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Expansions in Complex Bases

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Abstract. Beginning with a seminal paper of Rényi, expansions in noninteger real bases have been widely studied in the last forty years. They turned out to be relevant in various domains of mathematics, such as the theory of finite automata, number theory, fractals or dynamical systems. Several results were extended by Daróczy and Kátai for expansions in complex bases. We introduce an adaptation of the so-called greedy algorithm to the complex case, and we generalize one of their main theorems.

Beginning with a seminal paper by Rényi [8], expansions in non-integer real bases have been widely studied in the last forty years. The so-called beta-expansions of Rényi, obtained by a greedy algorithm, were characterized algebraically by Parry [7]. Subsequently, they turned out to be relevant in various domains of mathematics, such as the theory of finite automata, number theory, fractals or dynamical systems; see, *e.g.*, Daróczy, Járay and Kátai [1], Frougny and Solomyak [5], Sidorov [9].

Although the expansions in non-integer bases are not unique in most cases, Erdős, Horváth and Joó [3] discovered a surprising uniqueness property of certain expansions, based on combinatorial obstacles. These unique expansions were then characterized algebraically in [4, 6].

Expansions in complex bases were first studied by Daróczy and Kátai [2]. Given a nonzero complex number q, they denoted by E the set of all sums of the form

$$\sum_{k=-M}^{\infty} \frac{\varepsilon_k}{q^k}, \quad M = 1, 2, \dots, \ \varepsilon_k \in \{0, 1\}, \tag{1}$$

and they investigated the set of q's for which $E = \mathbb{C}$.

The purpose of this paper is to continue these studies. First, we introduce a natural generalization of the greedy expansions in the complex case, and we give a sufficient condition ensuring that the greedy algorithm provides an expansion of the more restricted form

$$\sum_{k=1}^{\infty} \frac{\varepsilon_k}{q^k}, \quad \varepsilon_k \in \{0, 1\}.$$

Next we introduce a variant of the greedy algorithm, similar to that used in [2], and we give a sufficient condition ensuring that for an arbitrarily given number $R > 0$, every complex number of modulus less than or equal to R has an expansion of the form (1).

Our proofs are based on various geometrical constructions. Some open questions are also formulated in the paper.

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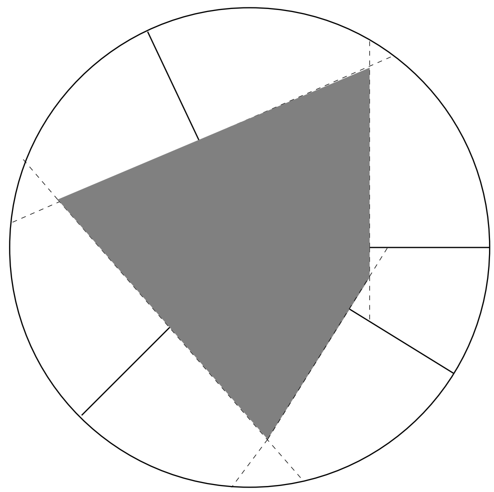

Figure 1: The polygon in a case with $m = 4$

1 Greedy Expansions

Fix a complex number q of modulus $|q| > 1$ and set $\omega := q/|q|$. Starting with a complex number $z = z_0$, we define a sequence $z_0, z_1, \dots$ by the recursive formula

$$z_k := \begin{cases} z_{k-1} - q^{-k} & \text{if } |z_{k-1} - q^{-k}| < |z_{k-1}|, \\ z_{k-1} & \text{otherwise,} \end{cases}$$

$k = 1, 2, \dots$. It is clear that $|z_0| \geq |z_1| \geq \cdots$. If $z_k \to 0$, then putting

$$\varepsilon_k := \begin{cases} 1 & \text{if } |z_k| < |z_{k-1}|, \\ 0 & \text{otherwise,} \end{cases}$$

we obtain an expansion of z:

$$\sum_{k=1}^{\infty} \frac{\varepsilon_k}{q^k} = z.$$

The purpose of this section is to establish the following.

