Can. J. Math., Vol. XXXVII, No. 5, 1985, pp. 872-892

SINGULARITY OF MONOMIAL CURVES IN A^3 AND GORENSTEIN MONOMIAL CURVES IN A^4

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Let $2 \leqq s \in \mathbf{N}$ and $\{n_1, \dots, n_s\} \subseteq \mathbf{N}^*$. In 1884, J. Sylvester **[13]** published the following well-known result on the singularity degree δ of the monomial curve whose corresponding semigroup is $S: = \langle n_1, \dots, n_s \rangle$: If $s = 2$, then

$$\delta = \frac{1}{2}(n_1 - 1)(n_2 - 1).$$

Let $K: = -\mathbf{Z}\backslash S$ and

$$a_i: = \min\{a \in \mathbf{N}^* | a n_i \in \sum_{j \neq i} \mathbf{N} \cdot n_j\}$$

for all $1 \leqq i \leqq s$. We introduce the invariant

$$\kappa: = \text{card } K\backslash S - \text{card } S\backslash K = 2\delta - 1$$

of S involving a correction term to the Milnor number 2δ **[4]** of S. As a modified version and extension of Sylvester's result to all monomial space curves, we prove the following theorem: If $s = 3$, then

$$\kappa = (a_1 - 1)n_1 + (a_2 - 1)n_2 + (a_3 - 1)n_3 - a_1 a_2 a_3.$$

We prove similar formulas for $s = 4$ if S is symmetric.

0. Basic invariants of monomial curves. Let A be a field and B a monomial curve over A in $\mathbf{A}^s$, $s \in \mathbf{N}^*$; that is, there exists a set $\{n_1, \dots, n_s\} \subseteq \mathbf{N}^*$ with $\gcd(n_1, \dots, n_s) = 1$ such that

$$B \cong A[[X_1, \dots, X_s]]/\mathfrak{P},$$

where

$$\mathfrak{P}: = \ker(A[X_1, \dots, X_s] \to C: = A[[S]])$$
$$X_i \mapsto t^{n_i},$$

and S is the numerical semigroup

Received December 23, 1983. This work, which was supported by the Studienstiftung des deutschen Volkes and Purdue University, is included in the author's 1983 Ph.D. thesis.

$$\langle n_1, \ldots, n_s \rangle : = \mathbf{N}n_1 + \ldots + \mathbf{N}n_s.$$

Let Φ denote the isomorphism

$$\text{Quot } B \xrightarrow{\sim} \text{Quot } C$$

induced by $B \xrightarrow{\sim} C$. As usual, we call

$$d: = d(B) = d(S) = \mu(\mathfrak{P}) - \text{codim } \mathfrak{P}$$

the deviation of B (*of* S), and

$$m: = m(B) = m(S) = \min\{n_1, \ldots, n_s\}$$

the multiplicity of B (*of* S).

If $I \subseteq \mathbf{Z}$, then I is called a *fractional* S*-ideal* if and only if $I \neq \emptyset$ and $S + I \subseteq I$ (cf. **[8]**). For a fractional S-ideal I let

$$G(I) = I \setminus (M + I)$$

denote the (unique) minimal system of generators of I and define

$$\mu(I): = \text{card } G(I).$$

Further let

$$I - J: = \{z \in \mathbf{Z} | z + J \subseteq I\}$$

for a fractional S-ideal J, $I^{-1}: = S - I$, and $I^{\vee}: = -\mathbf{Z} \setminus I$.

Fundamental fractional S-ideals are S, *the maximal ideal* $M: = M(S): = S \setminus \{0\}$, M^{-1}, and the *canonical ideal* $K: = K(S): = S^{\vee}$. Let

$$\begin{aligned} r: = r(B) = r(S) &= \dim_A \mathfrak{m}_B^{-1}/B \\ &= \dim_A \mathfrak{m}_B^{-1}/(\mathfrak{m}_B^{-1} \cap B) - \dim_A B/(B \cap \mathfrak{m}_B^{-1}) \end{aligned}$$

denote *the type of* B (*of* S). The canonical ideal

$$\Phi^{-1}\left(\sum_{x \in K} At^x\right)$$

of B we will shortly denote by $\mathfrak{k}$.

Here we will also be interested in the invariant

$$\begin{aligned} \kappa: = \kappa(B): = \kappa(S): &= \dim_A \mathfrak{k}/(\mathfrak{k} \cap B) - \dim_A B/(B \cap \mathfrak{k}) \\ &= \text{card } K \setminus S - 1 \end{aligned}$$

of B (of S), which we will use as a measure of the singularity of B. Note that $\kappa + 1$ is the Milnor number **[4]** and $\frac{1}{2}(\kappa + 1)$ is the singularity degree of B. We have

$$r(B) = 1 \Leftrightarrow B \text{ is Gorenstein},$$
$$d(B) = 0 \Leftrightarrow B \text{ is a complete intersection, and}$$
$$\kappa(B) = -1 \Leftrightarrow B \text{ is regular.}$$

For us, the study of the singularity of B is the study of $\mathfrak{k}/(\mathfrak{k} \cap B)$ and the computation of $\kappa(B)$ in terms of a minimal system of generators of $\mathfrak{P}$. By computing a basis of $\mathfrak{m}_B^{-1}/B$, we will be able to achieve our goal in case $s = 3$ (Section 1) and in case $s = 4$ if B is Gorenstein (Section 2). Let

$$z^{<} := \min M^{-1}\backslash S \quad \text{and} \quad z^{>} := \max M^{-1}\backslash S.$$

Denote $z^{<}$ and $z^{>}$ by z if B is Gorenstein. Note that $z^{>} + 1$ is the conductor of S.

Proposition 1. *If $S \subset \mathbf{N}$, then $M^{-1} \subseteq \mathbf{N}$ is also a numerical semigroup. Moreover, S is of the form $\widetilde{M}^{-1}$ for some numerical subsemigroup $\widetilde{S}$ of S, which, if $S \subset \mathbf{N}$, can be chosen to have the same multiplicity as S.*

Proof. Let $S \subset \mathbf{N}$ and take $\widetilde{S} := \{0\} \cup (S + m) \subseteq S$. Then

$$\widetilde{M}^{-1} = \{z \in \mathbf{Z} | z + (S + m) \subseteq \widetilde{S}\} \supseteq S.$$

On the other hand $z \in \widetilde{M}^{-1}$ implies $z + m = 0$ or $z + m = s + m$ for some $s \in S$, and hence $z \in S$. But $z = -m < 0$ is not possible, because $\widetilde{M}^{-1} \subseteq \mathbf{N}$. Therefore $S = \widetilde{M}^{-1}$.

Proposition 2. $G(K) = -M^{-1}\backslash S$.

Proof.

$$G(K) = \{-y | y \in \mathbf{Z}\backslash S \text{ and there does not exist an } s \in M \text{ such that } -y = s + (-x) \text{ for some } x \in \mathbf{Z}\backslash S\}.$$

Hence

$$-G(K) = \{y \in \mathbf{Z}\backslash S | y + s \in S \text{ for all } s \in M\} = M^{-1}\backslash S.$$

Corollary. $\mathbf{Z}\backslash S = \bigcup_{y \in M^{-1}\backslash S} \{y - s | s \in S\}.$

In particular, $\mu(K) = 1$ if and only if S is symmetric (cf. **[6]** for the definition of a symmetric numerical semigroup).

