# THE ORDER OF MAGNITUDE OF THE $m$TH COEFFICIENTS OF CYCLOTOMIC POLYNOMIALS

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*To Robert Rankin on the occasion of his 70th birthday*

**1. Introduction.** We define the $n$th cyclotomic polynomial $\Phi_n(z)$ by the equation

$$\Phi_n(z) = \prod_{\substack{r=1\\(r,n)=1}}^{n} (z - e(r/n)) \qquad (e(\alpha) = e^{2\pi i\alpha}) \tag{1}$$

and we write

$$\Phi_n(z) = \sum_{m=0}^{\phi(n)} a(m,n) z^m, \tag{2}$$

where $\phi$ is Euler's function.

Erdös and Vaughan [**3**] have shown that

$$|a(m,n)| < \exp((\tau^{1/2} + o(1))m^{1/2}) \tag{3}$$

uniformly in $n$ as $m \to \infty$, where

$$\tau = \prod_p \left(1 - \frac{2}{p(p+1)}\right),$$

and that for every large $m$

$$\log \max_n |a(m,n)| \gg \left(\frac{m}{\log m}\right)^{1/2}. \tag{4}$$

Vaughan [**8**] has obtained a sharper bound for infinitely many $m$; that is

$$\limsup_{n\to\infty} \left(m^{-1/2}(\log m)^{1/4} \log \max_n |a(m,n)|\right) > 0. \tag{5}$$

Erdös and Vaughan conjectured that

$$\log \max_n |a(m,n)| = o(m^{1/2}) \tag{6}$$

as $m \to \infty$. In this paper we prove this, and more. In particular we obtain the exact order of magnitude of

$$L(m) = \log \max_n |a(m,n)|, \tag{7}$$

namely that

$$L(m) \asymp m^{1/2}(\log m)^{-1/4} \tag{8}$$

as $m \to \infty$.

From (10) below it can be seen that if $p_1 > p_2 > m$, $(p_1p_2, n) = 1$, then $a(m,n) = a(m, p_1p_2n)$. Hence the definition of $L(m)$ would be unchanged if we were to replace $\max_n$ by $\limsup_{n\to\infty}$.

The upper bound here stems from Theorem 1 below, a theorem of independent interest concerning exponential sums with multiplicative coefficients.

THEOREM 1. *Let $\mathcal{P}$ be a set of prime numbers and let $\mathcal{M}$ denote the set of m all of whose prime divisors are in $\mathcal{P}$. Then, for $X \geq 1$, we have*

$$\sum_{\substack{m \leq X \\ m \in \mathcal{M}}} \mu(m) e(m\alpha) \ll X(\log 2X)^{-1/2}$$

*uniformly in $\mathcal{P}$ and $\alpha \in \mathbb{R}$.*

This estimate is best possible, as can be seen by taking

$$\mathcal{P} = \{p : p \equiv \pm 2 (\text{mod } 5)\} \quad \text{and} \quad \alpha = 1/5.$$

In this case standard methods can be used to show that

$$\#\{m : m \leq X, \mu(m) \neq 0, m \in \mathcal{M}\} \sim cX(\log X)^{-1/2}$$

as $X \to \infty$, where $c$ is a suitable positive constant, and that

$$\#\{m : m \leq X, m \in \mathcal{M}, \mu(m) \neq 0, m \equiv k (\text{mod } 5)\} \sim \tfrac{1}{4} cX(\log X)^{-1/2}$$

for $k = 1, 2, 3, 4$. Since $\mu(m) = \left(\frac{m}{5}\right)$ for $m \in \mathcal{M}$, the sum in question is

$$\tfrac{1}{4} cX(\log X)^{-1/2} \sum_{k=1}^{4} \left(\frac{k}{5}\right) e\left(\frac{k}{5}\right) + o(X(\log X)^{-1/2}) = \left(\frac{\sqrt{5}}{4} c + o(1)\right) X(\log X)^{-1/2}.$$

The upper bound for $L(m)$ in (8) is deduced from Theorem 1 in two steps.

THEOREM 2. *For each $z$ with $|z| < 1$ we have*

$$\log |\Phi_n(z)| \ll (1 - |z|)^{-1} \left(\log \frac{2}{1 - |z|}\right)^{-1/2}.$$

THEOREM 3. *We have*

$$L(m) \ll m^{1/2} (\log 2m)^{-1/4}.$$

To complement this we also prove the following result.

