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RECTANGULARITY VERSUS PIECEWISE RECTANGULARITY OF PRODUCT SPACES

BY
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Dedicated to my father, Professor Mitsuru Tsuda on the occasion of his 60th birthday.

ABSTRACT. We shall discuss relations between rectangularity and piecewise rectangularity of product spaces. In particular, we show that for each positive integer n there exists an n-dimensional, collectionwise normal, non-piecewise rectangular product $X \times Y$ which satisfies the inequality $\dim (X \times Y) \leqq \dim X + \dim Y$.

0. **Definitions**. A subset of the product space $X \times Y$ is said to be (*piecewise*) *cozero rectangular* if it is (a closed and open subset of the set) of the form $U \times V$, where U and V are cozero sets of X and Y, respectively. The product space $X \times Y$ is said to be (*piecewise*) *rectangular* if any finite cozero cover of it has a σ-locally finite refinement consisting of (piecewise) cozero rectangular subsets [8, 12, 13, 14, 15]. All spaces in this note are assumed to be Tychonoff. By the dimension $\dim X$ of a space X we mean the covering dimension of it [7]. In particular, we say that X is *strongly zero-dimensional* when $\dim X = 0$. For the undefined terminology refer to [6, 7].

1. **Introduction**. In [12] Pasynkov introduced the notion of a rectangular product and announced that the following inequality is valid for every rectangular product (see [14] for precise proof).

$$(*) \dim (X \times Y) \leqq \dim X + \dim Y.$$

The Pasynkov's theorem is relatively strong, but it is known that

(i) there exist *non-rectangular strongly zero-dimensional* products which satisfy the inequality (*) [9, 11, 19, 21, 22].

Moreover, Ohta [11] showed a *machine* to produce normal non-rectangular products $X \times Y$ which satisfy the inequality (*) for every normal, non-paracompact (not necessarily strongly zero-dimensional) space X.

In [15] Pasynkov extended his result for every piecewise rectangular product (see [16] for its detailed proof). One of the remarkable consequences from it is that the

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piecewise rectangularity of products is a necessary and sufficient condition for the validity of the inequality (*) when the factor spaces are *strongly zero-dimensional*. Hence, all the examples in (i) are included in the class of piecewise rectangular products.

In this note we construct at first the following example, using the method due to Ohta.

EXAMPLE 1. *There exists a one-dimensional, countably paracompact, collectionwise normal, non-piecewise rectangular product which satisfies the inequality* (*).

Next, we construct higher dimensional, connected ones.

EXAMPLE 2. *For every pair of positive integers* (m, n) *there exist an n-dimensional connected countably compact normal space* X_n *and an m-dimensional connected stratifiable space* S_m *such that* $X_n \times S_m$ *is* $(m + n)$*-dimensional, countably paracompact, collectionwise normal, and non-piecewise rectangular.*

2. **Preliminary lemmas**. We begin with the following easy but useful lemma (for its proof see [9]).

LEMMA 0. *A product space is (piecewise) rectangular if and only if every cozero set of it is a* σ*-locally finite union of (piecewise) cozero rectangular sets.*

We need also the following result due to Terasawa [2, Lemma 1].

LEMMA 1. *Every product with a compact factor is rectangular.*

Next, we show a lemma which will be used in the final section.

LEMMA 2. *Let* $X \times Y$ *be strongly zero-dimensional. Then, for every compact spaces* K *and* C, $S \times T$ *is piecewise rectangular, where* $S = X \times K$ *and* $T = Y \times C$.

PROOF. Let G be a cozero set in $S \times T$. Then, using Lemma 1 twice (to the products $(X \times Y) \times K$ and $(X \times Y \times K) \times C$) we obtain a σ-locally finite collection $\mathcal{U} = \{U \times V \times W\}$, where U, V, and W are cozero sets in $X \times Y$, K and C, respectively, whose union is equal to the set G. Since $X \times Y$ is strongly zero-dimensional, the set U is the union of countably many closed and open sets U_i of $X \times Y$ [22, Theorem 1]. Then, since $U_i \times V \times W$ is a closed and open subset of the rectangular cozero set $(X \times V) \times (Y \times W)$ in $S \times T$, the set G is a σ-locally finite union of piecewise cozero rectangular sets. Hence, $S \times T$ is piecewise rectangular by Lemma 0. This completes the proof.

Finally, we show a lemma which gives a sufficient condition for the coincidence of rectangularity and piecewise rectangularity.