If char $A = 0$, then using a theorem of Seidenberg **[12]**, one has

$$\mathfrak{m}_B^{-1} \cong \operatorname{Der}_A B \xrightarrow{\sim} \operatorname{Der}_A C = \sum_{d \in M^{-1}} At^{d+1}\frac{\partial}{\partial t},$$

$t^{d+1}\frac{\partial}{\partial t}$ being homogeneous of degree d. If $S \subset \mathbf{N}$, then $M^{-1} \subseteq \mathbf{N}$, so that the scalar multiples of the Euler derivation $t\frac{\partial}{\partial t}$ of C are the only homogeneous derivations of C of weight $\leqq 0$ (**[10]**, also **[14]**; concerning this, J. Wahl **[14]** proves a theorem for surfaces). We have

$$\operatorname{Der}_A C = \sum_{d \in G(M^{-1})} Ct^{d+1}\frac{\partial}{\partial t},$$

where $\left\{ t^{d+1}\frac{\partial}{\partial t} \middle| d \in G(M^{-1}) \right\}$ is a minimal system of generators of $\operatorname{Der}_A C$, and hence

$$\mu(\operatorname{Der}_A C) = \mu(M^{-1}).$$

Further

$$\left[t^{d+1}\frac{\partial}{\partial t}, t^{d'+1}\frac{\partial}{\partial t} \right] = (d + d')t^{d+d'+1}\frac{\partial}{\partial t} \quad \text{for all } d, d' \in M^{-1}.$$

In particular

$$\left[t\frac{\partial}{\partial t}, t^{d+1}\frac{\partial}{\partial t} \right] = dt^{d+1}\frac{\partial}{\partial t} \quad \text{for all } d \in M^{-1}.$$

If $S \subset \mathbf{N}$, then Proposition 2 shows

$$G(M^{-1}) = \{0\} \cup M^{-1}\backslash S = -G(K) \cup \{0\}$$

and hence $\mu(M^{-1}) = r + 1$, and as 0 corresponds to $t\frac{\partial}{\partial t}$, this means that the minimal generators of M^{-1} (the Euler derivation taken out) are reflected to the minimal generators of the canonical ideal K. This illustrates the distinguished role the Euler derivation plays among the elements of a minimal system of generators of $\operatorname{Der}_A B$.

PROPOSITION 3. *S is symmetric if and only if $z^< = z^>$, that is, $\mu(M^{-1}) \leqq 2$. In this case*

$$\operatorname{Der}_A C = Ct\frac{\partial}{\partial t} + Ct^{z+1}\frac{\partial}{\partial t} = Ct\frac{\partial}{\partial t} + At^{z+1}\frac{\partial}{\partial t}$$

if char $A = 0$.

Proof. "$\Rightarrow$" is true, because $z^> - z^< \in M$ implies

$$z^> = z^< + (z^> - z^<) \in S,$$

and "$\Leftarrow$" because of the corollary (cf. **[6]**, **[10]**).

1. Singularity of monomial curves in $\mathbf{A}^3$. For the computation of a basis of $\mathfrak{m}_B^{-1}/B$, one needs the important invariants of S

$$a_i: = \min\{a \in \mathbf{N}^* | an_i \in \sum_{\substack{j=1\\ j\neq i}}^{s} \mathbf{N} \cdot n_j\}, \quad 1 \leqq i \leqq s,$$

for $s > 1$, which were introduced by S. Johnson **[9]**. (Let $a_1: = 1$ if $s = 1$.) Theorems 1, 2, and 3 were essentially known to him; however, he did not take the ideal-theoretic point of view and did not distinguish clearly the cases I and II of the following theorem. This was later done by J. Herzog **[6]**.

THEOREM 1 (Relations of monomial curves in $\mathbf{A}^3$). *If* $s = 3$, *then precisely one of the following two cases does occur*:

(I) $\mathfrak{P} = (X_i^{a_i} - X_j^{a_j}, X_k^{a_k} - X_i^{a_{ki}}X_j^{a_{kj}})$

for some $(i, j, k) \in S_3$ *and some* $a_{ki}, a_{kj} \in \mathbf{N}$; $d = 0$.

(II) $\mathfrak{P} = (X_1^{a_1} - X_2^{a_{12}}X_3^{a_{13}}, X_2^{a_2} - X_3^{a_{23}}X_1^{a_{21}}, X_3^{a_3} - X_1^{a_{31}}X_2^{a_{32}})$

with unique $a_{12}, a_{13}, a_{23}, a_{21}, a_{31}, a_{32} \in \mathbf{N}^*$ *having the property*

$$a_i = a_{ji} + a_{ki} \quad \textit{for all } (i, j, k) \in S_3; \ d = 1.$$

Definition. Let $\mathfrak{J}$ be a graded fractional B-ideal and $x \in \mathfrak{J}$ a fractional monomial in $x_1, \ldots, x_s$. Then we will call the unique basis $\omega_x(\mathfrak{J})$ of $\mathfrak{J}/Bx$, consisting of residue classes of fractional monomials in $x_1, \ldots, x_s$, *the Apéry-basis of* $\mathfrak{J}$ *with respect to* x. Let $\omega_x: = \omega_x(B)$ for all $x \in B$.

THEOREM 2 (Weights of monomial curves in $\mathbf{A}^3$). *Let* $s = 3$. *Then in case*

(I) $n_i = a_j a_k, n_j = a_k a_i$, *and* $n_k = a_i a_{kj} + a_{ki} a_j$,

(II) $n_i = a_j a_k - a_{kj} a_{jk}$ *for all* $(i, j, k) \in S_3$.

Proof. The Apéry-bases of B with respect to $x_1, \ldots, x_s$ look as follows.

In case I one has

$$A[[X_h, X_k]]/(X_h^{a_h}, X_k^{a_k}) \cong B/Bx_g \xrightarrow{\sim} A[[S]]/(t^{n_g})$$

for all $\{g, h\} = \{i, j\}$. Hence

$$\omega_{x_g} = \{(x_h^{\beta} x_k^{\gamma})^- | 0 \leqq \beta \leqq a_h - 1 \text{ and } 0 \leqq \gamma \leqq a_k - 1\}$$

for all $\{g, h\} = \{i, j\}$, and therefore $n_i = a_j a_k$ and $n_j = a_k a_i$ (and $n_k = a_{ki} a_j + a_{kj} a_i$).

As for ω_{x_k}, note that

$$A[[X_i, X_j]]/(X_i^{a_i} - X_j^{a_j}, X_i^{a_{ki}} X_j^{a_{kj}}) \cong B/Bx_k \xrightarrow{\sim} A[[S]]/(t^{n_k})$$

and hence

$$\omega_{x_k} = \{ (x_i^\alpha x_j^\beta)^- | \, 0 \leqq \alpha \leqq a_i - 1 \text{ and } 0 \leqq \beta \leqq a_{kj} - 1\} \cup \{ (x_i^\alpha x_j^\beta)^- | \, 0 \leqq \alpha \leqq a_{ki} - 1 \text{ and } a_{kj} \leqq \beta \leqq a_j + a_{kj} - 1\}.$$

Therefore $n_k = a_i a_{kj} + a_{ki} a_j$.