THEOREM 4. *For all sufficiently large m we have*

$$L(m) \gg m^{1/2} (\log m)^{-1/4}.$$

**2. Proof of Theorem 1.** We first of all observe that

$$\sum_{\substack{m \leq X \\ m \in \mathcal{M}}} 1 \leq \sum_{\substack{m \leq X \\ (m,P)=1}} 1,$$

where

$$P = \prod_{\substack{p < z \\ p \notin \mathcal{P}}} p$$

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and $z$ is a parameter at our disposal. Since

$$\sum_{\substack{w<p\le z\\ p\notin\mathcal{P}}} \frac{\log p}{p} \le \log\frac{z}{w} + O(1)$$

whenever $2\le w<z$, Theorem 2.2 of Halberstam and Richert [**4**] with $z=X$ gives

$$\sum_{\substack{m\le X\\ m\in\mathcal{M}}} 1 \ll X \prod_{\substack{p<X\\ p\notin\mathcal{P}}} \left(1-\frac{1}{p}\right).$$

Hence

$$\sum_{\substack{m\le X\\ m\in\mathcal{M}}} 1 \ll \frac{X}{\log X} \prod_{\substack{p\le X\\ p\in\mathcal{P}}} \left(1-\frac{1}{p}\right)^{-1}. \tag{9}$$

Let $\mathcal{N}$ denote the set of natural numbers $n$, none of whose prime factors are in $\mathcal{P}$. Then

$$\sum_{\substack{n\mid m\\ n\in\mathcal{N}}} \mu(m/n) = \begin{cases} \mu(m) & \text{when } m\in\mathcal{M},\\ 0 & \text{when } m\notin\mathcal{M}.\end{cases}$$

Therefore

$$\sum_{\substack{m\le X\\ m\in\mathcal{M}}} \mu(m)e(m\alpha) = \sum_{n\in\mathcal{N}} \sum_{r\le X/n} \mu(r)e(rn\alpha).$$

Davenport [**1**] has shown that for any fixed $h$

$$\sum_{r\le Y} \mu(r)e(r\beta) \ll Y(\log(2Y))^{-h}$$

uniformly in $\beta$. Hence

$$\sum_{\substack{m\le X\\ m\in\mathcal{M}}} \mu(m)e(m\alpha) \ll \sum_{\substack{n\le X\\ n\in\mathcal{N}}} \frac{X/n}{(\log(2X/n))^2}.$$

The terms with $n\le\sqrt{X}$ contribute $\ll X/\log X$. The remaining terms contribute

$$\ll \sum_{0\le k\le \frac{\log X}{2\log 2}} \sum_{\substack{\frac{1}{2}X2^{-k}<n\le X2^{-k}\\ n\in\mathcal{N}}} 2^k(k+1)^{-2}.$$

By (9) with $\mathcal{M}$ replaced by $\mathcal{N}$, that is $\mathcal{P}$ replaced by $c\mathcal{P}=\{p : p\notin\mathcal{P}\}$, we see that the above is

$$\ll \sum_{k\ge 0} (k+1)^{-2} \frac{X}{\log X} \prod_{\substack{p\le X\\ p\notin\mathcal{P}}} \left(1-\frac{1}{p}\right)^{-1}.$$

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Therefore, by (9),

$$\sum_{\substack{m\le X\\ m\in\mathcal{M}}} \mu(m)e(m\alpha) \ll \frac{X}{\log X}\min\Big(\prod(\mathcal{P}), \prod(c\mathcal{P})\Big),$$

where $\prod(\mathcal{A}) = \prod_{\substack{p\le X\\ p\in\mathcal{A}}}\left(1-\frac{1}{p}\right)^{-1}$. Now $\prod(\mathcal{P})\prod(c\mathcal{P}) \ll \log X$, so that at least one of $\prod(\mathcal{P})$ and $\prod(c\mathcal{P})$ is $\ll \sqrt{\log X}$.