LEMMA 3. *Let* $X \times Y$ *be locally connected. then,* $X \times Y$ *is rectangular if and only if it is piecewise rectangular.*

PROOF. It suffices to see "if" part. Let G be an arbitrary cozero set in the product space $X \times Y$. Since it is piecewise rectangular, there exists a σ-locally finite collection

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U whose union is equal to the set G, and each $U \in \mathcal{U}$ is a closed and open subset of some cozero rectangular set $V \times W$. We show that each U is a union of a σ-locally finite family of cozero rectangular sets. Indeed, since both of X and Y are locally connected, V and W are topological sums of its connected components $\{V_\alpha\}$ and $\{W_\beta\}$, respectively. (Note that $V_\alpha \times W_\beta$ is closed and open in V and W, and hence is a cozero set in $X \times Y$.) Hence, the set U is a sum of some subcollection of $\{V_\alpha \times W_\beta\}$, since each $V_\alpha \times W_\beta$ is connected and U is closed and open. Because V and W are cozero sets, there exist two collections of countably many cozero sets $\{V_i\}$ and $\{W_i\}$ such that

$$\bar{V}_i \subset V_{i+1} \subset V,\ \bar{W}_i \subset W_{i+1} \subset W,\ V = \cup V_i, \text{ and } W = \cup W_i.$$

Then,

$$\{(V_\alpha \times W_\beta) \cap (V_i \times W_i) : V_\alpha \times W_\beta \subset U\}_{i=1}^{\infty}$$

is a σ-locally finite family of cozero rectangular collection whose union is the set U. Since G is the union of $\mathcal{U}$ and $\mathcal{U}$ is σ-locally finite, the set G is also a union of a σ-locally finite family of cozero rectangular sets. This completes the proof by Lemma 0.

3. **Examples**. At first we construct Example 1.

(a) **The factor space** X. Let X be the well known Long line [20, p. 71]. Then, it is known that X is non-paracompact, countably compact, normal, connected, and locally connected of weight $w(X) = \omega_1$.

(b) **The factor space** Y. Let Y_0 be the set of all points in the ω_1 fold Tychonoff product of unit intervals I^{ω_1} consisting of points whose all but finitely many coordinates are equal to zero. (Note that the cardinality of Y_0 is ω_1.) Let y_0 be the point of I^{ω_1} whose all coordinates are equal to 1. Put

$$Y = Y_0 \cup \{y_0\}$$

$\mathcal{U}$ with the topology in which all points in Y_0 are isolated, and the neighborhoods of y_0 are the same as in the relative topology of I^{ω_1}. (In other words, the Hannerization $Y_{\{y_0\}}$ as in [6, Example 5.1.22].) Then, it is known [3, 4, 11] that Y is a σ-discrete stratifiable space and

(0) The point y_0 has a closure-preserving base $\mathcal{B}$ which is locally finite at every point of Y_0.

(c) **The product space** $X \times Y$. It is known [11, Theorem 1] that $X \times Y$ is collectionwise normal, non-rectangular. Since Y is σ-locally compact paracompact, the product satisfies the inequality (*) by [10, Theorem 1]. Moreover, it is one-dimensional, since it contains a unit interval. We shall show that it is also non-piecewise rectangular. (Note that it is *countably paracompact*, since X is countably compact and (0) holds.) It suffices to see that

(1) if $X \times Y$ is assumed to be piecewise rectangular, then $X \times Y$ is rectangular.

Suppose that the product is piecewise rectangular. Then, for any cozero set G of it there exists a σ-locally finite collection $\mathcal{U}$ whose union is equal to the set G and for each $U \in \mathcal{U}$ there exist cozero sets V and W such that U is closed and open in V and W. Since X is locally connected, V is the topological sum of its connected components $\{V_\lambda\}$. We show that

(2) if $U \cap (V_\lambda \times \{y_0\}) \neq \emptyset$, then $U \supset V_\lambda \times B$ for some neighborhood $B \subset W$ of y_0.

Indeed, since V_λ is connected and U is closed and open in $V \times W$, $U \supset V_\lambda \times \{y_0\}$ if $U \cap (V_\lambda \times \{y_0\} \neq \emptyset$. Then, for some $v \in V_\lambda$ take a neighborhood $B \subset W$ of y_0 such that $\{v\} \times B \subset U$. Since U is closed and open and V_λ is connected, (2) holds for this B. Hence, for each V_λ there exists a closed and open set B_λ such that $y_0 \in B$, $V_\lambda \times B_\lambda \subset U$. Therefore, by the proof of Lemma 3 there exists a σ-locally finite rectangular cozero collection $\mathcal{G}$ such that

$$G \cap (X \times \{y_0\}) \subset \cup \mathcal{G} \subset G.$$

Since $Y_0 = Y \backslash \{y_0\}$ is σ-discrete, the remaining set $G \cap (X \times Y_0)$ is a union of σ-locally finite cozero rectangular sets. Hence, (1) holds, and this completes the proof.

Next, we construct Example 2.