In case II one has

$$A[[X_j, X_k]]/(X_j^{a_{ij}} X_k^{a_{ik}}, X_j^{a_j}, X_k^{a_k}) \rightrightarrows A[[S]]/(t^{n_i})$$

for all $(i, j\, k) \in S_3$. Hence

$$\omega_{x_i} = \{ (x_j^\beta x_k^\gamma)^- | \, 0 \leqq \beta \leqq a_{ij} - 1 \text{ and } 0 \leqq \gamma \leqq a_{ik} - 1\} \cup \{ (x_j^\beta x_k^\gamma)^- | \, 0 \leqq \beta \leqq a_{ij} - 1 \text{ and } a_{ik} \leqq \gamma \leqq a_k - 1\} \cup \{ (x_j^\beta x_k^\gamma)^- | \, a_{ij} \leqq \beta \leqq a_j - 1 \text{ and } 0 \leqq \gamma \leqq a_{ik} - 1\}$$

for all $(i, j, k) \in S_3$, and therefore

$$n_i = a_j a_{ik} + a_{ij} a_k - a_{ij} a_{ik} \quad \text{for all } (i, j, k) \in S_3.$$

In case II, for all $(i, j, k) \in S_3$ the integers

$$\alpha_{ij} := a_{ij} n_j - a_{ji} n_i \quad \text{and} \quad z_{ij} := a_{ki} n_i + a_j n_j - n_1 - n_2 - n_3$$

depend only on sign(i, j, k). Hence $a := |\alpha_{ij}|$ is completely independent of $(i, j, k) \in S_3$, and one gets two numbers $z^+ := z_{ij}$ and $z^- := z_{ji}$ for any $(i, j, k) \in A_3$.

THEOREM 3 ($\mathfrak{m}^{-1}/B$ for monomial curves in $\mathbf{A}^3$). *Let $s = 3$. Then in case*

(I) $\mathfrak{m}_B^{-1}/B = A(x_l^{a_l} x_k^{a_k}/x_1 x_2 x_3)^- \cong At^z$

with

$$z = a_l n_l + a_k n_k - n_1 - n_2 - n_3 \quad \textit{for all } l \in \{i, j\}; \ r = 1.$$

(II) $\mathfrak{m}_B^{-1}/B = A(x_i^{a_{ki}} x_j^{a_j}/x_1 x_2 x_3)^- + A(x_j^{a_{kj}} x_i^{a_i}/x_1 x_2 x_3)^-$

$$\cong At^{z^+} + At^{z^-} \textit{ for all } (i, j, k) \in A_3, \textit{ and}$$

$$z^< = \min\{z^+, z^-\} = z^> - a$$

with

$$z^> = \max\{z^+, z^-\}; \ r = 2.$$

Proof. Using the notation of the proof of Theorem 2, we see that in case I we have

$$\{ (x_h^{a_h - 1} x_k^{a_k - 1})^- \}$$

as A-basis of the socle of B/Bx_g, and in case II we have

$$\{ (x_j^{a_{ij}-1} x_k^{a_k-1})^-, (x_j^{a_j-1} x_k^{a_{ik}-1})^- \}$$

as A-basis of the socle of B/Bx_i. As for all $b \in B$, $(\bar{x} \mapsto \overline{x/b})$, $x \in B$, gives an isomorphism of the socle of B/Bb and $\mathfrak{m}_B^{-1}/B$, we get the assertion.

Let e denote the Euler derivation $\sum_{i=1}^{s} n_i x_i \frac{\partial}{\partial x_i}$ of B. Assuming char $A = 0$, we can also write Theorem 3 as

THEOREM 3′ (Module of derivations of monomial curves in $\mathbf{A}^3$). *Let $s = 3$. Then in case*

$$\text{(I)} \quad \mathrm{Der}_A B = Be + B\Big(n_g x_h^{a_h-1} x_k^{a_k-1} \frac{\partial}{\partial x_g} + n_h x_g^{a_g-1} x_k^{a_k-1} \frac{\partial}{\partial x_h}$$
$$+ n_k x_g^{a_{kg}-1} x_h^{a_h+a_{kh}-1} \frac{\partial}{\partial x_k}\Big) \cong Ct\frac{\partial}{\partial t} + Ct^{z+1}\frac{\partial}{\partial t}$$

for all $\{g, h\} = \{i, j\}$ such that $a_{kg} \neq 0$.

$$\text{(II)} \quad \mathrm{Der}_A B = Be + B \sum_{\substack{(i,j,k) \in S_3 \\ \mathrm{sign}(i,j,k) = +1}} n_i x_j^{a_{ij}-1} x_k^{a_k-1} \frac{\partial}{\partial x_i}$$
$$+ B \sum_{\substack{(i,j,k) \in S_3 \\ \mathrm{sign}(i,j,k) = -1}} n_i x_j^{a_{ij}-1} x_k^{a_k-1} \frac{\partial}{\partial x_i}$$
$$\cong Ct\frac{\partial}{\partial t} + Ct^{z^+ + 1}\frac{\partial}{\partial t} + Ct^{z^- + 1}\frac{\partial}{\partial t}.$$

$$\Big[t^{z^< + 1}\frac{\partial}{\partial t}, t^{z^> + 1}\frac{\partial}{\partial t}\Big] = at^{\sum_{i=1}^{3}(a_i - 2)n_i + 1}\frac{\partial}{\partial t}$$
$$= ax_1^{a_1-1} x_2^{a_2-2} x_3^{a_3-2} e.$$

Remark 1. Note that in the complete intersection cases the derivations $t^{z+1}\frac{\partial}{\partial t}$ can be written as determinants called "trivial derivations" by J. Wahl **[14]**.

COROLLARY 1 (Canonical ideal of monomial curves in $\mathbf{A}^3$). *Let $s = 3$. Then in case*

(I) $\mathfrak{k} = B(x_1 x_2 x_3 / x_l^{a_l} x_k^{a_k}) \cong Ct^{-z}$ *for all $l \in \{i, j\}$.*

(II) $\mathfrak{k} = B(x_1x_2x_3/x_i^{a_{ki}}x_j^{a_j}) + B(x_1x_2x_3/x_j^{a_{kj}}x_i^{a_i})$
$\cong Ct^{-z^+} + Ct^{-z^-}$ *for all* $(i, j, k) \in A_3$.

Proof. See Theorem 3 and Proposition 2.

COROLLARY 2. *Let* $s = 3$. *Then in case*

(I) $z = a_1a_2a_3 + (a_ia_{kj} + a_{ki}a_j)a_k - a_ja_k - a_ka_i - (a_ia_{kj}a_{ki}a_j)$.

(II) $z^+ = a_1a_2a_3 + a_{31}a_{12}a_{23}$
$- (a_2a_{13} + a_{12}a_{23}) - (a_3a_{21} + a_{23}a_{31}) - (a_1a_{32} + a_{31}a_{12})$,
$z^- = a_1a_2a_3 + a_{21}a_{32}a_{13}$
$- (a_3a_{12} + a_{13}a_{32}) - (a_1a_{23} + a_{21}a_{13}) - (a_2a_{31} + a_{32}a_{21})$,

and

$$a = |a_{31}a_{12}a_{23} - a_{21}a_{32}a_{13}|.$$

Proof. See Theorem 3 and Theorem 2.

Remark 2. Note that the determination of $z^>$ was a problem posed by G. Frobenius occasionally in his lectures (cf. **[5**, C7**]** for references).

We now come to the modified version and extension of a result of J. Sylvester **[13]** on the singularity degree of numerical semigroups generated by two elements.

THEOREM 4 (Singularity of monomial curves in $\mathbf{A}^3$). *Let* $s = 3$. *Then in case*

(I) $\kappa = a_1a_2a_3 + (a_ia_{kj} + a_{ki}a_j)a_k - a_ja_k - a_ka_i - (a_ia_{kj} + a_{ki}a_j)$.

