ON PERIODIC SOLUTIONS OF A SEMILINEAR
HYPERBOLIC PARABOLIC EQUATION

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Uniqueness and regularity of periodic solutions to the semilinear dissipative wave equation with small parameter $\varepsilon > 0$,

$$\varepsilon u_{tt} - \Delta u + u_t + g(u) = f(x, t) \quad \text{on} \quad \Omega \times R \quad \text{and} \quad u|_{\partial \Omega} = 0, \quad \Omega \subset R^N,$$

are investigated when $g(u)$ has a certain 'critical' nonlinearity.

1. INTRODUCTION

Let $\Omega$ be a bounded domain in the $N$-dimensional Euclidean space $R^N$ with smooth boundary $\partial \Omega$ and let us consider the periodicity problem for the semilinear wave equation with a dissipation:

$$\begin{cases}
u u_{tt} - \Delta u + \nu u_t + g(u) = f(x, t) \quad \text{on} \quad \Omega \times R, \\ u(x, t + \omega) = u(x, t) \quad \text{for} \quad (x, t) \in \Omega \times R \quad \text{and} \quad u|_{\partial \Omega} = 0 \quad \text{for} \quad t \in R,
\end{cases}$$

where $f(x, t)$ is an $\omega$-periodic function in $t$ and $g(u)$ is a nonlinear function satisfying

$$g(0) = 0 \quad \text{and} \quad 0 \leq g'(u) \leq k_0(1 + |u|^\alpha)$$

for some $\alpha \geq 0$. We assume $\nu = 1$ without loss of generality.

Concerning the existence of a 'weak' solution of the problem (1.1) it is well known that if $f \in L^2(\omega; L^2(\Omega))$ and $0 \leq \alpha \leq 2/(N - 2)$ $(0 \leq \alpha < \infty$ if $N = 1, 2)$, then the problem admits a solution $u$ in the class

$$C\left(\omega; \overset{\circ}{H}_1(\Omega)\right) \cap C^1(\omega; L^2(\Omega))$$

(see Clements [1], Nakao [4], et cetera), where we denote by $C(\omega; X)$, $X$ : Banach space, the space of $\omega$-periodic continuous $X$-valued functions on $R = (-\infty, +\infty)$. Similar notations will be used freely.

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Concerning the uniqueness and regularity, however, there remain some important open problems. First of all the uniqueness of weak solutions in the class (1.3) is not known. Moreover, it is not known whether the solution is classical or not for the case $\alpha = 2/(N-2)$ ($N \geq 3$), which is different from the case of initial-boundary value problems (see Sather [7], Wahl [9], et cetera). Of course, if $0 < \alpha < 2/(N-2)$ the regularity problem becomes easier and has been solved (see Kato and Nakao [2]).

Our purpose in this paper is to discuss the uniqueness of the weak solution for the case $0 \leq \alpha \leq 2/(N-2)$ as well as the regularity of it for the ‘critical’ case $\alpha = 2/(N-2)$ in some restricted situations.

The problem we consider is in fact:

\[
\begin{cases}
\varepsilon u_{tt} + Au + u_t + g(u) = f(x, t) & \text{on } \Omega \times R, \quad \varepsilon > 0, \\
u(x, t + \omega) = u(x, t) & \text{and } D^\mu u |_{\partial \Omega} = 0, \quad 0 \leq |\mu| \leq m - 1,
\end{cases}
\]

where $A$ is a symmetric uniformly elliptic operator of order $2m$:

\[
Au = \sum_{0 \leq |\alpha| \leq m, 0 \leq |\beta| \leq m} (-1)^{|\alpha|} D^\alpha (a_{\alpha\beta}(x) D^\beta u)
\]

with smooth coefficients $a_{\alpha\beta}(x)$ ($a_{\alpha\beta} \in C^3$ is sufficient), and hence

\[
C \|u\|_{H^m}^2 \leq (Au, u) \leq C' \|u\|_{H^m}^2 \quad \text{for } u \in \hat{H}^m \cap H^m
\]

with some positive constants $C$ and $C'$. Observe that if $\varepsilon = 0$, the problem (1.4) is of parabolic type and for this equation the uniqueness is trivial and the regularity is easier. Motivated by these observations we shall investigate the uniqueness and the regularity problems for (1.4) under a smallness assumption on $\varepsilon > 0$. Moreover we shall derive relations between the solutions $u_\varepsilon$ of (1.4) and the solution $u_0$ of the reduced parabolic problem:

\[
(1.5) \quad u_t + Au + g(u) = f(x, t) \quad \text{on } \Omega \times R
\]

with the periodicity and the boundary conditions.

When we treat the problem (1.4) with small parameter $\varepsilon$ the equation is sometimes called the ‘hyperbolic parabolic’ type. Such an equation has been considered by several authors; in particular our problem is closely related to that in Section 3, Chapter 4, in the book by Vejvoda [8]. In [8], however, the nonlinear term $g(u)$ of the equation is replaced by $\mu g(u)$ with a small parameter $\mu$ and essentially only small amplitude solutions are considered. There a Fourier expansion method is employed, while here we use an energy method.
Recently, Milani [3] treated a similar problem for the quasilinear wave equation. But, in [3], it is assumed that the forcing term $f$ is small and the solutions treated are 'small', while we would emphasise again that we make no smallness assumption on $f$ and hence our solution may be 'large'. For other related works see Rabinowitz [6], Wahl [10] and the references cited in the book [8].