**3. Proofs of Theorems 2 and 3.** Theorem 2 is trivial when $n=1$ and so we may suppose that $n>1$. Then, by (1),

$$\Phi_n(z) = \prod_{d\mid n}(1-z^{n/d})^{\mu(d)}. \tag{10}$$

Let $N$ denote the squarefree kernel of $n$; $N = \prod_{p\mid n} p$. Then

$$\Phi_n(z) = \prod_{d\mid N}(1-(z^{n/N})^{N/d})^{\mu(d)} = \Phi_N(z^{n/N}).$$

Thus it suffices to establish Theorem 2 when

$$n>1, \qquad \mu(n)\ne 0. \tag{11}$$

In that case the formula (10) becomes

$$\Phi_n(z) = \prod_{d\mid n}(1-z^d)^{\mu(n/d)} = \exp\Big(\mu(n)\sum_{d\mid n}\mu(d)\log(1-z^d)\Big).$$

On expanding $\log(1-z^d)$ in powers of $z$ this gives

$$\Phi_n(z) = \exp\Big(-\mu(n)\sum_{m=1}^{\infty} c_m z^m\Big) \tag{12}$$

with

$$c_m = \frac{1}{m}\sum_{d\mid(m,n)} d\mu(d). \tag{13}$$

For an arbitrary real number $\alpha$ we have

$$\sum_{m\le X} c_m e(m\alpha) = \sum_{k\le X}\frac{1}{k}\sum_{\substack{d\le X/k\\ d\mid n}}\mu(d)e(dk\alpha).$$

Thus, by Theorem 1, when $X\ge 1$ we have

$$\sum_{m\le X} c_m e(m\alpha) \ll \sum_{k\le X}\frac{1}{k}\left(\frac{X/k}{(\log(2X/k))^{1/2}}\right) \ll X(\log 2X)^{-1/2}.$$

Theorem 2 now follows easily from (12) by partial summation.

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The proof of Theorem 3 is a straightforward application of Cauchy's inequalities for the coefficients of a power series followed by an appeal to Theorem 2 with $|z| = 1 - m^{-1/2}(\log 3m)^{-1/4}$.

**4. Preliminaries to the proof of Theorem 4.** The proof of Theorem 4 is based on a precise analysis of the behaviour of $\Phi_n(z)$ for particular choices of $n$. In Vaughan [**8**] it was shown that $\Phi_n(z)$ can be made large by choosing $n$ to be the product of primes $p \le M$ with $p \equiv \pm 2 \pmod 5$. However the argument given there is not precise enough to localize the behaviour of $a(m, n)$ with respect to $m$.

It is possible to obtain quite precise estimates for $a(m, n)$ by starting from (12), or more or less equivalently $\exp(F(z))$, where $F$ is given by (15) below, and to apply the saddle point method to

$$\frac{1}{2\pi i}\int_{\mathscr{C}} z^{-m-1}\exp(F(z))\,dz,$$

where $\mathscr{C}$ is a circle radius $\rho < 1$, centre 0, analogously to the simplest arguments used to estimate the partition function. However this gives rise to considerable technical complications as the Dirichlet series generating function $D(s, \alpha)$ occurring in (19) below has an algebraic singularity at $s = 1$. To avoid these complications we employ a method which gives less precise estimates, albeit sufficient for the purpose at hand. We obtain an asymptotic estimate for

$$\log|\Phi_n(\rho e(a/5))| \quad \text{as } \rho \to 1-$$

and a sharp upper bound for $\log|\Phi_n(\rho e(\alpha))|$ that is uniform in $\alpha$. These estimates then permit us to complete the proof by a combinatorial argument similar to that of §8 of Erdös and Vaughan [**3**].

Let $\mathscr{P} = \{p : p \equiv \pm 2 \pmod 5\}$, let $\mathscr{N}$ denote the set of natural numbers all of whose prime factors are in $\mathscr{P}$, and let

$$c(m) = \frac{1}{m}\sum_{\substack{d \mid m \\ d \in \mathscr{N}}} d\mu(d). \tag{14}$$

For technical ease we work with

$$F(z) = \sum_{m=1}^{\infty} c(m) z^m \tag{15}$$

rather than the series $\sum_{m=1}^{\infty} c_m z^m$ which occurs in (12). Clearly if $n = \prod_{p \le M,\, p \equiv \pm 2 \pmod 5} p$ with $M$ large, then $a(m, n)$ is the coefficient of $z^m$ in the power series expansion of $\exp(-\mu(n)F(z))$.