(II) $\kappa = a_1a_2a_3 + a_{31}a_{12}a_{23} + a_{21}a_{32}a_{13}$
$- (a_2a_3 - a_{32}a_{23}) - (a_3a_1 - a_{13}a_{31}) - (a_1a_2 - a_{21}a_{12})$.

Proof. Of course, in case I, the formula for κ follows from Corollary 2, since S is symmetric; but we want to illustrate here a way of computing κ which also works in non-Gorenstein cases; namely, we want to study $\mathfrak{k}/(\mathfrak{k} \cap B)$.

I. Consider the following two subcases:

(A) $X_k^{a_k} - X_l^{a_l} \in \mathfrak{P}$ for some (all) $l \in \{i, j\}$.

(B) $X_k^{a_k} - X_l^{a_l} \notin \mathfrak{P}$ for some (all) $l \in \{i, j\}$.

In case (A) let $b_{ki}: = a_i$ and $b_{kj}: = a_j$, and in case (B) choose $b_{ki}, b_{kj} \in \mathbf{N}^*$ with

$$X_k^{a_k} - X_i^{b_{ki}}X_j^{b_{kj}} \in \mathfrak{P}.$$

Define

$$I: = \mathbf{N}_{a_i+b_{ki}-2} \times \mathbf{N}_{a_j+b_{kj}-2} \times \mathbf{N}_{a_k-1}$$

and

$$X: = \sum_{(\alpha_i,\alpha_j,\alpha_k)\in I} A(x \cdot x_i^{\alpha_i} x_j^{\alpha_j} x_k^{\alpha_k}),$$

where

$$x: = x_1 x_2 x_3 / x_l^{a_l} x_k^{a_k} \quad \text{for all } l \in \{i, j\}.$$

If

$$I': = \{ (\alpha_i, \alpha_j, \alpha_k) \in I | \alpha_l < a_l - 1 \text{ for all } l \in \{i, j\} \text{ or } \alpha_k < a_k - 1 \},$$

then

$$\mathfrak{f}/(\mathfrak{f} \cap B) = \sum_{(\alpha_i,\alpha_j,\alpha_k)\in I'} A(x \cdot x_i^{\alpha_i} x_j^{\alpha_j} x_k^{\alpha_k})^- \cong X/(X \cap B).$$

Further define

$$Id_{ij}: = \{ (\alpha_i, \alpha_j, \alpha_k) \in I | \alpha_i \geqq a_i \text{ and } \alpha_j \leqq b_{kj} - 1 \},$$

$$Xd_{ij}: = \sum_{(\alpha_i,\alpha_j,\alpha_k)\in Id_{ij}} A(x \cdot x_i^{\alpha_i} x_j^{\alpha_j} x_k^{\alpha_k}),$$

and Xd_{ji} in the same way.

There are $(a_i + b_{ki} - 1)(a_j + b_{kj} - 1)a_k$ representations taken into consideration for the formation of the elements of generating X, and those elements having two representations are precisely the elements generating $Xd_{ij} = Xd_{ji} = :Xd$. Hence

$$\dim_A X = (a_i + b_{ki} - 1)(a_j + b_{kj} - 1)a_k - (b_{ki} - 1)(b_{kj} - 1)a_k.$$

Now define

$$(I \cap z + H)_{lm}: = \{ (\alpha_i, \alpha_j, \alpha_k) \in I | \alpha_l \geqq a_l - 1 \text{ and } \alpha_m \geqq a_m - 1 \}$$

and

$$(X \cap B)_{lm} = \sum_{(\alpha_i,\alpha_j,\alpha_k)\in(I\cap z+H)_{lm}} A(x \cdot x_i^{\alpha_i} x_j^{\alpha_j} x_k^{\alpha_k})$$

for all $(l, m, n) \in A_3$. Then

$$X \cap B = \sum_{(l,m,n)\in A_3} (X \cap B)_{lm} \quad \text{in case (A)},$$

and

$$X \cap B = (X \cap B)_{ik} + (X \cap B)_{jk} \quad \text{in case (B)},$$

and there are

$$b_{ki}(a_j + b_{kj} - 1) + (a_i + b_{ki} - 1)b_{kj} + b_{ki}b_{kj}a_k - 3b_{ki}b_{kj} + b_{ki}b_{kj}$$

representations taken into consideration for the formation of the elements generating $X \cap B$ in case (A), and

$$b_{ki}(a_j + b_{kj} - 1) + (a_i + b_{ki} - 1)b_{kj} - b_{ki}b_{kj}$$

representations in case (B). Those elements having two representations are precisely the elements generating

$$(X \cap B) \cap Xd = Y_{ij} + Y_{ji},$$

with

$$J_{ij}: = \{ (\alpha_i, \alpha_j, \alpha_k) \in I | \alpha_i \geqq a_i, \alpha_j \leqq b_{kj} - 2, \text{ and } \alpha_k = a_k - 1\},$$

$$Y_{ij}: = \sum_{(\alpha_i,\alpha_j,\alpha_k) \in J_{ij}} A(x \cdot x_i^{\alpha_i} x_j^{\alpha_j} x_k^{\alpha_k}),$$

and Y_{ji} defined in the same way. As $Y_{ij} = Y_{ji}$, we get

$$\dim_A X \cap B = b_{ki}(a_j + b_{kj} - 1) + (a_i + b_{ki} - 1)b_{kj} + b_{ki}b_{kj}a_k - 2b_{ki}b_{kj} - (b_{ki} - 1)(b_{kj} - 1)$$

in case (A), and

$$\dim_A X \cap B = b_{ki}(a_j + b_{kj} - 1) + (a_i + b_{ki} - 1)b_{kj} - b_{ki}b_{kj} - (b_{ki} - 1)(b_{kj} - 1)$$

in case (B). Therefore

$$\begin{aligned}\kappa &= \dim_A X - \dim_A X \cap B - 1 \\ &= (b_{ki} - 1)a_j a_k + a_i(b_{kj} - 1)a_k + a_i a_j a_k - b_{ki}(a_j - 1) \\ &\quad - (a_i - 1)b_{kj} - b_{ki}b_{kj}a_k + b_{ki}b_{kj} - b_{ki} - b_{kj} \\ &= 2a_1a_2a_3 - a_2a_3 - a_3a_1 - a_1a_2\end{aligned}$$

in case (A), and

$$\begin{aligned}\kappa &= (b_{ki} - 1)a_j a_k + a_i(b_{kj} - 1)a_k + a_i a_j a_k - b_{ki}(a_j - 1) \\ &\quad - (a_i - 1)b_{kj} - b_{ki} - b_{kj} \\ &= a_1a_2a_3 + (a_i b_{kj} + b_{ki}a_j)a_k - a_j a_k - a_k a_i - (a_i b_{kj} + b_{ki}a_j)\end{aligned}$$

in case (B).

II. Define

$$I: = \mathbf{N}_{a_1-2} \times \mathbf{N}_{a_2-2} \times \mathbf{N}_{a_3-2}$$

and

$$X^{\pm}: = \sum_{(\alpha_1,\alpha_2,\alpha_3)\in I} A(x^{\pm}x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}),$$

where

$$x^+: = x_1x_2x_3/x_i^{a_{ki}}x_j^{a_j} \text{ and } x^-: = x_1x_2x_3/x_j^{a_{kj}}\, x_i^{a_i}$$

for all $(i, j, k) \in A_3$. Then

$$\mathfrak{k}/(\mathfrak{k} \cap B) \cong X^+ + X^-.$$

Further define

$$I_{ij}: = \{(\alpha_1, \alpha_2, \alpha_3) \in I | \alpha_i \geqq a_{ji} \text{ and } \alpha_j \leqq a_{kj} - 2\}$$

and

$$X_{ij}^{\pm}: = \sum_{(\alpha_1,\alpha_2,\alpha_3)\in I_{ij}} A(x^{\pm}x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3})$$

for all $(i, j, k) \in S_3$ with $\mathrm{sign}(i, j, k) = \pm 1$.