(a) **The factor space** X_n. Let X be the Long line (that is the space X in Example 1). Put $X = \cup L_\alpha$, $L_\alpha \subset L_\beta$ for any $\alpha < \beta < \omega_1$, and each L_α is homeomorphic to the unit interval. Put $X_n = X \times I^{n-1}$, where I^k is the k-dimensional cube (I^0 is the one point set). Then, it is easy to see that X_n is n-dimensional, countably compact, normal, connected, and locally connected.

(b) **The factor space** S_m. At first, we enlarge the factor space Y in Example 1 to a space S. Let $J(\omega_1)$ be the hedgehog space of weight ω_1 [6, Example 4.1.5]. Let A be the set consisting of the origin o and all the end points 1_α of each segment I_α of it. (Note that $J(\omega_1) = \cup \{I_\alpha : \alpha < \omega_1\}$.) Then, A is a closed discrete subset of cardinality ω_1 in the metric space $J(\omega_1)$. Let $f: A \to Y$ be a *bijection* satisfying $f(o) = y_0$. Let S be the underlying set of the *adjunction space* of $J(\omega_1)$ and Y with respect to f. Then, by the definition of S, we may think of $Y \subset S = J(\omega_1)$ *as a set*. We define a topology on S as follows. Each point except y_0 has the same neighborhoods as in the ordinary adjunction topology (see [1, Definition 6.1]). We alter the topology only for the point y_0, so that our product space is *normal*. Namely, the point $y_0 = f(o)$ has the following collection $\mathcal{S}$ as its neighborhood base. Let $\{G_i\}$ be a countable *connected* neighborhood base of o in $J(\omega_1)$. For each $B \in \mathcal{B}$ and an integer i let

$$S(B) = \cup \{I_\alpha : f(1_\alpha) \in B\}, \text{ and } S(B, i) = S(B) \cup G_i.$$

Then, put

$$\mathcal{S} = \{S(B, i) : B \in \mathcal{B}, i \in \omega\}.$$

Note that S contains Y topologically as a closed set. Put

$$S_m = S \times I^{m-1}.$$

Then, it is easy to see that S_m is a connected, locally connected, stratifiable space (note that the collection $\mathcal{S}$ is closure-preserving, and that $S\backslash\{y_0\}$ is an F_σ-set by (0) and the definition of $\mathcal{S}$).

(c) **The product space** $X_n \times S_m$. Since S_m is σ-locally compact paracompact, the product satisfies the inequality (*) by [10, Theorem 1]. Moreover, it is $(m + n)$ -dimensional, since it contains $(m + n)$-dimensional cube. We shall show at first that $X \times S$ is *normal*. The basic idea of the proof is due to [5, Example 2]. We begin with the precise definition of the base $\mathcal{B}$ in (0). For a finite set F of ω_1, put

$$B(F) = \pi_F^{-1}(1_F),$$

where $\pi_F : Y \to I^F$ is the restriction of the natural projection from I^{ω_1} into $|F|$-dimensional factor I^F, and 1_F is the point of I^F whose all coordinates are equal to 1. Then

$$\mathcal{B} = \{B(F) : F \text{ is a finite subset of } \omega_1\}$$

is a closure-preserving neighborhood base of y_0 in Y, and it is locally finite in Y_0 [2, 5, 11]. Next, we show that

(3) for every open set $V \supset Z = X \times \{y_0\}$ there exists an open set $G \supset Z$ such that $\bar{G} \subset V$.

For each finite set $F \subset \omega_1$ let

$$V_F = \cup \{W : W \text{ is open in } X \text{ and } W \times S(B(F)) \subset V\}.$$

Then, we define a finite set $F_\alpha \subset \omega_1$ for each $\alpha < \omega_1$ inductively as follows.

(4) $L_\alpha \subset V_{F_\alpha}$, and $F_\alpha \neq F_\beta$ for any $\alpha < \beta$.

Indeed, it is possible, since L_α is compact, and the set $\{F_\beta : \beta < \alpha\}$ is countable for each α. For each V_α put

$$K_F = L_\alpha \text{ if } F = F_\alpha, \text{ and } K_F = \phi \text{ otherwise.}$$

Then, by (4) K_F is well-defined, and $\{K_F\}$ covers X. Since X is normal, for each V_F of non-empty K_F take an open set G_F such that

$$K_F \subset G_F \subset \overline{G_F} \subset V_F.$$

Then, put

$$\mathcal{H} = \{G_F \times S(B(F) : K_F \text{ is non-empty}\}, \text{ and put } H = \cup \mathcal{H}.$$

Then, since $\mathcal{B}$ is locally finite in Y_0, $\mathcal{H}$ is also locally finite, and hence is closure-preserving. Therefore, $Z \subset H \subset \bar{H} \subset V$. For each integer i let

$$V_i = \cup \{W : W \text{ is open in } X \text{ and } W \times G_i \subset V\}.$$

