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Exact dimension functions of the prime continued fraction Cantor set

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Abstract. We study the exact Hausdorff and packing dimensions of the prime Cantor set, Λ_P, which comprises the irrationals whose continued fraction entries are prime numbers. We prove that the Hausdorff measure of the prime Cantor set cannot be finite and positive with respect to any sufficiently regular dimension function, thus negatively answering a question of Mauldin and Urbański (1999) and Mauldin (2013) for this class of dimension functions. By contrast, under a reasonable number-theoretic conjecture we prove that the packing measure of the conformal measure on the prime Cantor set is in fact positive and finite with respect to the dimension function $\psi(r) = r^\delta \log^{-2\delta} \log(1/r)$, where δ is the dimension (conformal, Hausdorff, and packing) of the prime Cantor set.

Key words: continued fractions, prime numbers, fractal geometry, thermodynamic formalism and transfer operators, Hausdorff and packing measures
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1. *Introduction*

Iterated function systems (IFSs) have been studied intensively since the 1980s by several groups of researchers, including Bandt, Barnsley, Dekking, Falconer, Graf, Hata, Hutchinson, Mauldin, Schief, Simon, Solomyak, and Urbański. For a very selective sampling of such research see [2–4, 9, 12, 13, 20, 24, 26, 33, 35]. Much of the early research on IFSs focused on systems with a finite number of Euclidean similarities as generators. Since the 1990s the theory has been extended to handle systems with countably many conformal maps. Mauldin and Urbański were among the pioneers of this extension of IFS theory, first to the study of infinite conformal iterated function systems (CIFSs), and then to their generalizations, namely, conformal graph directed Markov systems (CGDMSs) that may be used to study Fuchsian and Kleinian group limit sets as well as Julia sets associated with holomorphic and meromorphic iteration; see [24, 26].

In particular, the CIFS/CGDMS framework may be leveraged to encode a variety of sets that appear naturally at the interfaces of dynamical systems, fractal geometry and Diophantine approximation. In particular, with an eye on the focus of our paper, one can encode real numbers via their continued fraction expansions leading to the Gauss continued fraction IFS, which is a prime example of an infinite CIFS whose generators are the Möbius maps $x \mapsto 1/(a+x)$ for $a \in \mathbb{N}$. Given any subset $A \subset \mathbb{N}$, let Λ_A denote the set of all irrationals $x \in [0, 1]$ whose continued fraction partial quotients all lie in A. Then Λ_A may be expressed as the limit set of the subsystem of the Gauss IFS that comprises the maps $x \mapsto 1/(a+x)$ for $a \in A$; see, for example, [7, 18, 25].

We focus on Λ_P, the *prime Cantor set* of our title, that is, the Cantor set of irrationals whose continued fraction entries are prime numbers. Let $\delta = \delta_P$ denote the common value [25, Theorems 2.7 and 2.11] for the Hausdorff and packing dimensions of Λ_P. Using a result due to Erdős [10] guaranteeing the existence of arbitrarily large two-sided gaps in the sequence of primes, Mauldin and Urbański [25, Corollaries 4.5 and 5.6] proved that despite there being a conformal measure and a corresponding invariant Borel probability measure for this CIFS, the δ-dimensional Hausdorff and packing measures were zero and infinity, respectively. (Such phenomena cannot occur in the setting of finite-alphabet Gauss IFSs, since their limit sets are Ahlfors regular.) This result led naturally to the surprisingly resistant problem, first stated by Mauldin and Urbański in [25, Problem 2 in §7] and later repeated by Mauldin in the 2013 Erdős centennial volume [23, Problem 7.1], of determining whether there was an appropriate dimension function with respect to which the Hausdorff and packing measures of Λ_P were positive and finite. The study of such dimension functions, called *exact dimension functions*, has offered mathematicians myriad challenges over the past century. A very selective sampling of results follows: for Liouville numbers see [27]; for Bedford–McMullen self-affine carpets see [28, 29]; for geometrically finite Kleinian limit sets see [34]; and for random recursive constructions, Brownian sample paths and beyond see [13, 38].

2. *Main theorems*

We start by stating our main results; precise definitions will follow in the next section. Let $\delta = \delta_P$ denote the common value [25, Theorems 2.7 and 2.11] for the Hausdorff and packing dimensions of Λ_P. If μ is a locally finite Borel measure on $\mathbb{R}$, then we let

$$\mathcal{H}^\psi(\mu) \overset{\text{def}}{=} \inf\{\mathcal{H}^\psi(A) : \mu(\mathbb{R} \setminus A) = 0\},$$
$$\mathcal{P}^\psi(\mu) \overset{\text{def}}{=} \inf\{\mathcal{P}^\psi(A) : \mu(\mathbb{R} \setminus A) = 0\}.$$

A function ψ is *doubling* if for all $C_1 \geq 1$, there exists $C_2 \geq 1$ such that for all x, y with $C_1^{-1} \leq x/y \leq C_1$, we have $C_2^{-1} \leq \psi(x)/\psi(y) \leq C_2$.

THEOREM 2.1. *Let $\mu = \mu_P$ be the conformal measure on Λ_P, and let ψ be a doubling dimension function such that $\Psi(r) = r^{-\delta}\psi(r)$ is monotonic. Then $\mathcal{H}^\psi(\mu) = 0$ if the series*

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$$\sum_{k=1}^{\infty} \frac{y^{(1-2\delta)/(1-\delta)}}{(\log y)^{\delta/(1-\delta)}} \upharpoonright_{y=\Psi(\lambda^{-k})} \tag{2.1}$$

diverges, and $=\infty$ *if it converges, for all (equivalently, for any) fixed* $\lambda > 1$.

Note that $1/2 < \delta \approx 0.657 < 1$ [7, Table 1 and §3], so the exponent in the numerator is negative.

The following corollary negatively resolves [23, Problem 7.1] and part of [25, Problem 2 in §7] for sufficiently regular dimension functions, for example Hardy L-functions [16, 17].

COROLLARY 2.2. *For any doubling dimension function* ψ *such that* $\Psi(r) = r^{-\delta}\psi(r)$ *is monotonic, we have* $\mathcal{H}^{\psi}(\Lambda_P) \in \{0, \infty\}$.

Proof. By way of contradiction suppose that $0 < \mathcal{H}^{\psi}(\Lambda_P) < \infty$. Then $\mathcal{H}^{\psi} \upharpoonright \Lambda_P$ is a conformal measure on Λ_P and therefore a scalar multiple of μ_P, and thus $\mathcal{H}^{\psi}(\Lambda_P) = \mathcal{H}^{\psi}(\mathcal{H}^{\psi} \upharpoonright \Lambda_P) \in \{0, \infty\}$ by Theorem 2.1, a contradiction. □

Remark. Letting $\psi(r) = r^{\delta} \log^{s}(1/r)$ with $s > (1-\delta)/(2\delta - 1)$ gives an example of a function that satisfies the hypotheses of Theorem 2.1 such that the series (2.1) converges. For this function, we have $\mathcal{H}^{\psi}(\Lambda_P) \geq \mathcal{H}^{\psi}(\mu) = \infty$. This affirmatively answers part of [25, Problem 2 in §7].

THEOREM 2.3. *Let* μ *be the conformal measure on* Λ_P, *let* $\theta = 21/40$, *and let*

$$\phi(x) = \frac{\log(x) \log\log(x) \log\log\log\log(x)}{\log^2 \log\log(x)}. \tag{2.2}$$

Then

$$\mathcal{P}^{\psi}(\mu) = \infty \quad \textit{where } \psi(r) = r^{\delta}\phi^{-\delta}(\log(1/r)). \tag{2.3}$$

$$\mathcal{P}^{\psi}(\mu) = 0 \quad \textit{where } \psi(r) = r^{\delta}\log^{-s}(1/r) \textit{ if } s > \theta\delta/(2\delta - 1) \tag{2.4}$$

We can get a stronger result for packing measure by assuming the following conjecture.

Conjecture 2.4. Let p_n denote the nth prime, and let $d_n = p_{n+1} - p_n$. For each $k \geq 1$ let

$$R_k \stackrel{\text{def}}{=} \limsup_{n\to\infty} \frac{\min(d_{n+1}, \ldots, d_{n+k})}{\log^2(p_n)}.$$

Then $0 < R_k < \infty$ for all $k \in \mathbb{N}$.