Proposition 1.1 *Assume that* $\arg \omega/\pi$ *is irrational. If* $|q|$ *is sufficiently close to* 1*, then* $z_k \to 0$ *for every complex number* z_0 *of modulus* $|z_0| \leq 1/|q|$.

We need some preliminary results.

Lemma 1.2 *There exists an integer* $m \geq 3$ *such that the closed convex polygon*

$$\{z \in \mathbb{C} : |z| \leq |z - \omega^{-i}| \quad \textit{for } i = 1, \dots, m\}$$

belongs to the unit disc. (See Figure 1.*)*

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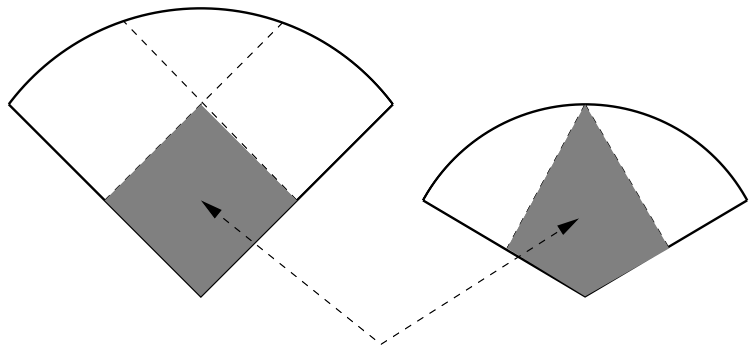

Figure 2: Part of the polygon in an angular sector

Note that $|z| \leq |z-\omega^{-i}|$ is the half-plane containing 0 whose boundary is the line equidistant from 0 and $z-\omega^{-i}$.

Proof It suffices to find a sufficiently large m such that all angular sectors defined by the halflines joining 0 to $\omega^{-1}, \dots, \omega^{-m}$ have opening angles $\leq 2\pi/3$; see Figure 2.

If $0 < |\arg \omega| \leq 2\pi/3$, then we let m be the smallest positive integer satisfying

$$m \geq \frac{2\pi}{|\arg \omega|}.$$

Clearly we obtain $m-1$ sectors of opening angle $|\arg \omega| \leq 2\pi/3$ and one sector of opening angle $2\pi - (m-1)|\arg \omega| \leq |\arg \omega| \leq 2\pi/3$.

If $2\pi/3 < |\arg \omega| < \pi$, then $0 < |\arg \omega^2| \leq 2\pi/3$. Applying the already proved part of the lemma, there exists a positive integer m' such that the halflines generated by the vectors $\omega^{-2}, \omega^{-4}, \dots, \omega^{-2m'}$ have opening angles $\leq 2\pi/3$. Then, of course, the property remains valid by adding the additional halflines generated by the vectors $\omega^{-1}, \omega^{-3}, \dots, \omega^{-2m'-1}$. Hence the lemma follows with $m = 2m'$. ■

Remarks

- The above proof also shows that a point z of the polygon lies on the unit circle only if $|z| \geq |z-\omega^{-i}|$ for at least two different $i \in \{1, \dots, m\}$.
- The lemma and its proof remain valid under the weaker assumption that $\arg \omega/\pi$ is not an integer, *i.e.,* $\omega \neq \pm 1$.

For the remainder of this section let us fix an integer $m \geq 3$ satisfying the conclusion of Lemma 1.2.