There are $2(a_1 - 1)(a_2 - 2)(a_3 - 1)$ representations taken into consideration for the formation of the elements generating $X^+ + X^-$. As $X_{ij}^+ = X_{ji}^-$ for all $(i, j, k) \in A_3$, those elements having two representations are precisely the elements generating

$$X^+ \cap X^- = \sum_{(i,j,k)\in A_3} X_{ij}^+,$$

and therefore

$$\begin{aligned}
\kappa &= \dim_A(X^+ + X^-) - 1 \\
&= 2(a_1 - 1)(a_2 - 1)(a_3 - 1) \\
&\quad - \sum_{(i,j,k)\in A_3} (a_{ki} - 1)(a_{kj} - 1)(a_k - 1) - 1 \\
&= 2a_1a_2a_3 - \sum_{(i,j,k)\in A_3} a_{ki}a_{kj}a_k \\
&\quad - \sum_{(i,j,k)\in A_3} ((a_{ij}a_k + a_{ik}a_j) - a_{ij}a_{ik}) \\
&= a_1a_2a_3 + a_{31}a_{12}a_{23} + a_{21}a_{32}a_{13} - \sum_{(i,j,k)\in A_3} (a_ja_k - a_{kj}a_{jk}).
\end{aligned}$$

COROLLARY 1. *In case* I, $n_i = a_j a_k$, $n_j = a_k a_i$, *and* $n_k = a_i a_{kj} + a_{ki} a_j$.

Proof. There exists an $\alpha \in \mathbf{N}^*$ such that $n_i = \alpha a_j$ and $n_j = \alpha a_i$. Hence there exists a $\lambda \in \mathbf{N}^*$ such that $\lambda\alpha = a_k$, since $\gcd(n_1, n_2, n_3) = 1$. Now $\kappa = z$ shows that $\lambda = 1$.

COROLLARY 2. *Let* $s = 3$. *Then*

$$\kappa = \sum_{\sigma=1}^{s} (a_\sigma - 1)n_\sigma - \prod_{\sigma=1}^{s} a_\sigma$$

with n_1, n_2, n_3 *as in Theorem* 2.

Remark 3. It is a result of M. Schaps **[11]** that space curves are always smoothable. As they are also unobstructed by **[7**, 3.2**]**, we have by **[3**, 4.1**]**

$$t^1 = \kappa + r$$

computed for all monomial space curves.

Very useful for the construction of examples is the following lemma, a weaker version of which is stated in **[9]**.

LEMMA **[9]**. *Assume* $s = 3$. 1. *Let* $b_1, b_2, b_3 \in \mathbf{N}^*$ *and* $b_{\nu\lambda}, b_{\nu\mu} \in \mathbf{N}$ *for some* $(\lambda, \mu, \nu) \in S_3$. *Assume*

$$X_\lambda^{b_\lambda} - X_\mu^{b_\mu} \in \mathfrak{P}, \quad X_\nu^{b_\nu} - X_\lambda^{b_{\nu\lambda}} X_\mu^{b_{\nu\mu}} \in \mathfrak{P}, \quad \textit{and}$$

$$n_\lambda = b_\mu b_\nu \quad \textit{and} \quad n_\mu = b_\nu b_\lambda.$$

Then we are in case I, *and we have* $(b_\lambda = a_\lambda$ *or* $b_\mu = a_\mu)$ *and* $b_\nu = a_\nu$. *If* $\nu = k$ *in Theorem* 1, *then we also have* $b_\lambda = a_\lambda$ *and* $b_\mu = a_\mu$.

2. *Let* $b_i, b_{ij}, b_{ik} \in \mathbf{N}^*$ *for all* $(i, j, k) \in A_3$. *Assume*

$$X_i^{b_i} - X_j^{b_{ij}} X_k^{b_{ik}} \in \mathfrak{P}, \quad b_i = b_{ji} + b_{ki}, \quad \textit{and}$$

$$n_i = b_j b_k - b_{kj} b_{jk} \quad \textit{for all } (i, j, k) \in A_3.$$

Then we are in case II, *and we have* $b_i = a_i$ *and hence* $b_{ij} = a_{ij}$ *for all* $(i, j, k) \in S_3$.

Proof. 1. If we were in case II, then we would have

$$n_\lambda = b_\mu b_\nu \geqq a_\mu a_\nu > a_\mu a_\nu - a_{\nu\mu} a_{\mu\nu} = n_\lambda.$$

Hence we are in case I. If $\nu \in \{i, j\}$, then let $\{\nu, \rho\} = \{i, j\}$, and we get $b_k = a_k$ and $b_\nu = a_\nu$ from

$$n_\rho = b_k b_\nu \geqq a_k a_\nu = n_\rho.$$

And if $\nu = k$, then

$$n_\lambda = b_\mu b_\nu \geqq a_\mu a_\nu = n_\lambda$$

$$n_\mu = b_\nu b_\lambda \geqq a_\nu a_\lambda = n_\mu$$

show $b_\mu = a_\mu$, $b_\nu = a_\nu$, and $b_\lambda = a_\lambda$.

2. If we were in case I, then

$$b_{ij}b_k + b_{kj}b_{ik} = n_i = a_j a_k = n_i = b_j b_{ik} + b_{ij}b_{jk}$$

would show $b_{ij} < a_j$ and $b_{ik} < a_k$, and we would get the contradiction

$$(b_i - a_i)n_i + (a_j - b_{ij})n_j = b_{ik}n_k.$$

Hence we are in case II.

The rest follows from **[9]**. Note that the assumption

$$\gcd(n_1, n_2) = \gcd(n_2, n_3) = \gcd(n_3, n_1) = 1$$

in **[9]** is not necessary.

Remark 4. Note that in part 1 of the lemma one can have $b_\lambda \neq a_\lambda$ or $b_\mu \neq a_\mu$. Consider for example $\langle 6, 9, 8 \rangle$. Here

$$\mathfrak{P} = (X_1^3 - X_2^2, X_3^3 - X_1^4), X_1^4 - X_3^3 \in \mathfrak{P}, X_2^2 - X_1^3 \in \mathfrak{P},$$

$$6 = 3 \cdot 2, 8 = 2 \cdot 4, \text{ and } a_1 = 3;$$

but $b_1 = 4$.

Example (Pythagorean monomial space curves). Let $s = 3$. We call B a *Pythagorean monomial curve* if and only if B is not a complete intersection and there exists an $(i, j, k) \in S_3$ such that

$$b: = a_{ki} = a_j = a_k \quad \text{and} \quad a: = a_{ji} = a_{kj} = a_{jk}.$$

These have $a_i = a + b$, $a_{ij} = a_{ik} = b - a$, and hence

$$\mathfrak{P} = (X_i^{a+b} - X_j^{b-a}X_k^{b-a}, X_j^b - X_k^a X_i^a, X_k^b - X_i^b X_j^a).$$

As $n_i = b^2 - a^2$, $n_j = 2ab$, and $n_k = a^2 + b^2$, we get

$$\kappa = b^3 - a^3 + 2ab^2 - 2b(a + b);$$

and a and b are positive natural numbers of opposite parity with $b > a$ and $\gcd(a, b) = 1$.