Remark. The case $k = 1$ of Conjecture 2.4 is known as the Cramér–Granville conjecture. Early heuristics led Harald Cramér to conjecture that it is true with $R_1 = 1$ [8]. Applying Cramér's heuristics to the case $k \geq 2$ of Conjecture 2.4 yields the prediction that $R_k = 1/k$. Specifically, assume that each integer n has probability $1/\log(n)$ of being prime. Under this assumption, for $m \leq n$ the probability that no integers in an interval $(n, n+m]$ are prime is approximately $(1 - 1/\log(n))^m \asymp \exp(-m/\log(n))$. Thus, the probability that $d_n > m$ is approximately $\exp(-m/\log(p_n))$, since $d_n > m$ if and only if the interval $(p_n, p_n + m]$ has no primes. So the probability that $\min(d_{n+1}, \ldots, d_{n+k}) > m$

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is approximately $\exp(-km/\log(p_n))$. Now fix a constant $C > 0$. The probability that $\min(d_{n+1}, \ldots, d_{n+k}) \geq C \log^2(p_n)$ is approximately $\exp(-kC \log^2(p_n)/\log(p_n)) = p_n^{-kC}$. Now, by the Borel–Cantelli lemma, the probability is 1 that $\min(d_{n+1}, \ldots, d_{n+k}) \geq C \log^2(p_n)$ for infinitely many n if and only if the series $\sum_n p_n^{-kC}$ diverges, which by the prime number theorem is true if and only if $C \leq 1/k$. It follows (under this probabilistic model) that $R_k = 1/k$, where R_k is as in Conjecture 2.4.

However, improved heuristics now suggest that $R_1 = 2e^{-\gamma}$, where γ is the Euler–Mascheroni constant; see [14, 15, 30]. So perhaps an appropriate correction would be $R_k = 2e^{-\gamma}/k$.

THEOREM 2.5. *If the cases $k = 1, 2$ of Conjecture 2.4 are true, then $\mathcal{P}^\psi(\mu) \in (0, \infty)$, where ψ is given by the formula*

$$\psi(r) = r^\delta \log^{-2\delta} \log(1/r). \tag{2.5}$$

Note that the cases $k = 1, 2$ of Conjecture 2.4 correspond to information about the lengths of one-sided and two-sided gaps in the primes, respectively.

Question 2.6. Determine whether (2.5) is an exact dimension function for the prime Cantor set, that is, whether $0 < \mathcal{P}^\psi(\Lambda_P) < \infty$.

2.1. *Outline of the proofs.* The basic idea of the proofs is to use the Rogers–Taylor–Tricot density theorem (Theorem 3.5), which relates the Hausdorff and packing measures of a measure μ to the upper and lower densities

$$\overline{D}_\mu^\psi(x) \stackrel{\text{def}}{=} \limsup_{r \searrow 0} \frac{\mu(B(x, r))}{\psi(r)},$$

$$\underline{D}_\mu^\psi(x) \stackrel{\text{def}}{=} \liminf_{r \searrow 0} \frac{\mu(B(x, r))}{\psi(r)}.$$

at μ-almost every point $x \in \mathbb{R}$. We use the Roger–Taylor–Tricot density theorem as applied to the conformal measure μ. The next step is to estimate these densities using a global measure formula (Theorem 4.5), which relates the μ-measure of a ball $B(x, r)$ to the μ-measure of certain cylinders contained in that ball. Here, a 'cylinder' is a set of the form $[\omega] = \phi_\omega([0, 1])$, where ϕ_ω is a composition of elements of the Gauss IFS (cf. §3). This allows us to estimate $\overline{D}_\mu^\psi(x)$ and $\underline{D}_\mu^\psi(x)$ in terms of certain sets $J_{k,\alpha,\epsilon} \subset E$, where E is the set of primes (see Proposition 4.6 for more details). Specifically, $\overline{D}_\mu^\psi(x)$ and $\underline{D}_\mu^\psi(x)$ can be estimated in terms of certain sets $S_{\alpha,1}$ and $S_{\alpha,-1}$, respectively.

Next, we need to estimate the μ-measure of x such that $\overline{D}_\mu^\psi(x) = 0$ (respectively, $\underline{D}_\mu^\psi(x) = 0$). This is done via Lemma 4.4, which relates the μ-measure of $S_{\alpha,1}$ (respectively, $S_{\alpha,-1}$) to the μ-measures of $J_{k,\alpha,1}$ (respectively, $J_{k,\alpha,-1}$). Specifically, the former is 0 if and only if the latter series converges. So the next thing we need to do is estimate the series $\sum_k \mu(J_{k,\alpha,\epsilon})$; this is done in Lemma 4.8 for the case of a general Gauss IFS $(\phi_a)_{a\in E}$. Finally, in §5 we perform further computations in the case where E is the set of primes, yielding Theorems 2.1, 2.3, and 2.5.

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2.2. *Layout of the paper.* In §3 we introduce preliminaries such as the concept of Gauss IFSs and Hausdorff and packing dimensions, as well as the Rogers–Taylor–Tricot theorem and its corollary. In §4 we prove some results that hold in the general setting of Gauss IFSs, which are used to prove our main theorems but may also be interesting in their own right. Finally, in §5 we specialize to the case of the prime Gauss IFS, allowing us to prove our main theorems.

3. *Preliminaries and notation*

Convention 3.1. In what follows, $A \lesssim B$ means that there exists a constant $C > 0$ such that $A \leq CB$. $A \asymp B$ means $A \lesssim B \lesssim A$. $A \lesssim_+ B$ means there exists a constant C such that $A \leq B + C$. $A \lesssim_{+,\times} B$ means that there exist constants C_1, C_2 such that $A \leq C_1 B + C_2$.

Convention 3.2. All measures and sets are assumed to be Borel, and measures are assumed to be locally finite. Sometimes we restate these hypotheses for emphasis.

Recall that the *continued fraction expansion* of an irrational number $x \in (0, 1)$ is the unique sequence of positive integers (a_n) such that

$$x = [0; a_1, a_2, \ldots] \stackrel{\text{def}}{=} \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}}$$

Given $E \subset \mathbb{N}$, we define the set Λ_E to be the set of all irrationals in (0, 1) whose continued fraction expansions lie entirely in E. Equivalently, Λ_E is the image of $E^{\mathbb{N}}$ under the *coding map* $\pi : \mathbb{N}^{\mathbb{N}} \to (0, 1)$ defined by $\pi((a_n)) = [0; a_1, a_2, \ldots]$.

The set Λ_E can be studied dynamically in terms of its corresponding *Gauss iterated function system*, that is, the collection of maps $\Phi_E \stackrel{\text{def}}{=} (\phi_a)_{a \in E}$, where

$$\phi_a(x) \stackrel{\text{def}}{=} \frac{1}{a + x}.$$

(The Gauss IFS Φ_E is a special case of a *conformal iterated function system* (see, for example, [7, 24, 25]), but in this paper we deal only with the Gauss IFS case.) Let $E^* = \bigcup_{n \geq 0} E^n$ denote the collection of finite words in the alphabet E. For each $\omega \in E^*$, let $\phi_\omega = \phi_{\omega_1} \circ \cdots \circ \phi_{\omega_{|\omega|}}$, where $|\omega|$ denotes the length of ω. Then

$$\pi(\omega) = \lim_{n \to \infty} \phi_{\omega \upharpoonright [1,n]}(0).$$

Equivalently, $\pi(\omega)$ is the unique intersection point of the *cylinder sets* $[\omega \upharpoonright [1, n]]$, where

$$[\omega] \stackrel{\text{def}}{=} \phi_\omega([0, 1]).$$

Next, we define the *pressure* of a real number $s \geq 0$ to be

$$\mathbb{P}_E(s) \stackrel{\text{def}}{=} \lim_{n \to \infty} \frac{1}{n} \log \sum_{\omega \in E^n} \|\phi'_\omega\|^s,$$

where $\|\phi'_\omega\| \stackrel{\text{def}}{=} \sup_{x \in [0,1]} |\phi'_\omega(x)|$. The Gauss IFS Φ_E is called *regular* if there exists $\delta = \delta_E \geq 0$ such that $\mathbb{P}_E(\delta_E) = 0$. The following result was proven in [24].