Lemma 1.3 *If $|z_k| \geq |q|^{-k-1}$ for some $k \geq 0$, then there exists an integer $j \in \{k+1, \dots, k+m\}$ such that $|z_j| < |z_{j-1}|$.*

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Proof Assume on the contrary that $|z_k| = |z_{k+1}| = \cdots = |z_{k+m}|$. Then it would follow from our algorithm that $z_k = z_{k+1} = \cdots = z_{k+m}$, and then that $|z_k| \le |z_k - q^{-j}|$ for all $j = k+1, \dots, k+m$. Since $|q| > 1$, these relations imply that

$$|z_k| \le |z_k - |q|^{-k-1}\omega^{-k-1}|$$

and that

$$|z_k| < |z_k - |q|^{-k-1}\omega^{-j}|$$

for $j = k+2, \dots, k+m$. Applying the preceding lemma, we conclude that $|z_k| \le |q|^{-k-1}$. Moreover, we cannot have equality here, by a remark following the proof of the lemma. Thus $|z_k| < |q|^{-k-1}$, contradicting our assumption. ■

Now let us establish another geometrical result.[1]

Lemma 1.4 *For every $\varepsilon > 0$ there exists $0 < \alpha < 1$ (close to 1) such that if a complex number w satisfies the inequalities*

$$\alpha^2 \le |w| \le \alpha^{-2} \quad \textit{and} \quad \alpha^2 \le |w-1| \le \alpha^{-2},$$

then either

$$|w - e^{\pi i/3}| < \varepsilon/2 \quad \textit{and} \quad |w - 1 - e^{2\pi i/3}| < \varepsilon/2,$$

or

$$|w - e^{-\pi i/3}| < \varepsilon/2 \quad \textit{and} \quad |w - 1 - e^{-2\pi i/3}| < \varepsilon/2.$$

Proof It follows from our assumptions that 0, 1 and w are close to the vertices of an equilateral triangle. Hence the conclusion follows; see Figure 3. ■

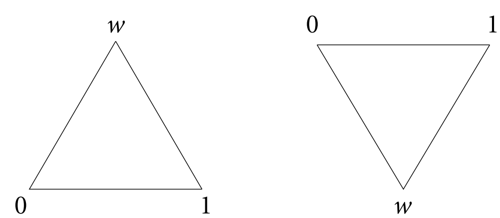

Figure 3: Proof of Lemma 1.4

Now let us denote by ε the distance of the point $e^{2i\pi/3}$ from the following finite union of $2m$ halflines:

$$\bigcup_{\ell=1}^{m} \{z \in \mathbb{C} : \arg z = \pm i\pi/3 - \ell \arg\omega\}.$$

Since $\frac{\arg\omega}{\pi}$ is irrational by assumption, $\varepsilon > 0$.

Let us choose $0 < \alpha < 1$ to this ε according to the preceding lemma. Finally, as $|q|$ is arbitrarily close to 1, let us assume in the sequel that $\alpha|q|^{2m} \le 1$.[2]

[1] We use α^2 in order to avoid the use of $\sqrt{\alpha}$ in Lemma 1.5 below.
[2] Since $|q| > 1$ and $2m > 1$, this implies that $\alpha|q| < 1$.

Lemma 1.5 *If $\alpha^{-1}|q|^{-k} \geq |z_{k-1}| \geq |q|^{-k}$, $|z_k| \geq |q|^{-k-1}$ and $|z_{k-1}| > |z_k| > \alpha|z_{k-1}|$ for some $k \geq 1$, then $|z_j| \leq \alpha|z_{j-1}|$ for some $j \in \{k+1, \dots, k+m\}$.*

Proof Assuming on the contrary that $|z_j| > \alpha|z_{j-1}|$ for all $j = k+1, \dots, k+m$, by Lemma 1.3 there exists $j \in \{k+1, \dots, k+m\}$ such that

$$z_k = z_{k+1} = \cdots = z_{j-1}, \quad z_j = z_{j-1} - q^{-j}, \quad |z_{j-1}| > |z_j| > \alpha|z_{j-1}|.$$