2. STATEMENTS OF THE RESULTS

The function spaces we use are all familiar and the definition of them will be omitted. But, we note that $||-||$ denotes $L^2$-norm on $\Omega$. We also assume $0 < \varepsilon \leq 1$ without loss of generality. Our first result reads as follows.

**Theorem 1.** Suppose that $f \in L^2(\omega; L^2(\Omega))$ and set

\[
\l_0 \equiv \left( \int_0^\omega ||f(t)||^2 \, dt \right)^{1/2}.
\]

Concerning the nonlinear term we assume that $g$ belongs to $C^1(R)$ and satisfies

\[
g(0) = 0 \quad \text{and} \quad 0 \leq g'(u) \leq k_0(1 + |u|^\alpha)
\]

for some $0 \leq \alpha \leq 2m/(N - 2m)$ ($0 \leq \alpha < \infty$ if $1 \leq N \leq 2m$).

Then, there exists a constant $C_0$ independent of $f$ and $g$ such that if

\[
\lambda(\varepsilon, \l_0) \equiv C_0k_0^2(1 + \l_0^2\varepsilon) \varepsilon < 1,
\]

the problem (1.4) has a unique solution $u$ in the class

\[
C(\omega; \hat{H}_m(\Omega)) \cap C^1(\omega; L^2(\Omega)).
\]

As a special case of Theorem 1 we have:

**Corollary 1.** If $g(u)$ is at most of linear growth, that is, (2.2) is satisfied with $\alpha = 0$, there exists a constant $C_1$ independent of $g$ and $f$ such that if

\[
0 < \varepsilon \leq C_1/k_0^2,
\]

the solution of (1.4) is unique in the class (2.4).

The solution $u \equiv u_\varepsilon$ of the problem (1.4) converges to the solution $u_0$ of the parabolic problem (1.5) as $\varepsilon \to 0$ in the following sense:
THEOREM 2. Under the assumption (2.3) we let \( u_\varepsilon \) and \( u_0 \) be the solutions of (1.4) and (1.5), respectively, and set \( w_\varepsilon(t) \equiv u_\varepsilon(t) - u_0(t) \). Then, there exists a constant \( C_2 \equiv C_2(d_0) \) such that

\[
\int_0^\infty \| w_\varepsilon(t) \|_{H_m}^2 \, dt \leq C_2 \varepsilon.
\]

If we assume, in addition, \( f \in W^{1,2}(\omega; L^2(\Omega)) \) and set

\[
d_1 \equiv \left( \int_0^\infty \| f(t) \|^2 \, dt \right)^{1/2},
\]

the solution \( u_\varepsilon(t) \) belongs to

\[W^{2,2}(\omega; L^2(\Omega)) \cap W^{1,2}(\omega; H_m) \cap L^2(\omega; H_{2m})\]

and the estimate:

\[
\int_0^\infty \left\{ \| \partial_t w_\varepsilon(t) \|^2 + \| w_\varepsilon(t) \|_{H_m}^2 + \| w_\varepsilon(t) \|_{H_{2m}}^2 \right\} \, dt \leq C_3 \varepsilon^2
\]

holds, where \( C_3 \) is a constant depending on \( d_0 \) and \( d_1 \).

Next, we shall state our result concerning the regularity of the solution of (1.4).

THEOREM 3. Let \( 2m < N < 4m \). Suppose that \( f \) belongs to

\[W^{3,2}(\omega; L^2(\Omega)) \cap C^1(\omega; H_m) \cap C(\omega; H_{2m})\]

and set

\[
d_i \equiv \left( \int_0^\infty \left\| \frac{\partial}{\partial t} f(t) \right\|^2 \, dt \right)^{1/2} \quad (i = 0, 1, 2, 3),
\]

\[
M_1 \equiv \sup_t \| D_i f(t) \| \quad \text{and} \quad M_2 \equiv \sup_t \left\{ \| f(t) \|_{H_{2m}} + \| f_t(t) \|_{H_m} \right\}.
\]

Concerning \( g(u) \), suppose that \( g \) belongs to \( C^{2m+1}(R) \) and satisfies, in addition to (1.2),

\[
|g^{(i)}(u)| \leq k_0 \left( 1 + |u|^\alpha \right)^{1-i}, \quad i = 1, 2,
\]

with \( \alpha = 2m/(N - 2m) \).

Then, under the assumption (2.3), the solution \( u \) of Theorem 1 belongs in fact to

\[C^4(\omega; L^2(\Omega)) \cap \bigcap_{j=0}^{3} C^j(\omega; H_m \cap H_{4m-jm})\]
and the following estimate holds:

\[
\int_0^\omega \left\| \left( \frac{\partial}{\partial t} \right)^4 u(t) \right\|^2 dt + \sup_t \left\{ \varepsilon \left\| \left( \frac{\partial}{\partial t} \right)^4 u(t) \right\|^2 + \sum_{j=0}^3 \left\| \left( \frac{\partial}{\partial t} \right)^j u(t) \right\|^2_{H_{4m-jm}} \right\} \leq C_4 < \infty
\]  

(2.13)

where \( C_4 \) is a constant depending on \( d_i (0 \leq i \leq 3) \) and \( M_i (i = 1, 2) \), but independent of \( \varepsilon \).