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PROPOSITION 3.3. [24, Theorem 3.5] *Let Φ_E be a regular (Gauss) IFS. Then there exists a unique measure $\mu = \mu_E$ on Λ_E such that*

$$\mu_E(A) = \sum_{a\in E} \int_{\phi_a^{-1}(A)} |\phi_a'(x)|^{\delta_E} \, d\mu_E(x)$$

for all $A \subset [0, 1]$.

The measure μ appearing in Proposition 3.3 is called the *conformal measure* of Φ_E, and δ_E is called the *conformal dimension* of Φ_E. Recall that the *bounded distortion property* (cf. [24, (2.9)]) states that

$$|\phi_\omega'(x)| \asymp \|\phi_\omega'\| \quad \text{for all } \omega \in E^* \text{ and } x \in [0, 1].$$

This implies that the measure of a cylinder set $[\omega]$ satisfies

$$\mu(\omega) \stackrel{\text{def}}{=} \mu([\omega]) \asymp \|\phi_\omega'\|^\delta$$

and that

$$\mu(\omega\tau) \asymp \mu(\omega)\mu(\tau) \quad \text{for all } \omega, \tau \in E^*. \tag{3.1}$$

Convention 3.4. We write $\mu(A) = \sum_{\omega\in A} \mu(\omega)$ for all $A \subset E^*$, and $\mu(A) = \mu(\pi(A))$ for all $A \subset E^{\mathbb{N}}$.

The aim of this paper is to study the Hausdorff and packing measures of the measure μ_P, where $P \subset \mathbb{N}$ is the set of primes. To define these quantities, let $\psi : (0, \infty) \to (0, \infty)$ be a *dimension function*, that is, a continuous increasing function such that $\lim_{r\to 0} \psi(r) = 0$. Then the *$\psi$-dimensional Hausdorff measure* of a set $A \subset \mathbb{R}$ is

$$\mathcal{H}^\psi(A) \stackrel{\text{def}}{=} \lim_{\epsilon\searrow 0} \inf \bigg\{ \sum_{i=1}^\infty \psi(\text{diam}(U_i)) : (U_i)_1^\infty \text{ is a countable cover of } A \text{ with } \text{diam}(U_i) \le \epsilon \text{ for all } i \bigg\}$$

and the *ψ-dimensional packing measure* of A is defined by the formulas

$$\widetilde{\mathcal{P}}^\psi(A) \stackrel{\text{def}}{=} \lim_{\epsilon\searrow 0} \sup \bigg\{ \sum_{j=1}^\infty \psi(\text{diam}(B_j)) : \begin{array}{l} (B_j)_1^\infty \text{is a countable disjoint collection of balls} \\ \text{with centers in } A \text{ and with } \text{diam}(B_j) \le \epsilon \text{ for all } j \end{array} \bigg\}$$

and

$$\mathcal{P}^\psi(A) \stackrel{\text{def}}{=} \inf \bigg\{ \sum_{i=1}^\infty \widetilde{\mathcal{P}}^\psi(A_i) : A \subset \bigcup_{i=1}^\infty A_i \bigg\}.$$

A special case is when $\psi(r) = r^s$ for some $s > 0$, in which case we write $\mathcal{H}^\psi = \mathcal{H}^s$ and $\mathcal{P}^\psi = \mathcal{P}^s$.

If μ is a locally finite Borel measure on $\mathbb{R}$, then we let

$$\mathcal{H}^\psi(\mu) \stackrel{\text{def}}{=} \inf\{\mathcal{H}^\psi(A) : \mu(\mathbb{R} \setminus A) = 0\},$$

$$\mathcal{P}^\psi(\mu) \stackrel{\text{def}}{=} \inf\{\mathcal{P}^\psi(A) : \mu(\mathbb{R} \setminus A) = 0\}.$$

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This is analogous to the definitions of the (upper) Hausdorff and packing dimensions of μ; see [11, Proposition 10.3].

Remark. The Hausdorff and packing dimensions of sets [11, §2.1] and the (upper) Hausdorff and packing dimensions of measures [11, Proposition 10.3] can be defined in terms of $\mathcal{H}^s$ and $\mathcal{P}^s$ as follows:

$$\dim_H(A) \stackrel{\text{def}}{=} \sup\{s \geq 0 : \mathcal{H}^s(A) > 0\}, \overline{\dim_H}(\mu) \stackrel{\text{def}}{=} \sup\{s \geq 0 : \mathcal{H}^s(\mu) > 0\},$$
$$\dim_P(A) \stackrel{\text{def}}{=} \sup\{s \geq 0 : \mathcal{P}^s(A) > 0\}, \overline{\dim_P}(\mu) \stackrel{\text{def}}{=} \sup\{s \geq 0 : \mathcal{P}^s(\mu) > 0\}.$$

It follows from [25, Theorems 2.7 and 2.11] and Theorems 2.1 and 2.3 above that

$$\dim_H(\Lambda_P) = \dim_P(\Lambda_P) = \overline{\dim_H}(\mu_P) = \overline{\dim_P}(\mu_P) = \delta_P.$$

For each point $x \in \mathbb{R}$ let

$$\overline{D}_\mu^\psi(x) \stackrel{\text{def}}{=} \limsup_{r\searrow 0} \frac{\mu(B(x,r))}{\psi(r)},$$
$$\underline{D}_\mu^\psi(x) \stackrel{\text{def}}{=} \liminf_{r\searrow 0} \frac{\mu(B(x,r))}{\psi(r)}.$$

THEOREM 3.5. (Rogers–Taylor–Tricot density theorem, [39, Theorems 2.1 and 5.4]; see also [32]) *Let μ be a positive and finite Borel measure on $\mathbb{R}$, and let ψ be a dimension function. Then for every Borel set $A \subset \mathbb{R}$,*

$$\mu(A) \inf_{x\in A} \frac{1}{\overline{D}_\mu^\psi(x)} \lesssim_\times \mathcal{H}^\psi(A) \lesssim_\times \mu(\mathbb{R}) \sup_{x\in A} \frac{1}{\overline{D}_\mu^\psi(x)}, \tag{3.2}$$
$$\mu(A) \inf_{x\in A} \frac{1}{\underline{D}_\mu^\psi(x)} \lesssim_\times \mathcal{P}^\psi(A) \lesssim_\times \mu(\mathbb{R}) \sup_{x\in A} \frac{1}{\underline{D}_\mu^\psi(x)}. \tag{3.3}$$

COROLLARY 3.6. *Let μ, ψ be as in Theorem 3.5. Then*

$$\mathcal{H}^\psi(\mu) \asymp_\times \operatorname*{ess\,sup}_{x\sim\mu} \frac{1}{\overline{D}_\mu^\psi(x)}, \tag{3.4}$$
$$\mathcal{P}^\psi(\mu) \asymp_\times \operatorname*{ess\,sup}_{x\sim\mu} \frac{1}{\underline{D}_\mu^\psi(x)}. \tag{3.5}$$

Here the implied constants may depend on μ and ψ, and $\operatorname{ess\,sup}_{x\sim\mu}$ *denotes the essential supremum with x distributed according to μ.*

Proof. We prove (3.4); (3.5) is similar. For the $\lesssim$ direction, take

$$A = \left\{x : \frac{1}{\overline{D}_\mu^\psi(x)} \leq \operatorname*{ess\,sup}_{y\sim\mu} \frac{1}{\overline{D}_\mu^\psi(y)}\right\}$$

in the right half of (3.2). A has full μ-measure, so $\mathcal{H}^\psi(\mu) \leq \mathcal{H}^\psi(A)$. For the $\gtrsim$ direction, let B be a set of full μ-measure, fix $t < \operatorname{ess\,sup}_{y\sim\mu}(1/\overline{D}_\mu^\psi(y))$, and let

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$$A = B \cap \left\{ x : \frac{1}{\overline{D}_\mu^\psi(x)} \geq t \right\}.$$

Then $\mu(A) > 0$. Applying the left half of (3.2), using $\mathcal{H}^\psi(A) \leq \mathcal{H}^\psi(B)$, and then taking the infimum over all B and supremum over t yields the $\gtrsim$ direction of (3.4). □

Remark. For a doubling dimension function ψ and a conformal measure $\mu = \mu_E$, the ess sup in (3.4)–(3.5) can be replaced by ess inf due to the ergodicity of the shift map σ with respect to μ [**24**, Theorem 3.8]. Indeed, a routine calculation shows that $\overline{D}_\mu^\psi(x) \asymp \overline{D}_\mu^\psi(\sigma(x))$ for all x, whence ergodicity implies that the function $x \mapsto \overline{D}_\mu^\psi(x)$ is constant μ-almost everywhere, and similarly for $x \mapsto \underline{D}_\mu^\psi(x)$.