Since $\alpha^{-1}|q|^{-k} \geq |z_{k-1}| \geq |q|^{-k}$ and $\alpha^{-1}|q|^{-k} > |z_{k-1} - q^{-k}| > \alpha|q|^{-k}$, by applying Lemma 1.4 we have

$$\left| z_k - q^{-k}e^{2\pi i/3} \right| < \frac{\varepsilon}{2}|q|^{-k} \quad \text{or} \quad \left| z_k - q^{-k}e^{-2\pi i/3} \right| < \frac{\varepsilon}{2}|q|^{-k}. \tag{2}$$

Similarly, using the inequalities $1 \leq j-k \leq m$ and $\alpha|q|^m \leq 1$, we have

$$\alpha^{-2}|q|^{-j} \geq \alpha^{-1}|q|^{-k} > |z_{j-1}| > \alpha|q|^{-k} > \alpha|q|^{-j}$$

and

$$\alpha^{-2}|q|^{-j} \geq \alpha^{-1}|q|^{-k} > |z_{j-1} - q^{-j}| > \alpha^2|q|^{-k} \geq \alpha^2|q|^{-j}.$$

Then applying Lemma 1.4, we obtain that

$$\left| z_{j-1} - q^{-j}e^{\pi i/3} \right| < \frac{\varepsilon}{2}|q|^{-j} \quad \text{or} \quad \left| z_{j-1} - q^{-j}e^{-\pi i/3} \right| < \frac{\varepsilon}{2}|q|^{-j}. \tag{3}$$

Since $z_k = z_{j-1}$ and $|q|^{-j} < |q|^{-k}$, applying the triangle inequality we deduce from (2) and (3) that

$$\left| q^{-k}e^{2\pi i/3} - q^{-j}e^{\pi i/3} \right| < \varepsilon|q|^{-k} \quad \text{or} \quad \left| q^{-k}e^{2\pi i/3} - q^{-j}e^{-\pi i/3} \right| < \varepsilon|q|^{-k}.$$

Putting $\ell = j-k$, we then conclude that

$$\left| e^{2\pi i/3} - q^{-\ell}e^{\pi i/3} \right| < \varepsilon \quad \text{or} \quad \left| e^{2\pi i/3} - q^{-\ell}e^{-\pi i/3} \right| < \varepsilon,$$

and this contradicts the choice of ε. ■

End of the proof of Proposition 1.1 Since $|z_0| \geq |z_1| \geq |z_2| \geq \cdots$, it is sufficient to prove that $|z_k| \leq |q|^{-k}$ infinitely often. This is obviously satisfied if $|z_{k-1}| \leq |q|^{-k}$ infinitely often. We may thus assume that $|z_{k-1}| > |q|^{-k}$ for all sufficiently large k. Since $|z_0| \leq |q|^{-1}$ by assumption, then there exists a largest integer i satisfying $|z_{i-1}| \leq |q|^{-i}$. Then we have $|z_i| \leq |q|^{-i}$, and

$$|z_j| > |q|^{-j-1} \quad \text{for all } j \geq i. \tag{4}$$

We will construct a strictly increasing sequence of indices k satisfying

$$|z_k| \leq |q|^{-k}. \tag{5}$$

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The index $k = i$ satisfies this condition. Now let $k \geq i$ be an arbitrary integer satisfying (5). Since $|z_k| > |q|^{-k-1}$ by (4), applying Lemma 1.3 there exists an integer $k+1 \leq j \leq k+m$ such that $|z_j| < |z_{j-1}|$. Observe that

$$|z_{j-1}| \leq |z_k| \leq \frac{1}{|q|^k} \leq \frac{1}{\alpha |q|^j},$$

because $\alpha |q|^{j-k} \leq \alpha |q|^m \leq 1$.