**REMARK 1.** By the Sobolev's imbedding theorem: \( H_{2m} \subset C(\Omega) \), we see \( u \in C(\omega; C^{2m}(\Omega)) \cap C^1(\omega; C^m(\Omega)) \cap C^2(\omega; C(\Omega)) \), that is, our solution is classical.

**REMARK 2.** The assumption \( N < 4m \) is made for simplicity. We could treat the case \( N \geq 4m \) by carrying out a more careful analysis.

**COROLLARY 2.** Under the assumptions on \( f \) and \( g \) in Theorem 3 the parabolic problem (1.5) has a unique solution \( u_0 \) in the class

\[
W^{4,2}(\omega; L^2(\Omega)) \cap \bigcap_{j=0}^3 C^j\left( \omega; H_{4m-jm} \right)
\]

and the estimate

\[
\int_0^\omega \left\| \left( \frac{\partial}{\partial t} \right)^4 u_0(t) \right\|^2 dt + \sum_{j=0}^3 \left\| \left( \frac{\partial}{\partial t} \right)^j u_0(t) \right\|^2_{H_{4m-jm}} \leq C_4 < \infty
\]

(2.14)

holds.

**THEOREM 4.** Let \( u_\varepsilon(t) \) and \( u_0(t) \) be the solutions of the problem (1.4) and (1.5), respectively, and set \( w = u_\varepsilon - u_0 \). Then, under the assumptions of Theorem 3 the following estimates hold:

\[
\sup_t \left\{ \varepsilon \left\| w_\varepsilon(t) \right\|^2 + \left\| w(t) \right\|^2_{H_m} \right\} \leq C(d_0, d_1)\varepsilon^2,
\]

(2.15)

\[
\int_0^\omega \left\{ \left\| w_\varepsilon(t) \right\|^2 + \left\| w(t) \right\|^2_{H_{2m}} \right\} dt + \sup_t \left\{ \varepsilon \left\| w_{\varepsilon t}(t) \right\|^2 + \left\| w_t(t) \right\|^2_{H_m} \right\} \leq C(d_0, d_1, d_2)\varepsilon^2,
\]

(2.16)

\[
\int_0^\omega \left\{ \left\| \left( \frac{\partial}{\partial t} \right)^3 w(t) \right\|^2 + \left\| w_\varepsilon(t) \right\|^2_{H_{2m}} \right\} dt \leq C(d_0, d_1)\varepsilon^2.
\]

(2.17)
\[\begin{align*}
&+ \sup_t \left\{ \epsilon \left\| \left( \frac{\partial}{\partial t} \right)^3 w(t) \right\|^2 + \| w_{tt}(t) \|^2_{H_m^2} \right\} \leq C(d_0, d_1, d_2, d_3)\epsilon^2, \\
\int_0^\omega \| w_{tt}(t) \|^2_{H_m^2} \, dt \leq C(d_0, d_1, d_2, d_3)\epsilon^2, \\
\sup_t \{ \| w(t) \|^2_{H_m^2} + \| w(t) \|^2_{H_m^2} \} \leq C(d_0, d_1, d_2, d_3, M_1, M_2)\epsilon^2, \\
\text{and} \\
\sup_t \| w_{tt} \|^2_{H_m^2} \leq C(d_0, d_1, d_2, d_3)\sqrt{\epsilon}.
\end{align*}\]

**Remark 3.** When \( 1 \leq N \leq 2m \), the assertions of Theorem 3 and Theorem 4 are valid without the smallness condition (2.3) on \( \epsilon \).

### 3. Proof of Theorem 1

First we shall prepare the following *a priori* estimates, which are in fact sufficient for the proof of the existence of a weak solution. In what follows we denote by \( C \) general constants independent of \( \epsilon \) which may be different from line to line. To clarify the dependence on some quantity \( q \) we use the notation \( C(q) \) et cetera.

**Proposition 3.1.** Let \( u(t) \) be a solution of (1.4) in the class (2.4). Then we have

\[\begin{align*}
(3.1) \quad \int_0^\omega \| u_t(t) \|^2 \, dt \leq d_0^2, \\
\text{and} \\
(3.2) \quad \sup_t \left\{ \epsilon \| u_t(t) \|^2 + \| A^{1/2} u(t) \|^2 \right\} \leq C d_0^2
\end{align*}\]

where \( \| A^{1/2} u \| \) is defined by

\[\| A^{1/2} u \| = \left( \int \sum_{0 \leq |\alpha| \leq m} \int \sum_{0 \leq |\beta| \leq m} a_{\alpha\beta} D^\alpha u \cdot D^\beta u \, dx \right)^{1/2}\]

which is equivalent to \( \| u \|_{H_m^2} \).