Terminological note. If ψ is a dimension function such that $\mathcal{H}^\psi(A)$ (respectively, $\mathcal{H}^\psi(\mu)$) is positive and finite, then ψ is called an *exact Hausdorff dimension function* for A (respectively, μ). Similar terminology applies to packing dimension.

4. *Results for regular Gauss IFSs*

In this section we consider a regular Gauss IFS Φ_E and state some results concerning $\mathcal{H}^\psi(\mu_E)$ and $\mathcal{P}^\psi(\mu_E)$, given appropriate assumptions on E and ψ. Throughout the section we will make use of the following assumptions, all of which hold for the prime Gauss IFS Φ_P.

Assumption 4.1. The set $E \subset \mathbb{N}$ satisfies an asymptotic law

$$\#(E \cap [N, 2N]) \asymp f(N), \tag{4.1}$$

where f is regularly varying with exponent $s \in (\delta, 2\delta)$. (A function f is said to be *regularly varying with exponent* s if for all $a > 1$, we have $\lim_{x\to\infty}(f(ax)/f(x)) = a^s$.) For example, if E is the set of primes, then by the prime number theorem $f(N) = N/\log(N)$ satisfies (4.1), and f is regularly varying with exponent $s = 1 \in (\delta, 2\delta)$, since $1/2 < \delta_P < 1$.

Assumption 4.2. There exists $\lambda > 1$ such that for all $0 < r \leq 1$,

$$\mu(\{a \in E : \lambda^{-1} r < \|\phi_a'\| \leq r\}) \asymp \mu(\{a \in E : \|\phi_a'\| \leq r\}).$$

For example, if E is the set of primes, then this assumption follows from the prime number theorem via a routine calculation showing that both sides are $\asymp r^\delta/\log(1/r)$.

Assumption 4.3. The Lyapunov exponent $-\sum_{a\in E} \mu(a) \log \|\phi_a'\|$ is finite. Note that this is satisfied when E is the set of primes, since $\mu(a) \log \|\phi_a'\| \asymp a^{-2\delta} \log(a)$ and $\delta > 1/2$.

For each $k \in \mathbb{N}$, let

$$W_k \stackrel{\text{def}}{=} \{\omega \in E^* : \lambda^{-(k+1)} < \|\phi_\omega'\| \leq \lambda^{-k}\}. \tag{4.2}$$

Note that although the sets $([\omega])_{\omega\in W_k}$ are not necessarily disjoint, there is a uniform bound (depending on λ) on the multiplicity of the collection, that is, there exists a constant C independent of k such that $\sup_x \{\#\{\omega \in W_k : x \in [\omega]\}\} \leq C$.

LEMMA 4.4. *Assume that Assumption 4.2 holds. Let* $\mathcal{J} = (J_k)_1^\infty$ *be a sequence of subsets of E, and let*

$$\Sigma_{\mathcal{J}} \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} \mu(J_k),$$

$$S_{\mathcal{J}} \stackrel{\text{def}}{=} \{\omega \in E^{\mathbb{N}} : \text{there exist infinitely many } (n,k) \text{ such that } \omega \upharpoonright n \in W_k, \omega_{n+1} \in J_k\}.$$

Then $\mu(S_{\mathcal{J}}) > 0$ *if* $\Sigma_{\mathcal{J}} = \infty$, *and* $\mu(S_{\mathcal{J}}) = 0$ *otherwise.*

Proof. For each $k \in \mathbb{N}$, let

$$A_k = \bigcup \{[\omega a] : \omega \in W_k, a \in J_k\}.$$

We claim that

(1) $\mu(A_k) \asymp \mu(J_k)$ and that

(2) the sequence $(A_k)_1^\infty$ is quasi-independent, meaning that $\mu(A_k \cap A_\ell) \lesssim \mu(A_k)\mu(A_\ell)$ whenever $k \neq \ell$.

Proof of (1). Since the collection $([\omega])_{\omega \in W_k}$ has bounded multiplicity, we have

$$\mu(A_k) \asymp \sum_{\omega \in W_k} \sum_{a \in J_k} \mu(\omega a) \asymp \sum_{\omega \in W_k} \mu(\omega)\mu(J_k)$$

and

$$\begin{aligned}
\sum_{\omega \in W_k} \mu(\omega) &\asymp \sum_{\omega \in W_k \|\phi'_{\omega \upharpoonright |\omega|-1}\| > \lambda^{-k}} \mu(\omega) \quad (\text{since}([\omega])_{\omega \in W_k} \text{has bounded multiplicity}) \\
&\asymp \sum_{\omega \in E^* \|\phi'_\omega\| > \lambda^{-k}} \sum_{a \in E \omega a \in W_k} \mu(\omega)\mu(a) \\
&\asymp \sum_{\omega \in E^* \|\phi'_\omega\| > \lambda^{-k}} \mu(\omega) \sum_{a \in E \|\phi'_{\omega a}\| \le \lambda^{-k}} \mu(a) \quad (\text{by Assumption 4.2}) \\
&\asymp \sum_{\omega \in E^* \ \|\phi'_\omega\| > \lambda^{-k}} \sum_{a \in E \|\phi'_{\omega a}\| \le \lambda^{-k}} \mu(\omega a) = \mu([0,1]) = 1.
\end{aligned}$$

Proof of (2). Let $k < \ell$. Then

$$\begin{aligned}
\mu(A_k \cap A_\ell) &= \sum_{\omega \in W_k} \sum_{a \in J_k} \sum_{\tau \in E^* \omega a \tau \in W_\ell} \sum_{b \in J_\ell} \mu(\omega a \tau b) \\
&\asymp \sum_{\omega \in W_k} \mu(\omega) \sum_{a \in J_k} \mu(a) \sum_{\tau \in E^* \omega a \tau \in W_\ell} \mu(\tau)\mu(J_\ell) \\
&\lesssim \mu(J_k)\mu(J_\ell) \asymp \mu(A_k)\mu(A_\ell),
\end{aligned}$$

where the $\lesssim$ in the last line is because the collection $\{[\tau] : \tau \in E^*, \omega a \tau \in W_\ell\}$ has bounded multiplicity.

Now if $\sum_k \mu(J_k) < \infty$, the convergence case of the Borel–Cantelli lemma completes the proof. If $\sum_k \mu(J_k) = \infty$, then (2) implies that condition [5, (3)] holds, so [5, Lemma DBC] completes the proof. □

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THEOREM 4.5. (Global measure formula for Gauss IFSs) *Let Φ_E be a regular Gauss IFS. Then for all $x = \pi(\omega) \in \Lambda_E$ and $r > 0$, there exists n such that*

$$[\omega \upharpoonright n+1] \subset B(x, Cr) \tag{4.3}$$

and

$$M(x,n,r) \le \mu(B(x,r)) \lesssim M(x,n,Cr), \tag{4.4}$$

where

$$M(x,n,r) \stackrel{\text{def}}{=} \sum_{\substack{a\in E \\ [(\omega\upharpoonright n)a]\subset B(x,r)}} \mu((\omega \upharpoonright n)a),$$

and where $C \ge 1$ is a uniform constant.

We call this theorem a 'global measure formula' due to its similarity to other global measure formulas found in the literature, such as [37, §7], [36, Theorem 2].