We suppose that $z = \rho e(\alpha)$ with $0 < \rho < 1$ and $\alpha \in \mathbb{R}$. Let

$$X = \left(\log \frac{1}{\rho}\right)^{-1}. \tag{16}$$

For large $X$ we define major and minor arcs as follows. When $1 \le a \le q \le (\log X)^3$ and

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$(a, q) = 1$, let the major arc $M(q, a)$ consist of the set of $z = \rho e(\alpha)$ with $|\alpha - a/q| \le (\log X)^4 q^{-1} X^{-1}$. Since $X$ is large, the major arcs are pairwise disjoint. We define the minor arcs $M$ to be the set of those $z$, with $|z| = \rho$, lying in no major arc $M(q, a)$.

**5. The minor arcs.** The treatment of the minor arcs is based on the following special case of Corollary 1 of Montgomery and Vaughan [**6**]. Note that, by (14), $c(m)$ is multiplicative and $|c(m)| \le 1$.

LEMMA 1. *Suppose that* $|\alpha - a/q| \le q^{-2}$, $(a, q) = 1$ *and* $2 \le R \le q \le M/R$. *Then*

$$\sum_{m=1}^{M} c(m)e(m\alpha) \ll \frac{M}{\log M} + MR^{-1/2}(\log R)^{3/2}.$$

From this we deduce the following result.

LEMMA 2. *Suppose that* $z \in m$. *Then*

$$F(z) \ll \frac{X}{\log X}.$$

*Proof.* We are given that $X$ is large and $z \in m$. By Dirichlet's theorem on diophantine approximation there are $\alpha, a, q$ with $z = \rho e(\alpha)$, $1 \le a \le q \le X/(\log X)^4$ and $|\alpha - a/q| \le (\log X)^4 X^{-1} q^{-1}$. Since $z \in m$ we further have $q > (\log X)^3$.

Let

$$S_n = \sum_{m=1}^{n} c(m)e(\alpha m).$$

Then, by (15),

$$F(z) = (1-\rho) \sum_{n=1}^{\infty} S_n \rho^n.$$

By (14), $|c(m)| \le 1$. Thus, by (16),

$$\sum_{n \le X \log X} S_n \rho^n + \sum_{n > X \log X} S_n \rho^n \ll \sum_{n \le X/\log X} n + \sum_{k>0} (k + [X \log X])\rho^{k+[X \log X]} \ll X^2/\log X.$$

When $X/\log X < n \le X \log X$ we have $(\log X)^3 < q < n/(\log X)^3$ and $|\alpha - a/q| < q^{-2}$. Hence, by Lemma 1 with $R = (\log X)^3$ we have

$$S_n \ll n/\log n.$$

Therefore

$$\sum_{X/\log X < n \le X \log X} S_n \rho^n \ll \sum_{n=1}^{\infty} \frac{n\rho^n}{\log X} \ll X^2/\log X.$$

Combining our estimates gives the desired conclusion.

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**6. The major arcs.** Let

$$\sigma_0 = 1 + \frac{1}{\log X}. \tag{17}$$

The argument of Satz 231 of Landau [**5**] shows that whenever $|\text{Im } w| < \frac{\pi}{2}$ one has

$$\frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} e^{-ws}\Gamma(s)\, ds = \exp(-e^w).$$

By Satz 229 of Landau [**5**],

$$\Gamma(s) \ll |s|^{\sigma - 1/2} e^{-(\pi/2)|t|} \qquad (1/2 \le \sigma \le 2). \tag{18}$$

This with (15) and a straightforward application of Satz 232 of Landau [**5**] shows that

$$F(\rho e(\alpha + \beta)) = \frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} D(s, \alpha) \left(\frac{X}{1 - 2\pi i X\beta}\right)^s \Gamma(s)\, ds,$$

where

$$D(s, \alpha) = \sum_{m=1}^{\infty} \frac{c(m)}{m^s} e(m\alpha).$$

Thus, in order to study $F(z)$ in the neighborhood of the point $\rho e(a/q)$, i.e., for $z \in M(q, a)$, we investigate the behaviour of $D(s, a/q)$. This investigation is dependent on replacing the additive character $e(am/q)$ by a linear combination of Dirichlet characters. This we accomplish in the following way. We have

$$D(s, a/q) = \sum_{d|q} \sum_{\substack{m=1 \\ (m,q)=q/d}}^{\infty} \frac{c(m)}{m^s} e(am/q).$$