**Proof:** The proof is given by a standard energy method and we sketch it briefly. Multiplying the equation (1.4) by \( u_t \) by \( u_t \) and integrating over \([0, \omega] \times \Omega\) we have

\[\begin{align*}
\int_0^\omega \| u_t(t) \|^2 \, dt = \int_0^\omega \int_{\Omega} u_t f \, dx \, dt \leq \left( \int_0^\omega \| u_t \|^2 \, dt \right)^{1/2} \left( \int_0^\omega \| f(t) \|^2 \, dt \right)^{1/2}
\end{align*}\]
which implies (3.1) immediately. Next, multiplying the equation (1.4) by \( u \) and integrating we get
\[
\int_0^\omega \left\{ \left\| A^{1/2}u(t) \right\|^2 + \int_\Omega g(u)u dx \right\} dt = \epsilon \int_0^\omega \left\| u_t(t) \right\|^2 dt + \int_\Omega \int f u \, dx \, dt
\]
and hence
\[
\int_0^\omega \left\{ \left\| A^{1/2}u(t) \right\|^2 + \int_\Omega g(u)u dx \right\} dt \leq C \epsilon^2.
\]
From (3.1) and (3.5) we see
\[
\int_0^\omega \left\{ \epsilon \left\| u_t(t) \right\|^2 + \left\| A^{1/2}u(t) \right\|^2 + \int_\Omega g(u)u dx \right\} dt \leq C \epsilon^2
\]
and hence, there exists \( t^* \in (0, \omega) \) such that
\[
\epsilon \left\| u_t(t^*) \right\|^2 + \left\| A^{1/2}u(t^*) \right\|^2 + \int_\Omega g(u(t^*))u(t^*) dx \leq C \epsilon^2.
\]
Thus, using the equation (1.4), we see
\[
\sup_t \left\{ \frac{1}{2} \left( \epsilon \left\| u_t(t) \right\|^2 + \left\| A^{1/2}u(t) \right\|^2 \right) + \int_\Omega \int_0^{u(t)} g(\eta) d\eta \, dx \right\}
\leq \frac{1}{2} \left( \epsilon \left\| u_t(t^*) \right\|^2 + \left\| A^{1/2}u(t^*) \right\|^2 \right) + \int_\Omega \int_0^{u(t^*)} g(\eta) d\eta \, dx
+ \int_0^\omega \int_\Omega |f u| \, dx \, dt \leq C \epsilon^2,
\]
which completes the proof of (3.2)

Theorem 1 is an immediate consequence of the following proposition.

**PROPOSITION 3.2.** Letting \( u \) and \( v \) be two solutions of the problem (1.4) in the class (2.4), we have the estimate for \( w = u - v \):
\[
\int_0^\omega \left\| A^{1/2}w(t) \right\|^2 dt \leq \lambda(\epsilon, d_0) \int_0^\omega \left\| A^{1/2}w(t) \right\|^2 dt
\]
where \( \lambda(\epsilon, d_0) \) is the constant given by (2.3).

**PROOF:** \( w = u - v \) satisfies the equation
\[
\epsilon w_{tt} + Aw + w_t + g(u) + g(v) = 0
\]
together with the periodicity and the boundary conditions. Multiplying the equation (3.9) by $w$ and integrating we have

\begin{equation}
\int_0^\omega \|A^{1/2}w(t)\|^2 \, dt \leq \varepsilon \int_0^\omega \|w_t(t)\|^2 \, dt,
\end{equation}

where the monotonicity of $g(u)$ is used essentially. Next, multiplying the equation by $w_t$ and integrating we have

\begin{equation}
\int_0^\omega \|w_t(t)\|^2 \, dt = - \int_0^\omega \int_\Omega (g(u) - g(v))w_t \, dx \, dt
\end{equation}

\begin{align*}
&\leq k_0 \left\{ \int_0^\omega \int_\Omega \left\{ 1 + (|u| + |v|)^2 \right\}^2 |w|^2 \, dx \, dt \right\}^{1/2} \left( \int_0^\omega \|w_t\|^2 \, dt \right)^{1/2} \\
&\leq C k_0 \left\{ \int_0^\omega \left( 1 + \|A^{1/2}u\| + \|A^{1/2}v\| \right)^2 \|A^{1/2}w\|^2 \, dx \right\}^{1/2} \left( \int_0^\omega \|w_t\|^2 \, dt \right)^{1/2},
\end{align*}

where we have used the Sobolev inequality:

$$\|u\|_{2(\alpha+1)} \leq C \|A^{1/2}u\| \quad \text{for} \quad u \in \mathring{H}_m \quad \text{if} \quad 0 \leq \alpha \leq 2m/(N - 2m),$$

$$0 \leq \alpha < \infty \quad \text{if} \quad 1 \leq N \leq 2m.$$ 

It follows from (3.10), (3.11) and (3.7) that

$$\int_0^\omega \|A^{1/2}w(t)\|^2 \, dt \leq C \varepsilon k_0^2 (1 + d_0^2\alpha) \int_0^\omega \|A^{1/2}w\|^2 \, dt.$$ 

\section*{4. Proof of Theorem 2}

For $w_\varepsilon = u_\varepsilon - u_0$ we have the equation:

\begin{equation}
\varepsilon u_{tt} + Aw + w_t + g(u) - g(u_0) = 0 \quad (u \equiv u_\varepsilon, w \equiv w_\varepsilon).
\end{equation}