Proof. Note that the first inequality $M(x,n,r) \le \mu(B(x,r))$ follows trivially from applying μ to both sides of the inclusion $\bigcup\{[\tau a] \subset B(x,r) : a \in E\} \subset B(x,r)$.

Given $x = \pi(\omega) \in \Lambda_E$ and $r > 0$, let $m \ge 0$ be maximal such that $B(x,r) \cap \Lambda_E \subset [\omega \upharpoonright m]$. By applying the inverse transformation $\phi^{-1}_{\omega\upharpoonright m}$ to the setup and using the bounded distortion property we may without loss of generality assume that $m = 0$, or equivalently that $B(x,r)$ intersects at least two top-level cylinders. We now divide into two cases.

- If $[\omega_1] \subset B(x,r)$, then we claim that

$$B(x,r) \cap \Lambda_E \subset \bigcup_{\substack{a\in E \\ [a]\subset B(x,Cr)}} [a]$$

which guarantees (4.4) with $n = 0$. Indeed, if $y = \pi(\tau) \in B(x,r) \cap \Lambda_E$, then $1/(\tau_1 + 1) \le \pi(\tau) \le \pi(\omega) + r \le 1/\omega_1 + r$ and thus

$$\begin{aligned} \operatorname{diam}([\tau_1]) \asymp \frac{1}{\tau_1^2} &\lesssim \max\left(\frac{1}{\omega_1^2}, r^2\right) \\ &\asymp \operatorname{diam}([\omega_1]) + r^2 \le 2r + r^2 \lesssim r. \end{aligned}$$

- If $[\omega_1]$ is not contained in $B(x,r)$, then one of the endpoints of $[\omega_1]$, namely $1/\omega_1$ or $1/(\omega_1+1)$, is contained in $B(x,r)$, but not both. Suppose that $1/\omega_1 \in B(x,r)$; the other case is similar. Now for all $N \in E$ such that $[(\omega_1 - 1) * 1 * N] \cap B(x,r) \ne \emptyset$ (where $*$ denotes concatenation), we have

$$r \ge d(1/\omega_1, [(\omega_1 - 1) * 1 * N]) \asymp 1/(\omega_1^2 N) \asymp d(1/\omega_1, \min([\omega_1 * N]))$$

and thus $[\omega_1 * N] \subset B(1/\omega_1, Cr) \subset B(x, (C+1)r)$ for an appropriately large constant C. Applying μ and summing over all such N gives

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$$\mu(B(x,r)) \le \mu([\omega_1] \cap B(x,r)) + \sum_{\substack{N \in E \\ [(\omega_1 - 1)*1*N] \cap B(x,r) \neq \emptyset}} \mu([(\omega_1 - 1) * 1 * N])$$
$$\lesssim \sum_{\substack{N \in E \\ [\omega_1 * N] \subset B(x,(C+1)r)}} \mu([\omega_1 * N])$$

which implies (4.4) with $n = 1$. On the other hand, since

$$r \ge d(x, 1/\omega_1) \asymp 1/(\omega_1^2 \omega_2) \ge 1/(\omega_1^2 \omega_2^2) \asymp \operatorname{diam}([\omega \restriction 2]),$$

we have $[\omega \restriction 2] \subset B(x, Cr)$ as long as C is sufficiently large. □

Fix $\epsilon \in \{\pm 1\}$ (loosely speaking, $\epsilon = 1$ when we are trying to prove results about Hausdorff measure, and $\epsilon = -1$ when we are trying to prove results about packing measure), a real number $\alpha > 0$, and a doubling dimension function $\psi(r) = r^\delta \Psi(r)$. We will assume that Ψ is *ϵ-monotonic*, meaning that Ψ is decreasing if $\epsilon = 1$ and increasing if $\epsilon = -1$. Fix $\alpha > 0$, and for each $k \in \mathbb{N}$ let

$$J_{k,\alpha,\epsilon} \stackrel{\text{def}}{=} \{a \in E : \text{there exists } r \in [\|\phi_a'\|, 1] \text{ with } r^{-\delta}\mu(B([a], r)) \lesseqgtr \alpha\Psi(\lambda^{-k} r)\}.$$

Here $\lesseqgtr$ denotes $\ge$ if $\epsilon = 1$ and $\le$ if $\epsilon = -1$, and $\lambda > 1$ is as in Assumption 4.2. Write

$$S_{\alpha,\epsilon} \stackrel{\text{def}}{=} S_{\mathcal{J}_{\alpha,\epsilon}} \text{ for } \mathcal{J}_{\alpha,\epsilon} \stackrel{\text{def}}{=} (J_{k,\alpha,\epsilon})_{k=1}^\infty,$$

as defined in Lemma 4.4. Note that $S_{\alpha,1}$ grows smaller as α grows larger, while $S_{\alpha,-1}$ grows larger as α grows larger.

PROPOSITION 4.6. *For all $\omega \in E^{\mathbb{N}}$,*

$$\sup\{\alpha : \omega \in S_{\alpha,1}\} \asymp \overline{D}_\mu^\psi(\pi(\omega)),$$
$$\inf\{\alpha : \omega \in S_{\alpha,-1}\} \asymp \underline{D}_\mu^\psi(\pi(\omega)).$$

Proof. Let $x = \pi(\omega)$, fix $r > 0$, and let C, n, and $\tau = \omega \restriction n$ be as in the global measure formula. Write $\tau \in W_k$ for some k, as in (4.2). By the global measure formula, we have

$$\sum_{\substack{a \in E \\ [\tau a] \subset B(x,r)}} \mu(\tau a) \le \mu(B(x,r)) \lesssim \sum_{\substack{a \in E \\ [\tau a] \subset B(x,Cr)}} \mu(\tau a).$$

Now for each $\beta \ge 1$ let

$$\Theta_\beta \stackrel{\text{def}}{=} \frac{\mu(B(x, \beta r))}{\psi(\beta r)}.$$

Let $y = \pi(\sigma^n \omega)$, where $\sigma : E^{\mathbb{N}} \to E^{\mathbb{N}}$ is the shift map. Then there exist constants $C_2, C_3 > 0$ (independent of x, r, n, and k) such that for all $s > 0$,

$$B(x, C_2\lambda^{-k}s) \subset \phi_\tau(B(y,s)) \subset B(x, C_3\lambda^{-k}s).$$

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Taking $s = C_3^{-1}\lambda^k r$ and $s = C_2^{-1}C\lambda^k\beta r$, and using the bounded distortion property and the fact that $\mu(\tau) \asymp \lambda^{-\delta k}$, yields

$$\Theta_1 \lesssim \frac{1}{\psi(r)}\lambda^{-\delta k} \sum_{\substack{a\in E\\ [a]\subset B(y,C_2^{-1}C\lambda^k r)}} \mu(a),$$

$$\Theta_\beta \gtrsim_\beta \frac{1}{\psi(r)}\lambda^{-\delta k} \sum_{\substack{a\in E\\ [a]\subset B(y,C_3^{-1}\lambda^k \beta r)}} \mu(a).$$

Write $b = \omega_{n+1}$, so that $x \in [\tau b] \subset B(x, Cr)$ by (4.3) and thus by the bounded distortion property $y \in [b] \subset B(y, C_4\lambda^k r)$ for sufficiently large C_4. Then $R \stackrel{\text{def}}{=} 2C_4\lambda^k r \geq \operatorname{diam}([b])$. Thus,

$$\mu(B(y, R)) \leq \mu(B([b], R)) \leq \mu(B(y, 2R)),$$

so $\Theta_1 \lesssim \Xi \lesssim_\beta \Theta_\beta$ for some $C_5, C_6 > 0$, where

$$\Xi \stackrel{\text{def}}{=} \frac{1}{\psi(\lambda^{-k}R)}\lambda^{-\delta k} \sum_{\substack{a\in E\\ [a]\subset B([b],R)}} \mu(a) = \frac{1}{\Psi(\lambda^{-k}R)} R^{-\delta} \sum_{\substack{a\in E\\ [a]\subset B([b],R)}} \mu(a).$$