If $|z_j| \leq \alpha |z_{j-1}|$, then (5) is satisfied with $k := j$, because

$$|z_j| \leq \alpha |z_{j-1}| \leq \alpha \cdot \frac{1}{\alpha |q|^j} = \frac{1}{|q|^j}.$$

If $|z_j| > \alpha |z_{j-1}|$, then, since we also have $|z_{j-1}| > |q|^{-j}$ by (4), applying Lemma 1.5 for j in place of k we obtain that there exists an integer $j+1 \leq n \leq j+m$ satisfying $|z_n| \leq \alpha |z_{n-1}|$. Then $|z_n| \leq \alpha |z_{n-1}| \leq \alpha |z_k| \leq \alpha |q|^{-k} \leq |q|^{-n}$ because $\alpha |q|^{n-k} \leq \alpha |q|^{2m} \leq 1$. Hence (5) is satisfied with k replaced by n. ■

Remarks

- We have established a little more than stated in the proposition: if

$$|\arg \omega| \neq \frac{\pi}{j} \quad \text{and} \quad |\arg \omega| \neq \frac{\pi}{3j} \quad (\text{modulo } 2\pi)$$

for all $j = 1, \ldots, m$ and if $|q| > 1$ is sufficiently close to 1, then $z_k \to 0$ for every complex number z_0 such that $|z_0| \leq 1/|q|$. It would be interesting to further weaken the irrationality condition on ω.

- It would be interesting to investigate whether Proposition 1.1 remains valid for all complex numbers z_0, regardless of their modulus.

2 Existence of Expansions

It remains an open question whether *every* complex number has a greedy expansion if $|q|$ is sufficiently close to 1. In this section we show that a variant of the greedy algorithm always leads to a suitable expansion. Setting $\mathbb{C}_R := \{z \in \mathbb{C} : |z| \leq R\}$ for brevity, we are going to prove the following.

Proposition 2.1 *Fix a nonreal complex number ω of modulus 1 and a positive real number R. If $p > 1$ is sufficiently close to 1, then setting $q := p\omega$, every complex number $z \in \mathbb{C}_R$ has at least one expansion.*

This proposition extends an earlier theorem of Daróczy and Kátai [2]; they considered the case $R = 1$. Our proof for $R > 1$ is based on the following.

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Lemma 2.2 *Fix two real numbers $2^{-1/2} < \beta < 1$, $R > 1$ and a nonreal complex number ω of modulus 1. There is a nonnegative integer N such that for every $z \in \mathbb{C}_R$ there exists a finite sequence $\varepsilon_1, \dots, \varepsilon_N$ satisfying*

$$\Bigl| z - \sum_{k=1}^{N} \frac{\varepsilon_k}{\omega^k} \Bigr| \le \beta.$$

Proof Let us first assume that that ω is a root of unity, *i.e.*, $\omega^m = 1$ for a smallest integer $m \ge 3$. Then $\omega_{k+m} = \omega_k$ for all k. Hence it suffices to show that if M is sufficiently large, then every $z \in \mathbb{C}_R$ satisfies the inequality

$$\Bigl| z - \frac{a}{\omega^k} - \frac{b}{\omega^{k+1}} \Bigr| \le 2^{-1/2}$$

with a suitable index k and with suitable integers $0 \le a \le M$ and $0 \le b \le M$. This follows from the fact that if we choose k such that the argument of z is between the arguments of $1/\omega^k$ and $1/\omega^{k+1}$, then the points

$$\frac{a}{\omega^k} + \frac{b}{\omega^{k+1}}$$

form the vertices of a lattice of rhombuses of side 1, one of which contains z. Finally, since $\mathbb{C}_R$ is bounded, it is covered by a finite number of such rhombuses.

The case $\omega = i$ is particularly simple because the rhombuses are unit squares: if $M := \lceil R \rceil$ is the upper integer part of R, then for every $z \in \mathbb{C}_R$ there exist nonnegative integers $0 \le a \le M$, $0 \le b \le M$, $0 \le c \le M$, $0 \le d \le M$ (two of which are zero) such that $|z - (a + bi - c - di)| \le 2^{-1/2}$.