Multiplying the equation (4.1) by $w$ and using the monotonicity of $g(u)$ and the periodicity of $u$ and $w$ we see

\begin{equation}
\int_0^\omega \|A^{1/2}w(t)\|^2 \, dt \leq -\varepsilon \int_0^\omega \int_\Omega \varepsilon u_{tt}w \, dx \, dt
\end{equation}

\begin{align*}
&= \varepsilon \int_0^\omega \int_\Omega u_t w_t \, dx \, dt \\
&\leq \varepsilon \left( \int_0^\omega \|u_t(t)\|^2 \, dt \right)^{1/2} \left( \int_0^\omega \|w_t(t)\|^2 \, dt \right)^{1/2} \\
&\leq \varepsilon \varepsilon d_0 \left( \int_0^\omega \|w_t(t)\|^2 \, dt \right)^{1/2} \quad \text{(by (3.1))}.
\end{align*}
Moreover, we see easily
\[
\left( \int_0^\omega \|w(t)\|^2 dt \right)^{1/2} \leq \left( \int_0^\omega \|u(t)\|^2 dt \right)^{1/2} + \left( \int_0^\omega \|u_{tt}(t)\|^2 dt \right)^{1/2}
\]
and hence, by (4.2)
\[
\int_0^\omega \|A^{1/2}w(t)\|^2 dt \leq C d_0^2 \varepsilon,
\]
which is the first assertion of Theorem 2.

Next, for the proof of second part, we assume \( f \in W^{1/2}(\omega; L^2(\Omega)) \).

Then, we shall use the equation:
\[
(4.3) \quad \varepsilon u_{ttt} + Au_t + u_{tt} + g'(u)u_t = f_t
\]
to derive further \textit{a priori} estimates. In fact, the following proposition assures us that
the solution \( u \equiv u_\varepsilon \) belongs to
\[
C^2(\omega; L^2(\Omega)) \cap C^1 \left( \omega; \overset{o}{H}_m \right) \cap C \left( \omega; H_{2m} \cap \overset{o}{H}_m \right).
\]

**Proposition 4.1.** Under the assumption (2.3), we have the estimates
\[
(4.4) \quad \int_0^\omega \|u_{tt}(t)\|^2 dt \leq C \{ d_1^2 + k_0^2 (1 + d_0^{2\alpha}) d_0 d_1 \},
\]
and
\[
(4.5) \quad \sup_t \{ \varepsilon \|u_{tt}(t)\|^2 + \|A^{1/2}u_t(t)\|^2 \} \leq C d_1^2.
\]

**Proof:** Multiplying the equation (4.3) by \( u_{tt} \) and integrating we see easily
\[
\int_0^\omega \|u_{tt}(t)\|^2 dt \leq d_1^2 + 2 \int_0^\omega \int_\Omega |g'(u)||u_t||U_{tt}| \, dx \, dt
\]
\[
\leq d_1^2 + 2k_0 \int_0^\omega \int_\Omega (1 + |u|^\alpha) |u_t||u_{tt}| \, dx \, dt
\]
\[
\leq d_1^2 + Ck_0^2 \int_0^\omega \left( 1 + \left\| A^{1/2}w(t) \right\|^{2\alpha} \right) \left\| A^{1/2}u_t(t) \right\|^2 dt
\]
\[
+ \frac{1}{2} \int_0^\omega \|u_{tt}(t)\|^2 dt
\]
and hence, with the aid of (3.2),

\begin{equation}
\int_0^\omega \|u_{tt}(t)\|^2 dt \leq 2d_1^2 + Ck_0^2(1 + d_0^{2\alpha}) \int_0^\omega \|A^{1/2}u_t(t)\|^2 dt.
\end{equation}

On the other hand, multiplying the equation (4.3) by $u_t$ and integrating we see

\begin{equation}
\int_0^\omega \|A^{1/2}u_t(t)\|^2 dt \leq \varepsilon \int_0^\omega \|u_{tt}(t)\|^2 dt + \int_0^\omega \int_\Omega f_t u_t dx dt
\end{equation}

\begin{equation*}
\leq \varepsilon \int_0^\omega \|u_{tt}(t)\|^2 dt + \frac{1}{2} \int_0^\omega \|A^{1/2}u_t(t)\|^2 dt + Cd_1^2.
\end{equation*}

It follows from (4.6) and (4.7) that

\begin{equation}
\int_0^\omega \|u_{tt}(t)\|^2 dt
\end{equation}

\begin{equation*}
\leq Ck_0^2(1 + d_0^{2\alpha})\varepsilon \int_0^\omega \|u_{tt}(t)\|^2 dt + 2d_1^2 + Ck_0^2(1 + d_0^{2\alpha})d_0 d_1.
\end{equation*}

Thus, under the assumption $Ck_0^2(1 + d_0^{2\alpha})\varepsilon < 1/2$, which is equivalent to (2.3) by changing $C_0$ if necessary, we have the estimate (4.4).

Moreover, (4.5) follows immediately from (4.4) and (4.7). Finally we note that these estimates give

\begin{equation}
\int_0^\omega \left\{\varepsilon \|u_{tt}(t)\|^2 + \|A^{1/2}u_t(t)\|^2\right\} dt \leq C d_1^2 \quad (0 < \varepsilon \leq 1),
\end{equation}

which implies, as in the proof of (3.2), the estimate (4.5).

Now, we shall prove the second part of Theorem 2.