Applying the global measure formula again yields

$$\Theta_1 \lesssim \frac{1}{\Psi(\lambda^{-k}R)} R^{-\delta}\mu(B([b], R)) \lesssim \Theta_\beta$$

for some $C_7, C_8 > 0$, and thus

$$\Theta_1 \geq C_9\alpha \Rightarrow R^{-\delta}\mu(B([b], R)) \geq \alpha\Psi(\lambda^{-k}R) \Rightarrow \Theta_\beta \geq C_{10}\alpha,$$
$$\Theta_\beta \leq C_{11}\alpha \Rightarrow R^{-\delta}\mu(B([b], R)) \leq \alpha\Psi(\lambda^{-k}R) \Rightarrow \Theta_1 \leq C_{12}\alpha$$

for some $C_9, C_{10}, C_{11}, C_{12} > 0$ and for all $\alpha > 0$. It follows that

$$\overline{D}_\mu^\psi(\pi(\omega)) \geq C_{13}\alpha \Rightarrow \omega \in S_{\alpha,1} \Rightarrow \overline{D}_\mu^\psi(\pi(\omega)) \geq C_{14}\alpha,$$
$$\underline{D}_\mu^\psi(\pi(\omega)) \leq C_{15}\alpha \Rightarrow \omega \in S_{\alpha,-1} \Rightarrow \underline{D}_\mu^\psi(\pi(\omega)) \leq C_{16}\alpha,$$

since $\omega \in S_{\alpha,\epsilon}$ if and only if there exist infinitely many n, k, R such that $\omega \upharpoonright n \in W_k$, $R \in [\|\phi'_{\omega_{n+1}}\|, 1]$, and $R^{-\delta}\mu(B([b], R)) \lesseqgtr \alpha\Psi(\lambda^{-k}R)$, and $\limsup_{r\searrow 0} \Theta_1 = \limsup_{r\searrow 0} \Theta_\beta = \overline{D}_\mu^\psi(\pi(\omega))$ and $\liminf_{r\searrow 0} \Theta_1 = \liminf_{r\searrow 0} \Theta_\beta = \underline{D}_\mu^\psi(\pi(\omega))$. Taking the supremum (respectively, infimum) with respect to α completes the proof. □

So to calculate $\mathcal{H}^\psi(\mu)$ or $\mathcal{P}^\psi(\mu)$, we need to determine whether the series $\Sigma_{\alpha,\epsilon} \stackrel{\text{def}}{=} \sum_{k=1}^\infty \mu(J_{k,\alpha,\epsilon})$ converges or diverges for each $\alpha > 0$.

LEMMA 4.7. *Let $\epsilon = 1$, and suppose that Ψ is ϵ-monotonic. If $\sum_{k=1}^\infty \mu(J_{k,\alpha,\epsilon})$ converges (respectively, diverges) for all $\alpha > 0$, then $\mathcal{H}^\psi(\mu) = \infty$ (respectively, $= 0$); otherwise $\mathcal{H}^\psi(\mu)$ is positive and finite. If $\epsilon = -1$, the analogous statement holds for $\mathcal{P}^\psi(\mu)$.*

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Proof. By Corollary 3.6 (and the subsequent remark), it suffices to show that $\overline{D}^{\psi}_{\mu}(\pi(\omega)) = 0$ (respectively, $= \infty$) for a positive μ-measure set of ωs. By Proposition 4.6, this is equivalent to showing that $\sup\{\alpha : \omega \in S_{\alpha,1}\} = 0$ (respectively, $= \infty$), or equivalently that $\omega \notin S_{\alpha,1}$ (respectively, $\in S_{\alpha,1}$) for all $\alpha > 0$. For each α, to show this for a positive μ-measure set of ωs it suffices to show that $\mu(S_{\alpha,1}) = 0$ (respectively, > 0), which by Lemma 4.4 is equivalent to showing that $\sum_{k=1}^{\infty} \mu(J_{k,\alpha,1})$ converges (respectively, diverges). The cases $\epsilon = -1$ and where $\sum_{k=1}^{\infty} \mu(J_{k,\alpha,\epsilon})$ converges for some α but diverges for others are proven similarly. □

LEMMA 4.8. *Assume that Assumptions 4.1, 4.2, and 4.3 all hold, and that Ψ is ϵ-monotonic. Then there exists a constant $C \geq 1$ such that for all $\alpha > 0$ and $\epsilon \in \{\pm 1\}$, we have*

$$\Sigma'_{C^{\epsilon}\alpha,\epsilon} \lesssim_{+,\times} \Sigma_{\alpha,\epsilon} \lesssim_{+,\times} \Sigma'_{C^{-\epsilon}\alpha,\epsilon}$$

where

$$\Sigma'_{\alpha,-1} \stackrel{\text{def}}{=} \sum_{a\in E} \mu(a) \max_{1\leq x\leq a/3} \log(1/\Psi^{-1}(\alpha^{-1}x^{-\delta}\#(B(a,x)\cap E))),$$

$$\Sigma''_{\alpha} \stackrel{\text{def}}{=} \sum_{a\in E} \mu(a) \log(1/\Psi^{-1}(\alpha^{-1}F(a^{-1}))),$$

$$\Sigma'_{\alpha,1} \stackrel{\text{def}}{=} \Sigma'_{\alpha,-1} + \Sigma''_{\alpha}.$$

Here $F(r) = r^{\delta} f(r^{-1})$, where f is as in Assumption 4.1.

Proof. Indeed,

$$\begin{aligned}
&\sum_{k=1}^{\infty} \mu(J_{k,\alpha,\epsilon})\\
&\quad= \sum_{a\in E} \mu(a)\#\{k\in\mathbb{N} : \text{there exists } r \in [\|\phi'_a\|, 1] r^{-\delta}\mu(B([a], r)) \lesseqgtr \alpha\Psi(\lambda^{-k}r)\}\\
&\quad= \sum_{a\in E} \mu(a) \max\{k\in\mathbb{N} : \text{there exists } r \in [\|\phi'_a\|, 1] r^{-\delta}\mu(B([a], r)) \lesseqgtr \alpha\Psi(\lambda^{-k}r)\}\\
&\qquad(\text{since } \Psi \text{ is decreasing if } \epsilon = 1 \text{ and increasing if } \epsilon = -1)\\
&\quad\asymp_{+,\times} \sum_{a\in E} \mu(a) \max(0, \max_{r\in[\|\phi'_a\|,1]} \log_{\lambda}(r/\Psi^{-1}(\alpha^{-1}r^{-\delta}\mu(B([a], r)))))\\
&\quad\in \bigg[\sum_{a\in E} \mu(a) \log_{\lambda} \|\phi'_a\|, C\bigg] + \sum_{a\in E} \mu(a) \max_{r\in[\|\phi'_a\|,1]} \log_{\lambda}(1/\Psi^{-1}(\alpha^{-1}r^{-\delta}\mu(B([a], r))))\\
&\qquad(\text{since } r \leq 1 \lesssim 1/\Psi^{-1}(\alpha^{-1}r^{-\delta}\mu(B([a], r)))).
\end{aligned}$$

The first term is finite by Assumption 4.3. The second term can be analyzed by considering

$$\Sigma_{1,\alpha} \stackrel{\text{def}}{=} \sum_{a\in E} \mu(a) \max_{a^{-1}\leq r\leq 1} \log_{\lambda}(1/\Psi^{-1}(\alpha^{-1}r^{-\delta}\mu(B([a], r)))),$$

$$\Sigma_{2,\alpha} \stackrel{\text{def}}{=} \sum_{a\in E} \mu(a) \max_{\|\phi'_a\|\leq r\leq a^{-1}/3} \log_{\lambda}(1/\Psi^{-1}(\alpha^{-1}r^{-\delta}\mu(B([a], r)))).$$

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Then

$$\Sigma_{1,\alpha} + \Sigma_{2,\alpha} \lesssim_{+,\times} \Sigma_{\alpha,\epsilon} \lesssim_{+,\times} \Sigma_{1,3^\delta\alpha} + \Sigma_{2,3^{-\delta}\alpha}.$$