Turning to the case where ω is not a root of unity, set $\delta = \beta - 2^{-1/2}$, $M = \lceil R \rceil$, and choose positive integers $k_1 < k_2 < \cdots < k_{4M}$ such that

$$\begin{aligned}
|\omega^{-k_j} - 1| &< \delta/4M \quad \text{if } 1 \le j \le M,\\
|\omega^{-k_j} - i| &< \delta/4M \quad \text{if } M+1 \le j \le 2M,\\
|\omega^{-k_j} + 1| &< \delta/4M \quad \text{if } 2M+1 \le j \le 3M,\\
|\omega^{-k_j} + i| &< \delta/4M \quad \text{if } 3M+1 \le j \le 4M.
\end{aligned}$$

Using the integers $0 \le a \le M$, $0 \le b \le M$, $0 \le c \le M$, $0 \le d \le M$ mentioned in the preceding paragraph, it follows that

$$\begin{aligned}
\Bigl| z - \Bigl(\sum_{j=1}^{a} \frac{1}{\omega^{k_j}} \Bigr) - i \Bigl(\sum_{j=M+1}^{M+b} \frac{1}{\omega^{k_j}} \Bigr) + \Bigl(\sum_{j=2M+1}^{2M+c} \frac{1}{\omega^{k_j}} \Bigr) + i \Bigl(\sum_{j=3M+1}^{3M+d} \frac{1}{\omega^{k_j}} \Bigr) \Bigr| \\
\le 2^{-1/2} + 4M \cdot \frac{\delta}{4M} = \beta.
\end{aligned}$$

■

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Proof of Proposition 2.1 It follows from the preceding lemma that if $p > 1$ is sufficiently close to 1, then for every complex number $z \in \mathbb{C}_R$ there exists a finite sequence $\varepsilon_1, \dots, \varepsilon_N$ satisfying

$$\Big| z - \sum_{k=1}^{N} \frac{\varepsilon_k}{(p\omega)^k} \Big| \le 1.$$

Indeed, it suffices to choose $p > 1$ such that

$$\Big| 1 - \frac{1}{p^N} \Big| \le \frac{1-\beta}{N},$$

because then we have

$$\begin{aligned} \Big| z - \sum_{k=1}^{N} \frac{\varepsilon_k}{(p\omega)^k} \Big| &\le \Big| z - \sum_{k=1}^{N} \frac{\varepsilon_k}{\omega^k} \Big| + \sum_{k=1}^{N} \Big| \frac{1}{\omega^k} - \frac{1}{(p\omega)^k} \Big| \\ &\le \beta + \sum_{k=1}^{N} \Big| 1 - \frac{1}{p^k} \Big| \le 1. \end{aligned}$$

Therefore it is sufficient to establish that every complex number $z \in \mathbb{C}_1$ has at least one expansion such that $\varepsilon_1 = \cdots = \varepsilon_N = 0$.

If $\omega := e^{2\pi Ai/m}$ with two coprime positive integers A and m such that $m \ge 4$, then

$$\min_{1 \le j \le m} \Big| z - \frac{1}{\omega^j} \Big| \le 2\sin(\pi/8) \approx 0.765 < 1 \tag{6}$$

for all complex numbers z satisfying $|z| = 1$, because the worst case is when ω is a fourth root of unity and z is an eighth root of unity. If the argument of ω is an irrational multiple of π, then the same property holds true for a sufficiently large positive integer m because the powers $\omega, \omega^2, \omega^3, \dots$ are dense in the unit circle.

By continuity, for every real number $p > 1$, sufficiently close to 1, setting $q := p\omega$ we have

$$\min_{N+1 \le j \le N+m} \Big| z - \frac{1}{q^j} \Big| \le \frac{1}{|q|^{m+1}} \tag{7}$$

for all complex numbers z satisfying

$$\frac{1}{|q|^{m+1}} < |z| \le 1.$$

We claim that every complex number z satisfying $|z| \le 1$ has at least one expansion.