Multiplying the equation (4.1) by $w(t) \equiv u_\varepsilon - u_0$ and integrating over $\Omega \times [0, \omega]$ as is usual, we have

\begin{equation}
\int_0^\omega \|A^{1/2}w(t)\|^2 dt \leq -\varepsilon \int_0^\omega \int_\Omega u_{tt}w dx dt
\end{equation}

\begin{equation*}
\leq C\varepsilon \left(\int_0^\omega \|u_{tt}(t)\|^2 dt\right)^{1/2} \left(\int_0^\omega \|A^{1/2}w(t)\|^2 dt\right)^{1/2}
\end{equation*}

and hence, by (4.4)

\begin{equation}
\int_0^\omega \|A^{1/2}w(t)\|^2 dt \leq C(d_0, d_1)\varepsilon^2.
\end{equation}
Further, multiplying the equation (4.1) by \( w_t \) we see

\[
\int_0^\omega \|u_t(t)\|^2 \, dt \leq -\varepsilon \int_0^\omega \int_\Omega u_{tt} w_t \, dx \, dt + Ck_0 \int_0^\omega \int_\Omega (1 + |u| + |u_0|) |w| |w_t| \, dx \, dt
\]

\[
\leq C(d_0, d_1) \left\{ \varepsilon \left( \int_0^\omega \|u_{tt}(t)\|^2 \, dt \right)^{1/2} + \left( \int_0^\omega \|A^{1/2} w(t)\|^2 \, dt \right)^{1/2} \right\} \times \left( \int_0^\omega \|w(t)\|^2 \, dt \right)^{1/2}
\]

and we conclude from (2.6) and (4.4) that

\[
\int_0^\omega \|w_t(t)\|^2 \, dt \leq C(d_0, d_1)\varepsilon^2.
\]

Similarly, multiplying the equation (4.1) by \( Aw \) and integrating, we can prove

\[
\int_0^\omega \|Aw(t)\|^2 \, dt \leq C(d_0, d_1)\varepsilon^2.
\]

The proof of Theorem 2 is now complete.

5. PROOFS OF THEOREM 3 AND THEOREM 4

By standard arguments it suffices for the proof of Theorem 3 to derive the a priori estimate (2.13) for an assumed solution \( u \) in the class (2.12). For this we observe:

PROPOSITION 5.1. Let \( u(t) \) be the solution of (1.4) in the sense of Theorem 1 and let \( U(t) \) be a solution in the class \( C^1(\omega; L^2(\Omega)) \cap C \left( \omega; \tilde{H}_m \right) \) of the problem

\[
\begin{cases}
\varepsilon U_{tt} + AU + U_t + g'(u)U = F(x, t) & \text{on } \Omega \times R, \\
U(x, t + \omega) = U(x, t) & \text{and } U |_{\partial \Omega} = 0
\end{cases}
\]

with \( F \in L^2(\omega; L^2(\Omega)) \).

Then, under the assumption (2.3), the estimates

\[
\int_0^\omega \|U_t(t)\|^2 \, dt \leq C \{ 1 + k_2^2(1 + d_0^2) \} \int_0^\omega \|F(t)\|^2 \, dt,
\]

\[
\sup_t \left\{ \varepsilon \|U_t(t)\|^2 + \|A^{1/2} U(t)\|^2 \right\} \leq C \int_0^\omega \|F(t)\|^2 \, dt
\]

hold.
PROOF: The proof is essentially the same as that of (4.5) in Proposition 4.1 and is omitted. □

Using Proposition 5.1 and Proposition 4.1 we shall show:

PROPOSITION 5.2. Under the assumption (2.3) the solution $u(t)$ in the class (2.12) satisfies

\begin{align}
(5.3) & \quad \int_0^\varpi \|u_{ttt}(t)\|^2 \, dt \leq C_5(d_0, d_1, d_2) < \infty \\
(5.4) & \quad \sup_t \{\varepsilon \|u_{ttt}(t)\|^2 + \|A^{1/2}u_{tt}(t)\|^2 \} \leq C_6(d_0, d_1, d_2) < \infty,
\end{align}

and

\begin{align}
(5.5) & \quad \sup_t \|Au_t(t)\| \leq C_7(d_0, d_1, d_2) < \infty
\end{align}

for some constants $C_5, C_6, C_7$ depending on the quantities indicated but independent of $\varepsilon$.

PROOF: Setting $U = u_{tt}$, $U$ satisfies the equation

\begin{equation}
(5.6) \quad \varepsilon U_{tt} + AU + U_t + g'(u)U = -g''(u)(u_t)^2 + f_{tt}
\end{equation}

with the periodicity and the boundary conditions. Here,

\[
\int_\Omega \left| g''(u) \right|^2 |u_t|^4 \, dx \leq k_0^2 \int_\Omega \left( 1 + |u|^{\alpha-1} \right)^2 |u_t|^4 \, dx \\
\leq C k_0^2 \left( 1 + \|A^{1/2}u\|^{2\alpha-2} \right) \|A^{1/2}u_t\|^4 \\
\leq C k_0^2 (1 + d_0^2 \alpha^{-2}) d_1^4 \quad \text{(by (3.2) and (4.5)).}
\]