Moreover, by (14),

$$c(nq/d) = c(q/d)c(n, q/d),$$

where

$$c(n, r) = \frac{1}{n} \sum_{\substack{m|n \\ m \in \mathcal{N} \\ (m,r)=1}} m\mu(m).$$

Therefore

$$D(s, a/q) = \sum_{d|q} \frac{c(q/d)}{(q/d)^s} \frac{1}{\phi(d)} \sum_{\chi \bmod d} \chi(a)\tau(\bar{\chi})D(s, q/d, \chi), \tag{20}$$

where $\tau(\bar{\chi})$ is the Gauss sum

$$\tau(\bar{\chi}) = \sum_{r=1}^{d} \bar{\chi}(r)e(r/q) \tag{21}$$

and

$$D(s, r, \chi) = \sum_{n=1}^{\infty} \frac{c(n, r)}{n^s} \chi(n).$$

Now,

$$D(s, r, \chi) = L(1+s, \chi)G(s, r, \chi), \tag{22}$$

where

$$G(s, r, \chi) = \sum_{\substack{m \in \mathcal{N} \\ (m,r)=1}} \frac{\chi(m)\mu(m)}{m^s}.$$

Clearly $G(s, r, \chi)$ is of the form

$$G(s, \chi') = \sum_{m \in \mathcal{N}} \frac{\chi'(m)\mu(m)}{m^s},$$

where $\chi'$ is the character induced by $\chi$ and having modulus $q$. Thus, by (20) and (22),

$$D(s, a/q) = \sum_{d \mid q} \frac{c(q/d)}{(q/d)^s} \frac{1}{\phi(d)} \sum_{\chi \bmod d} \chi(a)\tau(\bar{\chi})L(1+s, \chi)G(s, \chi'). \tag{23}$$

As a function of $s$, $G(s, \chi')$ is regular and non-zero for $\sigma > 1$ and satisfies

$$G(s, \chi')^2 = \frac{L(s, \chi_5\chi')}{L(s, \chi_1\chi')} \prod_{p \equiv \pm 2 \pmod 5} \left(1 - \frac{\chi'(p)^2}{p^{2s}}\right), \tag{24}$$

where $\chi_5$ is the quadratic character modulo 5 and $\chi_1$ is the principal character modulo 5.

Let

$$T = \exp((\log X)^{1/2}), \tag{25}$$

$$\delta = \frac{1}{C(\log X)^{1/2}}, \tag{26}$$

where $C$ is a large constant. Let $\mathcal{A}$ denote the set of complex numbers $s = \sigma + it$ with either $1 - 2\delta < \sigma \le 1$ and $|t| \le 2T$ or $\sigma > 1$, and let $\mathcal{A}'$ denote the set of complex numbers $s$ with either $1 - 2\delta < \sigma \le 1$ and $0 < |t| \le 2T$ or $\sigma > 1$. Then by combining the general theory of $L$-functions as expounded in Davenport [**2**], for example, with the argument of Theorem 3.11 of Titchmarsh [**7**] it follows that, for each non-principal character $\chi$ to a modulus $q \le 5(\log X)^3$, $L(s, \chi)$ is regular and non-zero in $\mathcal{A}$ and satisfies

$$L(s, \chi) \ll_\varepsilon (q(1+|t|))^\varepsilon$$

and

$$L(s, \chi)^{-1} \ll_\varepsilon (q(1+|t|))^\varepsilon$$

uniformly in $\mathcal{A}$, for each fixed positive $\varepsilon$. Also, if $\chi$ is a principal character to a modulus $q$ with $q \le 5(\log X)^3$, then

$$L(s, \chi) - \frac{1}{s-1} \prod_{p \mid q} \left(1 - \frac{1}{p^s}\right)$$

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is regular in $\mathscr{A}$, $L(s, \chi)$ is non-zero in $\mathscr{A}$ and $L(s, \chi)$ satisfies

$$L(s,\chi)-\frac{1}{s-1}\prod_{p\mid q}\left(1-\frac{1}{p^s}\right)\ll_\varepsilon (q(1+|t|))^\varepsilon$$

and

$$L(s,\chi)^{-1}\ll (q(1+|t|))^\varepsilon$$

uniformly in $\mathscr{A}$.