Now for $r > 0$ sufficiently small we have

$$\begin{aligned}\mu(B(0,r)) &\geq \sum_{a \geq r^{-1}} \mu(a) \asymp \sum_{a \geq r^{-1}} a^{-2\delta} \\ &\asymp \sum_{k=0}^{\infty} \#(E \cap [2^k r^{-1}, 2^{k+1} r^{-1}])(2^k r^{-1})^{-2\delta} \\ &\asymp \sum_{k=0}^{\infty} f(2^k r^{-1}) 2^{-2k\delta} r^{2\delta} \asymp f(r^{-1}) r^{2\delta} \\ &\qquad \text{(since } f \text{ is regularly varying with exponent } s < 2\delta)\end{aligned}$$

and similarly for the reverse direction, giving

$$\mu(B(0,r)) \asymp r^{2\delta} f(r^{-1}).$$

When $r \geq a^{-1}$, we have $B(0,r) \cap [0,1] \subset B([a],r) \subset B(0,2r)$, so

$$\mu(B([a],r)) \asymp r^{2\delta} f(r^{-1})$$

and thus $\Sigma'_{1,C^\epsilon\alpha} \leq \Sigma_{1,\alpha} \leq \Sigma'_{1,C^{-\epsilon}\alpha}$, where

$$\Sigma'_{1,\alpha} \stackrel{\text{def}}{=} \sum_{a \in E} \mu(a) \max_{a^{-1} \leq r \leq 1} \log(1/\Psi^{-1}(\alpha^{-1} r^\delta f(r^{-1}))).$$

Since f is regularly varying with exponent $s > \delta$, the function $F(r) = r^\delta f(r^{-1})$ is monotonically decreasing for r sufficiently small, while Ψ^{-1} is decreasing (respectively, increasing) if $\epsilon = 1$ (respectively, $\epsilon = -1$). It follows that the maximum occurs at $r = a^{-1}$ (respectively, $r = 1$), corresponding to

$$\Sigma'_{1,\alpha} \asymp_+ \sum_{a \in E} \mu(a) \log(1/\Psi^{-1}(\alpha^{-1} F(a^{-1}))) \quad \text{if } \epsilon = 1 \tag{4.5}$$

and

$$\Sigma'_{1,\alpha} \asymp_+ \sum_{a \in E} \mu(a) \text{ const.} < \infty \quad \text{if } \epsilon = -1. \tag{4.6}$$

The latter series always converges, whereas the former series may either converge or diverge.

On the other hand, we have

$$\Sigma_{2,\alpha} \asymp \sum_{a \in E} \mu(a) \max_{a^{-2} \leq r \leq a^{-1}/3} \log(1/\Psi^{-1}(\alpha^{-1} r^{-\delta} \mu(B([a],r)))).$$

Using the change of variables $r = a^{-2}x$ and the fact that $\mu(b) \asymp \mu(a) \asymp a^{-2\delta}$ for all $b \in E$ such that $B([a],r) \cap [b] \neq \emptyset$, we get $\Sigma'_{2,C^\epsilon\alpha} \lesssim \Sigma_{2,\alpha} \lesssim \Sigma'_{2,C^{-\epsilon}\alpha}$, where

$$\Sigma'_{2,\alpha} \stackrel{\text{def}}{=} \sum_{a \in E} \mu(a) \max_{1 \leq x \leq a/3} \log(1/\Psi^{-1}(\alpha^{-1} x^{-\delta} \#(B(a,x) \cap E))) \tag{4.7}$$

and $C \geq 1$ is a constant. Combining (4.5), (4.6), and (4.7) yields the conclusion. □

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5. *Proofs of main theorems*

In this section we consider the Gauss IFS Φ_P, where P is the set of primes. We begin with a number-theoretic lemma.

LEMMA 5.1. *For all* $\delta < 1$,

$$\#(P \cap B(a, x)) \lesssim (x/a)^\delta f(a) \quad \textit{for } 1 \le x \le a/3 \tag{5.1}$$

where $f(N) = N/\log(N)$ *is as in Assumption 4.1.*

Proof. A well-known result of Hoheisel [19] (see also [6, Ch. V] for a book reference) states that there exists $\theta < 1$ such that

$$\#(P \cap [a, b]) \asymp \frac{b-a}{\log(a)} \quad \text{if } a^\theta \le b - a \le a. \tag{5.2}$$

(This result has seen numerous improvements (see [31] for a survey), but it does not matter very much for our purposes, although the lower bound does make a difference in our upper bound for the exact packing dimension. The most recent improvements we were aware of are $\theta = 6/11 + \epsilon$ for the upper bound [21] and $\theta = 21/40$ for the lower bound [1, pp. 562].)

It follows that

$$\#(P \cap B(a, x)) \lesssim \frac{x}{\log(a)} \quad \text{if } a^\theta \le x \le a/3.$$

In this case, since $\delta < 1$ and $x \le a$ we have

$$\frac{x}{\log(a)} = \frac{x}{a} f(a) \le \left(\frac{x}{a}\right)^\delta f(a),$$

and combining yields (5.1) in this case. On the other hand, if $1 \le x \le a^\theta$, then

$$\#(P \cap B(a, x)) \le 2x + 1 \lesssim (x/a)^\delta f(a),$$

since

$$x^{1-\delta} \le a^{(1-\delta)\theta} \lesssim a^{1-\delta}/\log(a),$$

demonstrating (5.1) for the second case. □

Thus, for appropriate $C \ge 1$,

$$\begin{aligned}
\Sigma'_{\alpha,-1} &\le \sum_{a\in P} \mu(a) \log(1/\Psi^{-1}(C\alpha^{-1} \max_{1\le x\le a/3} x^{-\delta}(x/a)^\delta f(a))) \\
&= \sum_{a\in P} \mu(a) \log(1/\Psi^{-1}(C\alpha^{-1} a^{-\delta} f(a))) \\
&= \sum_{a\in P} \mu(a) \log(1/\Psi^{-1}(C\alpha^{-1} F(a^{-1}))) \asymp_+ \Sigma''_{C^{-1}\alpha}.
\end{aligned}$$

It follows that $\Sigma'_{\alpha,1} \lesssim \Sigma''_{C^{-1}\alpha}$.

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Proof of Theorem 2.1. Set $\epsilon = 1$ and $E = P$. If Ψ is increasing, then $\mathcal{H}^{\psi}(\mu) \lesssim \mathcal{H}^{\delta}(\mu) = 0$ and the series (2.1) diverges, so we may henceforth assume that Ψ is decreasing, allowing us to use Lemma 4.8. Using the Iverson bracket notation

$$[\Phi] = \begin{cases} 1, & \Phi \text{ true}, \\ 0, & \Phi \text{ false}, \end{cases}$$

we have

$$\begin{aligned}
\Sigma''_{\alpha} &\asymp_{+} \sum_{a \in P} \mu(a) \sum_{k=1}^{\infty} [k \le \log_{\lambda}(1/\Psi^{-1}(\alpha^{-1} F(a^{-1}))] \\
&= \sum_{a \in P} \mu(a) \sum_{k=1}^{\infty} [F^{-1}(\alpha \Psi(\lambda^{-k})) \ge a^{-1}] \\
&\asymp \sum_{k=1}^{\infty} \sum_{\substack{a \in P \\ a \ge 1/F^{-1}(\alpha\Psi(\lambda^{-k}))}} a^{-2\delta} \\
&\asymp \sum_{k=1}^{\infty} \frac{x^{1-2\delta}}{\log x} \Big|_{x = 1/F^{-1}(\alpha\Psi(\lambda^{-k}))}
\end{aligned}$$

Now we have $f(x) = x/\log x$, thus $F(r) = r^{\delta-1}/\log(r^{-1})$ and $F^{-1}(x) \asymp (x \log x)^{1/(\delta-1)}$. So we can continue the computation as

$$\begin{aligned}
&\asymp \sum_{k=1}^{\infty} \frac{x^{1-2\delta}}{\log x} \Big|_{x = (y \log y)^{1/(1-\delta)}, y = \alpha\Psi(\lambda^{-k})} \\
&\asymp \sum_{k=1}^{\infty} \frac{(y \log y)^{(1-2\delta)/(1-\delta)}}{\log y} \Big|_{y = \alpha\Psi(\lambda^{-k})} \\
&\asymp \sum_{k=1}^{\infty} \frac{(y \log y)^{(1-2\delta)/(1-\delta)}}{\log y} \Big|_{y = \Psi(\lambda^{-k})} .
\end{aligned}$$

If this series converges for all $\alpha > 0$, then so do $\Sigma'_{\alpha,1}$ and (by Lemma 4.8) $\sum_k \mu(J_{k,\alpha,1})$, and thus by Lemma 4.7 we have $\mathcal{H}^{\psi}(\mu) = \infty$. On the other hand, if the series diverges, then so does $\sum_k \mu(J_{k,\alpha,1})$ for all $\alpha > 0$, and thus by Lemma 4.7 we have $\mathcal{H}^{\psi}(\mu) = 0$. $\square$

Proof of Theorem 2.3. We can get bounds on the exact packing dimension of μ_P by using known results about the distribution of primes. First, we state the strongest known lower bound on two-sided gaps.