Thus, applying Proposition 5.1 we obtain the estimates (5.3) and (5.4). Moreover, returning to the equation (4.3) we see easily

\[
\|Au_t(t)\| \leq C \left\{ \varepsilon \|u_{ttt}(t)\| + \|u_{ttt}(t)\| + \left( \int_\Omega |g'(u)u_t|^2 \, dx \right)^{1/2} + \|f_t(t)\| \right\} \\
\leq C_7(d_0, d_1, d_2, M_1) < \infty
\]

for some constant $C_7$. □

PROPOSITION 5.3. Under the assumption (2.3), the solution $u(t)$ in the class
Semilinear wave equation

(2.12) satisfies

\( \int_0^w \left( \frac{\partial}{\partial t} \right)^4 u(t) \| dt \leq C_0(d_0, d_1, d_2, d_3) < \infty, \)

(5.7)

\( \sup_t \left\{ \epsilon \left( \left( \frac{\partial}{\partial t} \right)^4 u(t) \right)^2 + \left\| A^{1/2} u_{ttt}(t) \right\|^2 \right\} \leq C_0(d_0, d_1, d_2, d_3) < \infty, \)

and

(5.8)

\( \sup_t \| A u_{ttt}(t) \| \leq C_{10}(d_0, d_1, d_2, d_3) < \infty. \)

PROOF: Setting \( U = u_{ttt} \), it satisfies the equation

\( U_{tt} + AU + U_t + g'(u)U = -3g''(u)u_t u_{tt} - g'''(u)(u_t)^3 + f_{ttt} \)

with the periodicity and the boundary conditions. Here,

\[ \int_\Omega \left| g''(u)u_t u_{ttt} \right|^2 \, dx \leq k_0^2 \int_\Omega \left( 1 + |u|^{a-2} \right)^2 |u_t|^2 |u_{ttt}|^2 \, dx \]

\[ \leq Ck_0^2 \left( 1 + \left\| A^{1/2} u \right\|^{2(a-2)} \right) \left\| A^{1/2} u_t \right\|^2 \left\| A^{1/2} u_{ttt} \right\|^2 \]

\[ \leq C(d_0, d_1, d_2) < \infty \quad \text{(by (3.2), (4.5) and (5.4)).} \]

Thus, applying Proposition 5.1 to (5.10), we get (5.7) and (5.8). Furthermore, returning to the equation (5.6) and using the estimates in hand we see easily

\[ \| A u_{ttt}(t) \| \leq \left\{ \epsilon \left( \left( \frac{\partial}{\partial t} \right)^4 u(t) \right)^2 + \| u_{ttt}(t) \| + \| g'(u)u_{ttt} \| + \| g''(u)u_t^2 \| + \| f_{ttt} \| \right\} \]

\[ \leq C(d_0, d_1, d_2, d_3) < \infty. \]

(Note that \( \sup_t \| f_{ttt}(t) \| \leq C(d_2 + d_3). \)

It remains to derive estimates for \( \| u(t) \|_{H^4m} \) and \( \| u_1(t) \|_{H^3m}. \)

PROPOSITION 5.4. Under the assumption (2.3) the solution \( u(t) \) in the class (2.12) satisfies further

(5.11) \( \| u(t) \|_{H^4m} \leq C(d_0, d_1, d_2, d_3, M_2) < \infty \)

and

(5.12) \( \| u_1(t) \|_{H^3m} \leq C(d_0, d_1, d_2, d_3, M_2) < \infty. \)
PROOF: To prove (5.11) we use the equation (1.4):

$$Au = -\varepsilon u_{tt} - u_t - g(u) + f(t).$$

Here, we see by (5.5) and (5.9)

$$\| -\varepsilon u_{tt} - u_t + f\|_{H^m_t} \leq C(d_0, d_1, d_2) + M_2.$$  

To estimate \(\|g(u)\|_{H^m_t}\), we denote by \(D^k\) any partial differentiations in \(x = (x_1, x_2, \ldots, x_N)\) of order \(k\). The estimation is standard and we sketch it briefly. First, notice that

$$D^m g(u(t)) = \sum_{k=1}^{2m} g^{(k)}(u) \sum_{\sigma \in S_k} (Du)^{\sigma_1} (D^2 u)^{\sigma_2} \cdots (D^m u)^{\sigma_{2m}}$$

where we set

$$S_k \equiv \left\{ \sigma \equiv (\sigma_1, \sigma_2, \ldots, \sigma_{2m}) \in N^{2m} \left| \begin{array}{l}
\sigma_1 + \sigma_2 + \ldots + \sigma_{2m} = k, \\
\sigma_1 + 2\sigma_2 + \ldots + 2m \sigma_{2m} = 2m
\end{array} \right. \right\}.$$  

We know that by the estimate (4.5) and (3.2)

$$\|Au(t)\| \leq \varepsilon \|u_{tt}(t)\| + \|u_t(t)\| + \|g(u)\| + \|f(t)\| \leq C(d_0, d_1) < \infty$$

and hence, by the assumption \(N < 4m\),

$$\|u(t)\|_{\infty} \leq C \|Au(t)\| \leq C(d_0, d_1) < \infty.$$  

Thus,

$$\|D^m g(u)\| \leq C(d_0, d_1) \sum_{k=1}^{2m} \sum_{\sigma \in S_k} \left\| (Du)^{\sigma_1} (D^2 u)^{\sigma_2} \cdots (D^m u)^{\sigma_{2m}} \right\|$$