Further $n_i^2 + n_j^2 = n_k^2$ showing

$$X_k^{n_k} - X_i^{n_i}X_j^{n_j} \in \mathfrak{P}.$$

In fact, we have the identity

$$X_k^{a^2 + b^2} - X_i^{b^2-a^2}X_j^{2ab}$$
$$= X_k^{a^2}((X_k^b)^{b-1} + (X_k^b)^{b-2}(X_i^b X_j^a) + \dots$$
$$+ (X_i^b X_j^a)^{b-1})(X_k^b - X_i^b X_j^a) - X_i^{b^2-a^2}X_j^{ab}((X_j^b)^{a-1}$$

$$+ (X_j^b)^{a-2}(X_k^a X_i^a) + \ldots + (X_k^a X_i^a)^{a-1})(X_j^b - X_k^a X_i^a)$$

$$= \left(X_k^{a^2} \sum_{l=0}^{b-1} (X_k^b)^{b-1-l}(X_i^b X_j^a)^l\right)(X_k^b - X_i^b X_j^a)$$

$$- \left(X_i^{b^2-a^2} X_j^{ab} \sum_{l=0}^{a-1} (X_j^b)^{a-1-l}(X_k^a X_i^a)^l\right)(X_j^b - X_k^a X_i^a).$$

Conversely, assume there exists a $k \in \{1, 2, 3\}$ such that

$$n_i^2 + n_j^2 = n_k^2 \quad \text{for some } i, j \in \{1, 2, 3\}\setminus\{k\}.$$

Choose i and j so that n_i is odd and n_j is even. Then, as is well known, there exist $a, b \in \mathbf{N}^*$ of opposite parity with $\gcd(a, b) = 1$ and $b > a$ such that

$$n_i = b^2 - a^2, n_j = 2ab \quad \text{and} \quad n_k = a^2 + b^2.$$

And as

$$(a + b)n_i = (b - a)n_j + (b - a)n_k$$

$$bn_j = an_k + an_i$$

$$bn_k = bn_i + an_j,$$

we get $a_{ki} = b = a_j = a_k$ and $a_{ji} = a = a_{kj} = a_{jk}$ by the lemma.

This shows that the Pythagorean monomial space curves are precisely those monomial space curves for which there exists an $(i, j, k) \in S_3$ such that $n_i^2 + n_j^2 = n_k^2$.

2. Singularity of Gorenstein monomial curves in $\mathbf{A}^4$. The next step is the calculation of κ if $s = 4$. This will be considerably more complicated than for monomial space curves, as it has been shown by H. Bresinsky **[1]** that there exist monomial curves in any $\mathbf{A}^s$, $s \geqq 4$, requiring arbitrarily large numbers of generators for their defining ideals.

However, our study of monomial curve singularities can still be carried out, provided one is able to divide the curves in question into subclasses, whose members have defining equations which one can survey. A division of this kind has been made for Gorenstein monomial curves in $\mathbf{A}^4$ by H. Bresinsky **[2]**.

Let us first treat the case that B is a complete intersection.

THEOREM 5. **[2]**. (Relations of monomial curves in $\mathbf{A}^4$ which are complete intersections). *If* $s = 4$ *and* $d = 0$, *then at least one of the following two cases does occur*:

(A) $\mathfrak{P} = (X_i^{a_i} - X_j^{a_j}, X_k^{a_k} - X_i^{a_{ki}} X_j^{a_{kj}}, X_l^{a_l} - X_i^{a_{li}} X_j^{a_{lj}} X_k^{a_{lk}})$

for some $(i, j, k, l) \in S_4$ *with* $a_{ki}, a_{kj}, a_{li}, a_{lj}, a_{lk} \in \mathbf{N}$.

(B) $\mathfrak{P} = (X_i^{a_i} - X_j^{a_j}, X_k^{a_k} - X_l^{a_l}, X_i^{b_i}X_j^{b_j} - X_k^{b_k}X_l^{b_l})$

for some $(i, j, k, l) \in S_4$ *with* $b_i, b_j, b_k, b_l \in \mathbf{N}$.

Remark 5. $\langle 8, 9, 10, 12\rangle$ is an example for case (A), $\langle 10, 14, 15, 21\rangle$ is an example for case (B), and $\langle 10, 12, 15, 18\rangle$ is an example for both cases.

The formulas in case (B) of the following theorem are also due to H. Bresinsky **[2]**.

THEOREM 6 (Weights of monomial curves in $\mathbf{A}^4$ which are complete intersections). *Let* $s = 4$ *and* $d = 0$. *Then in case*

(A) $n_i = a_ja_ka_l$, $n_j = a_ka_la_i$, $n_k = a_l(a_ia_{kj} + a_{ki}a_j)$, *and*

$$n_l = (a_ia_{kj} + a_{ki}a_j)a_{lk} + (a_ia_{lj} + a_{li}a_j)a_k.$$

(B) $n_i = a_j(a_kb_l + b_ka_l)$; $n_j = (a_kb_l + b_ka_l)a_i$;

$$n_k = a_l(a_ib_j + b_ia_j);\ n_l = (a_ib_j + b_ia_j)a_k.$$

Proof. As in Theorem 2 we consider the Apéry-bases of B with respect to $x_1, \ldots, x_s$.

In case (A) one has

$$B/Bx_i \cong A[[X_j, X_k, X_l]]/(X_j^{a_j}, X_k^{a_k}, X_l^{a_l})$$

if $a_{li} \neq 0$ and

$$B/Bx_i \cong A[[X_j, X_k, X_l]]/(X_j^{a_j}, X_k^{a_k}, X_l^{a_l} - X_j^{a_{lj}}X_k^{a_{lk}})$$

if $a_{li} = 0$. Both times

$$\omega_{x_i} = \{(x_j^\beta x_k^\gamma x_l^\delta)^- | 0 \leqq \beta \leqq a_j - 1, 0 \leqq \gamma \leqq a_k - 1, \text{ and } 0 \leqq \delta \leqq a_l - 1\}.$$

Therefore $n_i = a_ja_ka_l$; and the formula for n_j one gets by symmetry.

Further

$$B/Bx_k \cong A[[X_i, X_j, X_l]]/(X_i^{a_i} - X_j^{a_j}, X_i^{a_{ki}}X_j^{a_{kj}}, X_l^{a_l})$$

if $a_{lk} \neq 0$ and

$$B/Bx_k \cong A[[X_i, X_j, X_l]]/(X_i^{a_i} - X_j^{a_j}, X_i^{a_{ki}}X_j^{a_{kj}}, X_l^{a_l} - X_i^{a_{li}}X_j^{a_{lj}})$$

if $a_{lk} = 0$. Both times

$$\omega_{x_k} = \{(x_i^\alpha x_j^\beta x_l^\delta)^- | 0 \leqq \alpha \leqq a_i - 1, 0 \leqq \beta \leqq a_{kj} - 1 \text{ and } 0 \leqq \delta \leqq a_l - 1\}$$

$$\cup \{(x_i^\alpha x_j^\beta x_l^\delta)^- | 0 \leqq \alpha \leqq a_{ki} - 1, a_{kj} \leqq \beta \leqq a_j + a_{kj} - 1, \text{ and } 0 \leqq \delta \leqq a_l - 1\}$$

https://doi.org/10.4153/CJM-1985-047-8 Published online by Cambridge University Press

and therefore $n_k = a_l(a_i a_{kj} + a_{ki} a_j)$.