Therefore, by (24), when $\chi'$ is non-principal and is not induced by $\chi_5$, $G(s, \chi')$ has an analytic continuation throughout $\mathscr{A}$, and

$$L(1+s,\chi)G(s,\chi')\ll_\varepsilon (q(1+|t|))^\varepsilon$$

uniformly in $\mathscr{A}$. Let $\mathscr{C}$ denote the piecewise linear path with vertices $\sigma_0-i\infty, \sigma_0-iT$, $1-\delta-iT$, $1-\delta+iT$, $\sigma_0+iT$, $\sigma_0+i\infty$. Then

$$\left.\begin{aligned}\frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}\left(\frac{d}{q}\right)^s L(1+s,\chi)G(s,\chi')\left(\frac{X}{1-2\pi iX\beta}\right)^s\Gamma(s)\,ds\\ =\frac{1}{2\pi i}\int_{\mathscr{C}}\left(\frac{d}{q}\right)^s L(1+s,\chi)G(s,\chi')\left(\frac{X}{1-2\pi iX\beta}\right)^s\Gamma(s)\,ds\end{aligned}\right\}\qquad(27)$$

and by (18) this is

$$\ll q^\varepsilon\,|1-2\pi iX\beta|^{(1/2)+\varepsilon}\left(X^{1-\delta}+XT^{(1/2)+\varepsilon}\exp\left(-\frac{T}{|1-2\pi iX\beta|}\right)\right).$$

For $pe((a/q)+\beta)\in M(q,a)$ we have $q\le(\log X)^3$ and $|\beta|\le(\log X)^4q^{-1}X^{-1}$. Hence the expression above is

$$\ll X\exp(-c_1(\log X)^{1/2})$$

for a suitable positive constant $c_1$.

When $\chi'$ is principal or is induced by $\chi_5$ we observe that $G(s, \chi')$ has an analytic continuation throughout $\mathscr{A}'$ and that

$$L(1+s,\chi)G(s,\chi')\ll_\varepsilon (1+|s-1|^{-1/2})(q(1+|t|))^\varepsilon$$

holds uniformly in $\mathscr{A}'$. Now let $\mathscr{C}$ have vertices

$$\sigma_0-i\infty, \sigma_0-iT, 1-\delta-iT, 1-\delta-i\eta, 1+\eta-i\eta, 1+\eta+i\eta, 1-\delta+i\eta,$$
$$1-\delta+iT, \sigma_0+iT, \sigma_0+i\infty,$$

where $\eta$ is any small positive number. Then (27) holds once more. Moreover, by (18) again we obtain

$$\begin{aligned}\frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}\left(\frac{d}{q}\right)^s L(1+s,\chi)G(s,\chi')\left(\frac{X}{1-2\pi iX\beta}\right)^s\Gamma(s)\,ds\\ =\frac{1}{2\pi i}\int_{\mathscr{C}_\eta}\left(\frac{d}{q}\right)^s L(1+s,\chi)G(s,\chi')\left(\frac{X}{1-2\pi iX\beta}\right)^s\Gamma(s)\,ds\\ +O(X\exp(-c_2(\log X)^{1/2})),\end{aligned}$$

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where $\mathscr{C}_\eta$ has vertices

$$1-\delta-i\eta,\ 1+\eta-i\eta,\ 1+\eta+i\eta,\ 1-\delta+i\eta.$$

When $\chi'$ is principal we have, by (24),

$$G(s,\chi')^2=(s-1)H(s,\chi'),$$

where $H$ is regular and non-zero in $\mathscr{A}$, and

$$H(s,\chi')=\frac{L(s,\chi_5\chi')}{(s-1)L(s,\chi_1\chi')}\prod_{p\equiv\pm2 \pmod 5}\left(1-\frac{\chi'(p)^2}{p^{2s}}\right).$$

