrm{End}\left(E_y\right)$) but that preserve the added structure on E_y.

Alternatively, by the composition properties of parallel transport, we can see parallel transport around a closed curve starting at $x \in M$ as an element $g \in \mathrm{Aut}(E_x)$. If we take all the closed curves, this generates a subgroup of $\mathrm{Aut}(E_x)$ called $\mathrm{Hol}_{(x)}(\mathrm{D})$. It can be shown that, on a simply connected region, the holonomy depends on x only up to conjugation by a group element. Thus, it is customary to refer to the path-independent $\mathrm{Hol}(\mathrm{D})$ as the *the holonomy group* $\mathrm{Hol}(\mathrm{D})$. It can also be shown that, given a connection D that is compatible with the typical fiber structure on V, one can find a principal bundle (P, M, G), with a connection ω, such that the holonomy group is isomorphic (as a G-torsor) to the structure group G, and E is an associated bundle to P with D being the induced connection from ω (cf. Michor, 2008, Theorem 17.11).

To relate a covariant derivative given in (A.7) explicitly to the definition in (A.5), take a local section σ for $L(E)$, call it $\{e_i\}$, and represent the covariant derivative directly in terms of this frame. A linear transformation of E_x is an element of $E_x^* \otimes E_x$, and we can describe the extent to which the chosen basis is non-parallel along a certain direction by a 1-form valued on $E \otimes E^*$, which we write as

$$\omega^\sigma = \omega^{\sigma}{}^j_i \otimes e^i \otimes e_j, \tag{A.13}$$

where $\omega_i^j \in \Gamma(T^*U)$ are 1-forms on the space of vectors of M. Thus, for $X \in (T_x M)$,

$$\mathrm{D}_X e_j = \omega^{\sigma}{}^i_j(X) e_i. \tag{A.14}$$

Now, for some section of the real (or complex) vector bundle $\kappa \in \Gamma(E)$, we locally write $\kappa = \kappa^i e_i$, and the covariant derivative of κ becomes:

$$\mathrm{D}\kappa = d\kappa^j \otimes e_j + \kappa^i \omega^{\sigma}{}^j_i \otimes e_j. \tag{A.15}$$

Of course, under a change of frame, ω^σ given in (A.1.8) will transform in the familiar, inhomogeneous form (see (A.17) below). This change of frame gives a passive interpretation of gauge transformations. But we can formulate the corresponding active interpretation in terms of $\Delta(E)$ by considering two fibre-wise linearly isomorphic vector bundles, E, E', over M.

Two connections in two linearly isomorphic vector bundles are equivalent if they are related by conjugation by the linear isomorphism (here a diffeomorphism

$f : E \to E'$ such that $\pi_E \circ f = \pi'$, where f takes $\pi_E^{-1}(x) \to \pi_{E'}^{-1}(x)$ by a linear isomorphism). This relation guarantees that the following diagram commutes (for all $X \in \Gamma(TM)$):

$$\begin{array}{ccc} \Gamma(E) & \xrightarrow{\mathrm{D}_X} & \Gamma(E) \\ f \downarrow & & \downarrow f \\ \Gamma(E') & \xrightarrow[\tilde{D}x]{} & \Gamma(E\prime) \end{array}$$

Thus, we can represent the connection D under a bundle isomorphism, obtaining a new connection

$$\tilde{\mathrm{D}}_X(s) = f\mathrm{D}_X(f^{-1}s) \Rightarrow \tilde{\mathrm{D}}_X = f\mathrm{D}_X f^{-1} \tag{A.16}$$

or, equivalently, $f\mathrm{D}_X = \tilde{\mathrm{D}}_X f$. And of course, if D is related to ω and $\tilde{\mathrm{D}}$ is related to $\tilde{\omega}$ then the relationship between ω and $\tilde{\omega}$ is the familiar inhomogeneous one, as I will now show.

Over $\pi_E^{-1}(U) = \pi_{E\prime}^{-1}(U)$, the domain of a trivialization, we can set $\tilde{\mathrm{D}}_o = \mathrm{d}$, obtaining $f\mathrm{D}_X = \tilde{\mathrm{D}}_X f$. For any fixed choice of frame of E and $E\prime$,

$$f\mathrm{D}_X e_i = f\omega_i^k(X)e_k = \omega_i^k(X)f_k^j e_j = \left(\omega_i^l(X)f_l^k\right)e_k,$$

$$\tilde{\mathrm{D}}_X(fe_i) = \tilde{\mathrm{D}}_X\left(f_i^j e\right) = \left(df_i^k + f_i^j\tilde{\omega}_j^k(X)\right)e_k,$$

$$\therefore \quad df_i^k + f_i^j\tilde{\omega}_j^k(X) = \omega_i^l(X)f_l^k,$$

valid for all $X \in \Gamma\left(TM_{|}U\right)$. We then obtain

$$\tilde{\omega} = (df)f^{-1} + f\omega f^{-1}. \tag{A.17}$$

Cite this article: Gomes, Henrique. 2025. "Gauge Theory Without Principal Fiber Bundles." *Philosophy of Science* 92 (3):511–527. https://doi.org/10.1017/psa.2024.49