As for ω_{x_l}, note that

$$B/Bx_l \cong A[[X_i, X_j, X_k]]/(X_i^{a_i} - X_j^{a_j}, X_k^{a_k} - X_i^{a_{ki}} X_j^{a_{kj}}, X_i^{a_{li}} X_j^{a_{lj}} X_k^{a_{lk}}),$$

and hence

$$\begin{aligned}\omega_{x_l} = &\{(x_i^{\alpha} x_j^{\beta} x_k^{\gamma})^- | 0 \leqq \alpha \leqq a_i - 1, 0 \leqq \beta \leqq a_{kj} - 1, \\ &\qquad \text{and } 0 \leqq \gamma \leqq a_{lk} - 1\} \\ \cup &\{(x_i^{\alpha} x_j^{\beta} x_k^{\gamma})^- | 0 \leqq \alpha \leqq a_{ki} - 1, a_{kj} \leqq \beta \leqq a_j + a_{kj} - 1, \\ &\qquad \text{and } 0 \leqq \gamma \leqq a_{lk} - 1\} \\ \cup &\{(x_i^{\alpha} x_j^{\beta} x_k^{\gamma})^- | 0 \leqq \alpha \leqq a_i - 1, 0 \leqq \beta \leqq a_{lj} - 1 \\ &\qquad \text{and } a_{lk} \leqq \gamma \leqq a_k + a_{lk} - 1\} \\ \cup &\{(x_i^{\alpha} x_j^{\beta} x_k^{\gamma})^- | 0 \leqq \alpha \leqq a_{li} - 1, a_{lj} \leqq \beta \leqq a_j + a_{lj} - 1, \\ &\qquad \text{and } a_{lk} \leqq \gamma \leqq a_k + a_{lk} - 1\}.\end{aligned}$$

Therefore

$$n_l = (a_i a_{kj} + a_{ki} a_j) a_{lk} + (a_i a_{lj} + a_{li} a_j) a_k.$$

In case (B) one has

$$B/Bx_i \cong A[[X_j, X_k, X_l]]/(X_j^{a_j}, X_k^{a_k} - X_l^{a_l}, X_k^{b_k} X_l^{b_l}),$$

and hence

$$\begin{aligned}\omega_{x_i} = &\{(x_j^{\beta} x_k^{\gamma} x_l^{\delta})^- | 0 \leqq \beta \leqq a_j - 1, 0 \leqq \gamma \leqq a_k - 1, \\ &\qquad \text{and } 0 \leqq \delta \leqq b_l - 1\} \\ \cup &\{(x_j^{\beta} x_k^{\gamma} x_l^{\delta})^- | 0 \leqq \beta \leqq a_j - 1, 0 \leqq \gamma \leqq b_k - 1, \\ &\qquad \text{and } b_l \leqq \delta \leqq a_l + b_l - 1\}.\end{aligned}$$

Therefore $n_i = a_j(a_k b_l + b_k a_l)$; and the formulas for n_j, n_k, and n_l one gets by symmetry.

THEOREM 7 ($\mathfrak{m}^{-1}/B$ for monomial curves in $\mathbf{A}^4$ which are complete intersections). *Let $s = 4$ and $d = 0$. Then in case*

(A) $\mathfrak{m}_B^{-1}/B = A(x_j^{a_j} x_k^{a_k} x_l^{a_l}/x_1 x_2 x_3 x_4)^- \cong At^z$

with

$$z = a_j n_j + a_k n_k + a_l n_l - n_1 - n_2 - n_3 - n_4.$$

(B) $\mathfrak{m}_B^{-1}/B = A(x_j^{a_j} x_k^{b_k} x_l^{a_l + b_l}/x_1 x_2 x_3 x_4)^- \cong At^z$

with

$$z = a_j n_j + b_k n_k + (a_l + b_l) n_l - n_1 - n_2 - n_3 - n_4.$$

Proof. By the proof of Theorem 6, we have

$$\{ (x_j^{a_j-1}x_k^{a_k-1}x_l^{a_l-1})^- \}$$

as A-basis of the socle of B/Bx_i in case (A), and

$$\{ (x_j^{a_j-1}x_k^{b_k-1}x_l^{a_l+b_l-1})^- \}$$

in case (B).

Assuming char $A = 0$, we can also write Theorem 7 as

THEOREM 7′ (Module of derivations of monomial curves in $\mathbf{A}^4$ which are complete intersections). *Let $s = 4$ and $d = 0$. Then in case*

(A) *If $a_{lk} \neq 0$, then define*

$$x: = x_g^{a_{kg}+a_{lg}-1}x_h^{a_h+a_{kh}+a_{lh}-1}x_k^{a_{lk}-1}$$

for any $\{g, h\} = \{i, j\}$ such that $a_{kg} \neq 0$. Otherwise define

$$x: = x_g^{a_{lg}-1}x_h^{a_h+a_{lh}-1}x_k^{a_k+a_{lk}-1}$$

for any $\{g, h\} = \{i, j\}$ such that $a_{lg} \neq 0$. Then $x \in B$ is well-defined and

$$\begin{aligned}\mathrm{Der}_A\, B &= Be + B\Bigg(n_i x_j^{a_j-1}x_k^{a_k-1}x_l^{a_l-1}\frac{\partial}{\partial x_i} + n_j x_i^{a_i-1}x_k^{a_k-1}x_l^{a_l-1}\frac{\partial}{\partial x_j}\\ &\quad + n_k x_g^{a_{kg}-1}x_h^{a_h+a_{kh}-1}x_l^{a_l-1}\frac{\partial}{\partial x_k} + n_l x\frac{\partial}{\partial x_l}\Bigg)\\ &\cong Ct\frac{\partial}{\partial t} + Ct^{z+1}\frac{\partial}{\partial t}\end{aligned}$$

for all $\{g, h\} = \{i, j\}$ such that $a_{kg} \neq 0$.

(B)
$$\begin{aligned}\mathrm{Der}_A\, B &= Be + B(n_i x_j^{a_j-1}x_\mu^{b_\mu-1}x_\nu^{a_\nu+b_\nu-1}\frac{\partial}{\partial x_i}\\ &\quad + n_j x_i^{a_i-1}x_\mu^{b_\mu-1}x_\nu^{a_\nu+b_\nu-1}\frac{\partial}{\partial x_j}\\ &\quad + n_k x_\kappa^{b_\kappa-1}x_\lambda^{a_\lambda+b_\lambda-1}x_l^{a_l-1}\frac{\partial}{\partial x_k}\\ &\quad + n_l x_\kappa^{b\kappa-1}x_\lambda^{a_\lambda+b_\lambda-1}x_k^{a_k-1}\frac{\partial}{\partial x_l}\\ &\cong Ct\frac{\partial}{\partial t} + Ct^{z+1}\frac{\partial}{\partial t}\end{aligned}$$

for all $\{\kappa, \lambda\} = \{i, j\}$ such that $b_\kappa \neq 0$ and all $\{\mu, \nu\} = \{k, l\}$ such that $b_\mu \neq 0$.

Remark 6. Note that, as in Remark 1, the derivations $t^{z+1}\frac{\partial}{\partial t}$ are trivial derivations.

COROLLARY 1 (Canonical ideal of monomial curves in $\mathbf{A}^4$ which are complete intersections). *Let $s = 4$ and $d = 0$. Then in case*

(A) $\mathfrak{k} = B(x_1x_2x_3x_4/x_j^{a_j}x_k^{a_k}x_l^{a_l}) \cong Ct^{-z}$.

(B) $\mathfrak{k} = B(x_1x_2x_3x_4/x_j^{a_j}x_k^{b_k}x_l^{a_l+b_l}) \cong Ct^{-z}$.