Then, $V_i \subset V_{i+1}$ and $\{V_i\}$ is a countable open cover of X. Since X is countably

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compact, there exists an integer i such that $X = V_{i-1}$. Hence, $X \times \bar{G}_i \subset X \times G_{i-1} \subset V$. Put

$$G = \cup \{(G_F \times S(B(F), i) : K_F \neq \emptyset\}.$$

Then, G is an open neighborhood of Z, and

$$\bar{G} = \overline{\cup \mathcal{H}} \cup \overline{(\cup (G_F \times G_i))} \subset \bar{H} \cup (X \times \bar{G}_i) \subset V.$$

Hence, (3) holds. Now, we show that $X \times S$ is normal. Let A and B be disjoint closed sets in $X \times S$. Then, we show at first that, using (3), there exist disjoint open sets U_0 and V_0 such that

(5) $U_0 \supset A \cap Z$, $V_0 \supset B \cap Z$, and $\bar{U}_0 \cap \bar{V}_0 = \emptyset$.

Indeed, since X is normal and Z is homeomorphic to X, the exists an open set W in X such that

$$W \times \{y_0\} \supset A \cap Z, \text{ and } B \cap (\bar{W} \times \{y_0\}) = \emptyset.$$

Put

$$H = W \times S, \text{ and } V = X \times S \backslash (\bar{H} \cap B).$$

Then, V is an open neighborhood of Z, and hence we can apply (3) so that there exists an open neighborhood G of Z such that $\bar{G} \subset V$. Put $U_0 = H \cap G$. Then $\bar{U}_0 \cap B = \emptyset$. Hence, (5) holds, since we can obtain V_0 in a parallel way, using disjoint closed sets $\bar{U}_0$ and B. Since $S \backslash \{y_0\}$ is a σ-locally compact *metric* space, there exist disjoint open sets U_1 and V_1 in $X \times S \backslash Z = X \times (S \backslash \{y_0\})$ (hence, also open in $X \times S$) such that $U_1 \supset (A \cup \bar{U}_0) \backslash Z$, and $V_1 \supset (B \cup \bar{V}_0) \backslash Z$. Then, we obtain disjoint open neighborhoods $U_0 \cup U_1$ and $V_0 \cup V_1$ of closed sets A and B, respectively. This completes the proof that $X \times S$ is normal. It is not difficult to see that $X \times S$ is *collectionwise normal* in a parallel way to the above proof (cf. [11]). Because the above proof for the normality of $X \times S$ is valid for every countably compact space X which is the union of ω_1 compact subspaces, $X_n \times S_m$ is also *collectionwise normal*, since $X_n \times S_m$ is the product of S and a countably compact space $X \times I^{m+n-2}$.

Our space $X_n \times S_m$ is *not* rectangular by [11, Theorem 3], since $X_n \times S_m$ is normal, X_n is not paracompact, and S_m contains a ω_1-cofinal point $(y_0, 1, \ldots, 1)$ (see [11, Definition] for the definition of ω_1-cofinal point). Hence, $X_n \times S_m$ is *not piecewise rectangular* either by Lemma 3, since it is locally connected. This completes the proof.

4. **Concluding remarks**.

REMARK 1. The *use* of the space Y in Examples 1 and 2 for producing non-rectangular products is due to Ohta [11]. The *discovery* of Y goes back to Mr. and Mrs. Chiba [3], and it was established that the space Y has many interesting properties [3, 4, 5].

REMARK 2. One can show without difficulty, in a parallel way to Example 1, that the product space of the spaces $X \times I^{n-1}$ and $Y \times I^{m-1}$ for X and Y in Example 1 is non-piecewise rectangular, either. But, in this case the product space is neither connected nor locally connected.

REMARK 3. It is known that there exist normal non-piecewise rectangular product for every dimension which *do not* satisfy the inequality (*) [17, 23, 24, 25, 26].

REMARK 4. The proof of the properties of our example leads to a problem whether or not every piecewise rectangular product is rectangular *when it is non-strongly zero-dimensional*. We can show, however, that it is not the case: for every pair of non-negative integers (m, n) there exists an $(m + n)$-dimensional collectionwise normal, *non*-rectangular product which *is* piecewise rectangular. Indeed, let $X = [0, \omega_1)$, and Y be the space in Remark 1. Then, put $S = X \times I^{n-1}$ and $T = Y \times I^{m-1}$. Then, $S \times T$ is normal, and $S \times T$ is non-rectangular, since T has a ω_1-cofinal point. On the other hand, by Lemma 2 the product $S \times T$ is piecewise rectangular, since $X \times Y$ is strongly zero-dimensional.

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