Fix such a z. If it has a finite expansion, then we are done. Henceforth assume that z has no finite expansion. We are going to construct by induction two sequences of integers (n_k) and (j_k) such that

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- every n_k is a multiple of $m+1$,
- $0 \le n_1 < n_2 < n_3 < \cdots$,
- $N+1 \le j_k \le N+m$ for all $k = 1, 2, \dots$, and
- $$\frac{1}{|q|^{m+1+n_k}} < \Bigl| z - \sum_{\ell=1}^{k-1} \frac{1}{q^{n_\ell + j_\ell}} \Bigr| \le \frac{1}{|q|^{n_k}} \tag{8}$$

for all $k = 1, 2, \dots$.

It follows from the first three relations that $n_k \to \infty$ and that

$$N < n_1 + j_1 < n_2 + j_2 < \cdots.$$

(Let us note for further reference that we even have $n_k + j_k + 1 < n_{k+1} + j_{k+1}$ for all k.) Therefore the last inequality shows that we have a suitable expansion of z.

Turning to the construction of these sequences, let us rewrite (8) in the following equivalent way:

$$\frac{1}{|q|^{m+1}} < \Bigl| q^{n_k} \Bigl(z - \sum_{\ell=1}^{k-1} \frac{1}{q^{n_\ell + j_\ell}} \Bigr) \Bigr| \le 1. \tag{9}$$

Since $0 < |z| \le 1$ (z is different from zero because it has no finite expansion), there exists a nonnegative multiple n_1 of $m+1$ such that

$$\frac{1}{|q|^{m+1}} < |q^{n_1} z| \le 1.$$

This is just (9) for $k = 1$.

Continuing by induction, asume that (9) is satisfied for some $k \ge 1$. Thanks to (7), there exists an integer $N+1 \le j_k \le N+m$ such that

$$\Bigl| q^{n_k} \Bigl(z - \sum_{\ell=1}^{k-1} \frac{1}{q^{n_\ell + j_\ell}} \Bigr) - \frac{1}{q^{j_k}} \Bigr| \le \frac{1}{|q|^{m+1}}.$$

Since z has no finite expansion by assumption, the left-hand side is strictly positive, so that we have

$$0 < \Bigl| q^{n_k} \Bigl(z - \sum_{\ell=1}^{k} \frac{1}{q^{n_\ell + j_\ell}} \Bigr) \Bigr| \le \frac{1}{|q|^{m+1}}.$$

There exists therefore a multiple n_{k+1} of $m+1$ such that

$$\frac{1}{|q|^{m+1}} < \Bigl| q^{n_{k+1}} \Bigl(z - \sum_{\ell=1}^{k} \frac{1}{q^{n_\ell + j_\ell}} \Bigr) \Bigr| \le 1.$$

Comparing with the previous estimate we see that we have necessarily $n_{k+1} > n_k$. We conclude by observing that the last estimate is just (9) for $k+1$ instead of k.

In the remaining two cases $\omega = e^{\pm 2\pi i/3}$ the above proof can be modified as follows. We choose $m = 3$; we change (6) to

$$\min_{1\le j\le m}\Big|z-\frac{1}{\omega^j}\Big| < 1-\delta \quad\text{or}\quad \min_{1\le j\le m}\Big|z-\frac{1}{\omega^j}-\frac{1}{\omega^{j+1}}\Big| < 1-\delta,$$

and then (7) to

$$\min_{N+1\le j\le N+m}\Big|z-\frac{1}{q^j}\Big| \le \frac{1}{|q|^{m+1}} \quad\text{or}\quad \min_{N+1\le j\le N+m}\Big|z-\frac{1}{q^j}-\frac{1}{q^{j+1}}\Big| \le \frac{1}{|q|^{m+1}},$$

and in the construction we replace

$$\frac{1}{q^{n_k+j_k}}$$

by

$$\frac{1}{q^{n_k+j_k}}+\frac{1}{q^{n_k+j_k+1}}$$

in case of necessity. ∎

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