Thus there is a function $K(s,\chi')$, regular in $\mathscr{A}$, such that

$$G(s,\chi')=(s-1)^{1/2}K(s,\chi')$$

in $\mathscr{A}'$, and $K(s,\chi')^2=H(s,\chi')$ in $\mathscr{A}$. Moreover

$$K(s,\chi')\ll_\varepsilon (q(1+|t|))^\varepsilon$$

uniformly in $\mathscr{A}$. Using $\chi_0$ for the principal character modulo $d$ and $\chi_0'$ for the principal character modulo $q$ we find, on letting $\eta\to0+$, that

$$\begin{aligned}\frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}\left(\frac{d}{q}\right)^s L(1+s,\chi_0)G(s,\chi_0')\left(\frac{X}{1-2\pi iX\beta}\right)^s\Gamma(s)\,ds\\ =-\frac{1}{\pi}\int_{1-\delta}^{1}\left(\frac{d}{q}\right)^u L(1+u,\chi_0)(1-u)^{1/2}K(u,\chi_0')\left(\frac{X}{1-2\pi iX\beta}\right)^u\Gamma(u)\,du\\ +O(X\exp(-c_2(\log X)^{1/2}))\\ \ll_\varepsilon q^\varepsilon X(\log X)^{-3/2}.\end{aligned}$$

Therefore, by (19) and (23), the total contribution to $F(z)$, when $z\in M(q,a)$, from the characters *not* induced by $\chi_5$ is

$$\ll\sum_{d\mid q}\left(\frac{1}{\phi(d)}q^\varepsilon X(\log X)^{-3/2}+d^{1/2}X\exp(-c_1(\log X)^{1/2})\right)\ll X/\log X.$$

Thus we have established the following lemma.

LEMMA 3. *Suppose that* $5\nmid q$, $1\le a\le q\le(\log X)^3$, $(a,q)=1$, $X$ *is large and* $z\in M(q,a)$. *Then*

$$F(z)\ll X/\log X.$$

We now have to turn our attention to the situation when characters induced by $\chi_5$ occur. Then $5\mid q$ and such characters can only occur to moduli $d$ dividing $q$ with $5\mid d$. Let $\chi$ denote such a character. Then, much as in the case of the principal character above we find that for $s\in\mathscr{A}'$

$$G(s,\chi')=(s-1)^{-1/2}J(s,\chi'),$$

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where $J$ is regular in $\mathscr{A}$ and satisfies

$$J(s,\chi')^2=\frac{(s-1)L(s,\chi_5\chi')}{L(s,\chi_1\chi')}\prod_{p\equiv\pm2(\mathrm{mod}\ 5)}\left(1-\frac{\chi'(p)^2}{p^{2s}}\right).$$

Moreover, for $q\le(\log X)^3$, $d\mid q$, we have

$$\frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}\left(\frac{d}{q}\right)^s L(1+s,\chi)G(s,\chi')\left(\frac{X}{1-2\pi iX\beta}\right)^s\Gamma(s)\,ds$$
$$=\frac{1}{\pi}\int_{1-\delta}^{1}\left(\frac{d}{q}\right)^u L(1+u,\chi)(1-u)^{-1/2}J(u,\chi')\left(\frac{X}{1-2\pi iX\beta}\right)^u\Gamma(u)\,du$$
$$+O(X\exp(-c_3(\log X)^{1/2})).$$

By the formula at the bottom of page 67 of Davenport [**2**] we have $\tau(\bar\chi)=\mu(d/5)\chi_5(d/5)5^{1/2}$. Hence, by (19) and (23), when $z\in M(q,a)$ we have

$$F(z)=O(X/\log X)+\frac{\chi_5(a)5^{1/2}}{\pi}\sum_{5\mid d\mid q}\frac{c(q/d)\mu(d/5)\chi_5(d/5)}{\phi(d)}$$
$$\times\int_{1-\delta}^{1}\left(\frac{d}{q}\right)^u I(u,d,q)(1-u)^{-1/2}\left(\frac{X}{1-2\pi X\beta}\right)^u\Gamma(u)\,du,$$

where $I(u,d,q)$ is equal to

$$L(1+u,\chi_5)\left(\frac{(u-1)\zeta(u)}{L(u,\chi_5)}\right)^{1/2}\left(\prod_{p\mid d}\left(1-\frac{\chi_5(p)}{p^{1+u}}\right)\right)\left(\prod_{p\mid q}\frac{p^u-1}{p^u-\chi_5(p)}\right)^{1/2}\prod_{\substack{p\equiv\pm2(\mathrm{mod}\ 5)\\(p,q)=1}}(1-p^{-2u})^{1/2}.$$