$$\leq C(d_0, d_1) \sum_{k=1}^{2m} \sum_{\sigma \in S_k} \|Du\|_{p_1 \sigma_1} \|D^2 u\|_{p_2 \sigma_2} \cdots \|D^m u\|_{p_m \sigma_{2m}}$$

$$\leq C(d_0, d_1) \sum_{k=1}^{2m} \|u\|_{H^m_t} \leq C(d_0, d_1) < \infty,$$

where we should choose \(p_j\) \((j = 1, 2, \ldots, 2m)\) in such a way that

$$2 \leq p_j \leq \infty, \quad \sum_{j=1}^{2m} \frac{1}{p_j} = \frac{1}{2}$$

and

$$\begin{cases}
2N/\sigma_j (N - 4m + 2j) & \text{if } N > 4m - 2j, \\
< \infty & \text{if } N = 4m - 2j, \\
= \infty & \text{if } N < 4m - 2j.
\end{cases}$$
Such a choice of \( \{p_j\} \) is possible, since 
\[
\sum_{j=1}^{2m} \sigma_j(N - 4m + 2j) \frac{1}{2N} < \frac{1}{2} \quad \text{if} \quad \sigma_{2m} = 0.
\]

The estimate (5.11) follows from (5.13), (5.17) and the equation (1.4).

Next, using the equation (4.3): 
\[
A \epsilon_t = -\epsilon u_{ttt} - u_{tt} - g'(u)u_t + f_t
\]
we can derive (5.12) by a similar argument, the details being omitted (see [5]). \( \square \)

Now, all the estimates required for the proof of Theorem 3 have been derived and
the proof is complete.

6. PROOF OF THEOREM 4

Setting \( w = u_\epsilon - u_\theta \), we have 
\[
(5.18) \quad \epsilon w_{tt} + Aw + w_t + \int_0^1 g'(\theta u_\epsilon + (1 - \theta)u_\theta)d\theta w = -\epsilon u_{tttt}.
\]

Notice that the result of Proposition 5.1 is valid even if \( g'(u) \) in (5.1) is replaced by 
\[
\int_0^1 g'(\theta u_\epsilon + (1 - \theta)u_\theta)d\theta.
\]
Thus, we obtain (2.15) immediately.
(Note that \( \int_0^\infty \|u_{tttt}\|^2 dt \leq C(d_1) \).

Differentiating the equation (5.18) with respect to \( t \) we get 
\[
\epsilon w_{ttt} + Aw_t + w_{tt} + \int_0^1 g'(\theta u_\epsilon + (1 - \theta)u_\theta)d\theta w_t
\]
\[
(5.19) \quad = -\int_0^1 g''(\theta u_\epsilon + (1 - \theta)u_\theta)(\theta u_{tt} + (1 - \theta)u_{ttt})d\theta w - \epsilon u_{tttt}.
\]

Applying a variant of Proposition 5.1 to (5.19) we have (2.16). (See the proof of
Proposition 5.2.) Moreover, differentiating the equation (5.19) once more we get 
\[
\epsilon \left( \frac{\partial}{\partial t} \right)^4 w + Aw_{ttt} + w_{tttt} + \int_0^1 g'(\theta u_\epsilon + (1 - \theta)u_\theta)d\theta w_{ttt}
\]
\[
(5.20) \quad = -\int_0^1 g''(\theta u_\epsilon + (1 - \theta)u_\theta)(\theta u_{tt} + (1 - \theta)u_{ttt})^2 d\theta w

- \int_0^1 g''(\theta u_\epsilon + (1 - \theta)u_\theta)(\theta u_{tt} + (1 - \theta)u_{ttt})d\theta w

- 2\int_0^1 g''(\theta u_\epsilon + (1 - \theta)u_\theta)(\theta u_{tt} + (1 - \theta)u_{ttt})d\theta w_t

- \epsilon \left( \frac{\partial}{\partial t} \right)^4 u_\theta.
Applying a variant of Proposition 5.1 to (5.20) once more, and repeating similar estimations as in the proof of Proposition 5.3, we can prove (2.17) and (2.18). The estimate (2.19) follows from similar arguments as in the proof of Proposition 5.4, the details being omitted.

Finally, using the equation (5.20) we get

\[
\sup_t \|Aw_{tt}(t)\| \leq \sup_t \left\{ \varepsilon \left( \frac{\partial}{\partial t} \right)^4 u(t) \right\} + \|\omega_{ttt}(t)\| \\
+ \sup_t \int_0^1 |g'(\theta u_e (1 - \theta)u_0)| \, d\theta \|\omega_{tt}(t)\| \\
+ \sup_t \int_0^1 |g''(\theta u_e (1 - \theta)u_0)(\theta u_{et} + (1 - \theta)u_{ot})^2| \, d\theta \|\omega(t)\| \\
+ \sup_t \int_0^1 |g''(\theta u_e (1 - \theta)u_0)(\theta u_{ett} + (1 - \theta)u_{oet})| \, d\theta \|\omega(t)\| \\
+ 2\sup_t \int_0^1 |g''(\theta u_e (1 - \theta)u_0)(\theta u_{et} + (1 - \theta)u_{ot})| \, d\theta \|\omega_e(t)\| \\
\leq C(d_0, d_1, d_2, d_3)\sqrt{\varepsilon} \quad \text{(by (5.8))},
\]

which proves (2.20). The proof is complete.

References


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