Proof. Use Theorem 7 and Proposition 2.

COROLLARY 2 (Singularity of monomial curves in $\mathbf{A}^4$ which are complete intersections). *Let $s = 4$ and $d = 0$. Then in case*

(A) $$\kappa = z = \sum_{\sigma=1}^{s} (a_\sigma - 1)n_\sigma - \prod_{\sigma=1}^{s} a_\sigma$$

(B) $$\kappa = z = \sum_{\sigma=1}^{s} (a_\sigma - 1)n_\sigma - \prod_{\sigma=1}^{s} a_\sigma$$
$$+ (a_ib_j + b_ia_j - a_ia_j)(a_kb_l + b_ka_l - a_ka_l)$$
$$= (a_ia_j + a_ib_j + b_ia_j)(a_ka_l + a_kb_l + b_ka_l) - \prod_{\sigma=1}^{s} a_\sigma - \sum_{\sigma=1}^{s} n_\sigma$$

with $n_1, \ldots, n_s$ as in Theorem 6.

Proof. Use Theorem 7 and Theorem 6.

Question. Let $\mathfrak{P}$ be generated by binomials $X_i^{a_i} - X$ for some $i \in \{1, \ldots, s\}$ and some monomial $X \in A[[X_1, \ldots, X_s]]$.
Is

$$\kappa = \sum_{\sigma=1}^{s} (a_\sigma - 1)n_\sigma - \prod_{\sigma=1}^{s} a_\sigma?$$

Now we will treat the case that B is not a complete intersection.

THEOREM 8 **[2]** (Relations of deviation 2 Gorenstein monomial curves in $\mathbf{A}^4$). *Let $s = 4$, $d \neq 0$, and $r = 1$. Then*

$$\mathfrak{P} = (X_i^{a_i} - X_k^{a_{ik}}X_l^{a_{il}}, X_j^{a_j} - X_l^{a_{jl}}X_i^{a_{ji}}, X_k^{a_k} - X_i^{a_{ki}}X_j^{a_{kj}},$$
$$X_l^{a_l} - X_j^{a_{lj}}X_k^{a_{lk}}, X_i^{a_{ji}}X_k^{a_{lk}} - X_j^{a_{kj}}X_l^{a_{il}})$$

for some $(i, j, k, l) \in S_4$ with unique $a_{ik}, a_{il}, a_{jl}, a_{ji}, a_{ki}, a_{kj}, a_{lj}, a_{lk} \in \mathbf{N}^*$ having the properties

$$a_{ji} + a_{ki} = a_i, a_{kj} + a_{lj} = a_j, a_{lk} + a_{ik} = a_k, \quad and$$
$$a_{il} + a_{jl} = a_l; d = 2.$$

THEOREM 9 **[2]**. (Weights of deviation 2 Gorenstein monomial curves in $\mathbf{A}^4$). *Under the assumptions of Theorem* 8,

$$n_i = a_j a_k a_{il} + a_{kj} a_{ik} a_{jl}; \; n_j = a_k a_l a_{ji} + a_{lk} a_{jl} a_{ki};$$
$$n_k = a_l a_i a_{kj} + a_{il} a_{ki} a_{lj}; \; n_l = a_i a_j a_{lk} + a_{ji} a_{lj} a_{ik}.$$

Proof. Again we consider the Apéry-bases of B with respect to $x_1, \ldots, x_s$. One has

$$B/Bx_i \cong A[\,[X_j, X_k, X_l]\,]/(X_k^{a_{ik}} X_l^{a_{il}}, X_j^{a_j}, X_k^{a_k}, X_l^{a_l} - X_j^{a_{lj}} X_k^{a_{lk}}, X_j^{a_{kj}} X_l^{a_{il}}),$$

and hence

$$\omega_{x_i} = \{ (x_j^\beta x_k^\gamma x_l^\delta)^- | 0 \leqq \beta \leqq a_j - 1, 0 \leqq \gamma \leqq a_k - 1, \text{ and } 0 \leqq \delta \leqq a_{il} - 1\}$$
$$\cup \{ (x_j^\beta x_k^\gamma x_l^\delta)^- | 0 \leqq \beta \leqq a_{kj} - 1, 0 \leqq \gamma \leqq a_{ik} - 1, \text{ and } a_{il} \leqq \delta \leqq a_l - 1\}.$$

Therefore $n_i = a_j a_k a_{il} + a_{kj} a_{ik} a_{jl}$; and the formulas for n_j, n_k, and n_l one gets by symmetry.

Under the assumptions of Theorem 8,

$$z_{\lambda\mu\nu} := a_\lambda n_\lambda + a_\mu n_\mu + a_{\kappa\nu} n_\nu - n_1 - n_2 - n_3 - n_4$$

is independent of

$$(\kappa, \lambda, \mu, \nu) \in I := \{ (\kappa, \lambda, \mu, \nu) \in S_4 | \langle \kappa, \lambda, \mu, \nu \rangle = \langle i, j, k, l \rangle \},$$

and we have

THEOREM 10 ($\mathfrak{m}^{-1}/B$ for deviation 2 Gorenstein monomial curves in $\mathbf{A}^4$). *Under the assumptions of Theorem* 8,

$$\mathfrak{m}_B^{-1}/B = A(x_j^{a_j} x_k^{a_k} x_l^{a_{il}} / x_1 x_2 x_3 x_4)^- \cong At^z \quad with\ z = z_{jkl}.$$

Proof. By the proof of Theorem 9 we have

$$\{ (x_j^{a_j - 1} x_k^{a_k - 1} x_l^{a_{il} - 1})^- \}$$

as A-basis of the socle of B/Bx_i.

Assuming char $A = 0$, we can also write Theorem 10 as

THEOREM 10′ (Module of derivations of deviation 2 Gorenstein monomial curves in $\mathbf{A}^4$). *Under the assumptions of Theorem* 8,

$$\operatorname{Der}_A B = Be + B \sum_{(\kappa,\lambda,\mu,\nu) \in I} n_\kappa x_\lambda^{a_\lambda - 1} x_\mu^{a_\mu - 1} x_\nu^{a_{\kappa\nu} - 1} \frac{\partial}{\partial x_\kappa}$$

$$\cong Ct\frac{\partial}{\partial t} + Ct^{z+1}\frac{\partial}{\partial t}.$$

COROLLARY 1 (Canonical ideal of deviation 2 Gorenstein monomial curves in $\mathbf{A}^4$). *Under the assumptions of Theorem* 8,

$$\mathfrak{k} = B(x_1x_2x_3x_4/x_j^{a_{ij}}x_k^{a_{jk}}x_l^{a_{il}}) \cong Ct^{-z}.$$

Proof. Use Theorem 10 and Proposition 2.

COROLLARY 2 (Singularity of deviation 2 Gorenstein monomial curves in $\mathbf{A}^4$). *Under the assumptions of Theorem* 8,

$$\kappa = z = \sum_{\sigma=1}^{s} (a_\sigma - 1)n_\sigma - \prod_{\sigma=1}^{s} a_\sigma + a_{ki}a_{lj}a_{ik}a_{jl}$$

with $n_1, \ldots, n_s$ *as in Theorem* 9.

Proof. Use Theorem 10 and Theorem 9.

Remark 7. As in Remark 3, we have by [**3**, 4.2 and 4.1]

$$t^1 = \kappa + 1$$

computed for all Gorenstein monomial curves in $\mathbf{A}^4$.

Remark 8. Note that most of what we have said in this paper also makes sense for algebraic monomial curves.

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