THEOREM 5.2. [22] *Let p_n denote the nth prime and let $d_n = p_{n+1} - p_n$. For all k,*

$$\limsup_{n \to \infty} \frac{\min(d_{n+1}, \ldots, d_{n+k})}{\phi(p_n)} > 0$$

where ϕ is as in (2.2).

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Let $k = 2$ and let (n_ℓ) be a sequence along which the limit superior is achieved, and let $a_\ell = p_{n_\ell+2} \in P$ for some $\ell \in \mathbb{N}$, so that $P \cap B(a_\ell, x) = \{a_\ell\}$, where $x = c\phi(a_\ell)$ for some constant $c > 0$. Then we have

$$\Sigma'_{\alpha,-1} \gtrsim \sum_{a\in P} a^{-2\delta} \log(1/\Psi^{-1}(\alpha^{-1}x^{-\delta}\#(P \cap B(a, x))))$$
$$\geq \sum_{\ell\in\mathbb{N}} a_\ell^{-2\delta} \log(1/\Psi^{-1}(\alpha^{-1}c^{-\delta}\phi(a_\ell)^{-\delta})).$$

Let $\psi(r) = r^\delta\phi^{-\delta}(\log(1/r))$ as in (2.3), so that $\Psi(r) = \phi^{-\delta}(\log(1/r))$ and $\Psi^{-1}(x) = \exp(-\phi^{-1}(x^{-1/\delta}))$. Note that

$$\phi^{-1}(x) = \log(1/\Psi^{-1}(x^{-\delta})) \asymp e^x \frac{\log^2 x}{x \log\log x},$$

and thus, letting $\alpha = c^{-\delta}\gamma^{-\delta}$,

$$a^{-2\delta} \log(1/\Psi^{-1}(\alpha^{-1}c^{-\delta}\phi(a)^{-\delta})) = a^{-2\delta}\phi^{-1}(\gamma\phi(a))$$
$$\asymp a^{-2\delta}a^\gamma$$

so that $\Sigma'_{\alpha,-1}$ diverges for $\gamma > 2\delta$. Combining with Lemmas 4.7 and 4.8 demonstrates (2.3).

On the other hand, we use the lower bound of Hoheisel's theorem (5.2) to get an upper bound on the exact packing dimension. Let $\theta = 21/40$. Fix $a \in P$ and $1 \leq x \leq a/3$, and let $\theta = 21/40$ as in (5.1). If $x \geq a^\theta$, then we have

$$x^{-\delta}\#(P \cap B(a, x)) \gtrsim x^{-\delta}\frac{x}{\log(a)} \geq a^{-\theta\delta}\frac{a^\theta}{\log(a)} \gtrsim a^{-\theta\delta},$$

and if $x \leq a^\theta$, then we have

$$x^{-\delta}\#(P \cap B(a, x)) \geq x^{-\delta} \geq a^{-\theta\delta}.$$

Thus, for appropriate $c_2 > 0$,

$$\Sigma'_{\alpha,-1} \asymp \sum_{a\in P} a^{-2\delta} \max_{1\leq x\leq a/3} \log(1/\Psi^{-1}(\alpha^{-1}x^{-\delta}\#(P \cap B(a, x))))$$
$$\leq \sum_{a\in P} a^{-2\delta} \log(1/\Psi^{-1}(c_2\alpha^{-1}a^{-\theta\delta})).$$

Letting

$$\psi(r) = r^\delta \log^{-s}(1/r), \Psi(r) = \log^{-s}(1/r), \Psi^{-1}(x) = \exp(-x^{-1/s}),$$

we get

$$\Sigma'_{\alpha,-1} \lesssim \sum_{a\in P} a^{-2\delta}a^{\theta\delta/s} < \infty \quad \text{if } \theta\delta/s < 2\delta - 1$$

Combining with Lemmas 4.7 and 4.8 demonstrates (2.4). □

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Proof of Theorem 2.5. In what follows we assume the cases $k = 1, 2$ of Conjecture 2.4. From the case $k = 1$ of Conjecture 2.4, in particular from $R_1 < \infty$, it follows that the gaps between primes have size $d_n = O(\log^2(p_n))$, and thus for all $a \in P$ and $1 \le x \le a/3$, we have

$$x^{-\delta}\#(P \cap B(a, x)) \gtrsim x^{-\delta}\left(\frac{x}{\log^2(a)} + 1\right) \ge x^{-\delta}\left(\frac{x}{\log^2(a)}\right)^{\delta} = \frac{1}{\log^{2\delta}(a)}. \tag{5.3}$$

On the other hand, from the case $k = 2$, in particular $R_2 > 0$, it follows that for an appropriate constant $c > 0$ there exists an infinite set $I \subset P$ (that is, the set $\{p_{n+2} : \min(d_{n+1}, d_{n+2}) \ge c \log^2(p_{n+2})\}$ for an appropriate constant $c > 0$) such that for all $a \in I$ and $1 \le x = x_a = c \log^2(a) \le a/3$ we have $P \cap B(a, x) = \{a\}$ and thus

$$x^{-\delta}\#(P \cap B(a, x)) = x^{-\delta} \asymp \frac{1}{\log^{2\delta}(a)}. \tag{5.4}$$

Now let $\psi(r) = r^{\delta} \log^{-2\delta} \log(1/r)$ be as in (2.5), so that $\Psi(r) = \log^{-2\delta}(1/r)$ and $\Psi^{-1}(x) = \exp(-\exp(x^{-1/2\delta}))$. In particular, Ψ^{-1} is increasing. It follows that for appropriate $C_1 \ge 1 \ge C_2 > 0$,

$$\begin{aligned}[a \in I] \log\left(1/\Psi^{-1}\left(\alpha^{-1}\frac{C_1}{\log^{2\delta}(a)}\right)\right) &\underset{(5.4)}{\le} \max_{1\le x \le a/3} \log(1/\Psi^{-1}(\alpha^{-1}x^{-\delta}\#(B(a, x) \cap P))) \\ &\underset{(5.3)}{\le} \log\left(1/\Psi^{-1}\left(\alpha^{-1}\frac{C_2}{\log^{2\delta}(a)}\right)\right),\end{aligned}$$

and plugging into (4.7) yields

$$\sum_{a\in I} \mu(a) \log\left(1/\Psi^{-1}\left(\alpha^{-1}\frac{C_1}{\log^{2\delta}(a)}\right)\right) \le \Sigma'_{\alpha,-1} \le \sum_{a\in P} \mu(a) \log\left(1/\Psi^{-1}\left(\alpha^{-1}\frac{C_2}{\log^{2\delta}(a)}\right)\right).$$

We get

$$\sum_{a\in I} a^{-2\delta} a^{(\alpha C_1^{-1})^{1/2\delta}} \lesssim \Sigma'_{\alpha,-1} \lesssim \sum_{a\in P} a^{-2\delta} a^{(\alpha C_2^{-1})^{1/2\delta}}.$$

By choosing $\alpha > 0$ so that $(\alpha C_2^{-1})^{1/2\delta} < 2\delta - 1$, we get $\Sigma'_{\alpha,-1} < \infty$, and by choosing $\alpha > 0$ so that $(\alpha C_1^{-1})^{1/2\delta} > 2\delta$, we get $\Sigma'_{\alpha,-1} = \infty$. It follows then from Lemmas 4.7 and 4.8 that $\mathcal{P}^{\psi}(\mu)$ is positive and finite. □

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