Thus

$$F(z)=O(X/\log X)+\frac{\chi_5(a)5^{1/2}}{4\pi}\int_{1-\delta}^{1}(1-u)^{-1/2}\left(\frac{X}{1-2\pi iX}\right)^u B_1(u,q/5)\,du,$$

where

$$B_1(u,r)=\Gamma(u)L(1+u,\chi_5)\left(\frac{(u-1)\zeta(u)}{L(u,\chi_5)}\right)^{1/2}(1-5^{-u})^{1/2}\left(\prod_{p\equiv\pm2(\mathrm{mod}\ 5)}(1-p^{-2u})\right)^{1/2}B_2(u,r)$$

and

$$B_2(u,r)=\sum_{k\mid r}\frac{c(r/k)\mu(k)\chi_5(k)}{\phi(k)}\left(\frac{k}{r}\right)^u$$
$$\times\left(\prod_{p\mid k}\left(1-\frac{\chi_5(p)}{p^{1+u}}\right)\right)\prod_{\substack{p\mid r\\p\equiv\pm2(\mathrm{mod}\ 5)}}\left(1+\frac{1}{p^u}\right)^{-1}.$$

When $1-\delta\le u\le1$ we have $B_1(u,r)=B_1(1,r)+(1-u)B_1'(v,r)$ for some $v\in(u,1)$, and $B_1'(w,r)\ll_\varepsilon r^\varepsilon$ uniformly on $[u,1]$. Hence

$$F(z)=\frac{\chi_5(a)5^{1/2}}{4\pi}B_1(1,q/5)\int_{1-\delta}^{1}(1-u)^{-1/2}\left(\frac{X}{1-2\pi iX\beta}\right)^u du+O(X/\log X).$$

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We also have

$$B_1(1, r) = \frac{L(2, \chi_5)}{(L(1, \chi_5))^{1/2}} \frac{2}{\sqrt{5}} \Big( \prod_{p \equiv \pm 2 (\mathrm{mod}\ 5)} (1 - p^{-2}) \Big)^{1/2} B(r),$$

where

$$B(r) = \sum_{k \mid r} \frac{c(r/k)\mu(k)\chi_5(k)}{\phi(k)} \left(\frac{k}{r}\right) \Big( \prod_{p \mid k} \Big(1 - \frac{\chi_5(p)}{p^2}\Big)\Big) \prod_{\substack{p \mid r \\ p \equiv \pm 2 (\mathrm{mod}\ 5)}} \frac{p}{p+1}.$$

We can evaluate $B(r)$ by observing that it is multiplicative and satisfies

$$\begin{aligned} B(p) &= \frac{2}{p^2 - 1} \quad \text{when} \quad \chi_5(p) = -1, \\ B(p^k) &= -p^{2-2k} \quad \text{when} \quad k > 1 \quad \text{and} \quad \chi_5(p) = -1, \\ B(p^k) &= -p^{1-2k} \quad \text{when} \quad \chi_5(p) = +1, \\ B(5^k) &= 5^{-2k}. \end{aligned}$$

Thus

$$B(1) = 1, \qquad |B(r)| \le \frac{4}{3r} \qquad (r > 1).$$

We also observe that

$$\left| \int_{1-\delta}^{1} (1-u)^{-1/2} \left( \frac{X}{1 - 2\pi i X \beta} \right)^{u} du \right| \le \int_{1-\delta}^{1} (1-u)^{-1/2} X^u \, du$$

and that

$$\begin{aligned} \int_{1-\delta}^{1} (1-u)^{-1/2} X^u \, du &= X \int_{1-\delta}^{1} (1-u)^{-1/2} X^{-(1-u)} \, du \\ &= \frac{X}{(\log X)^{1/2}} \int_0^\infty v^{-1/2} e^{-v} \, dv + O\left( \frac{X}{\log X} \right) \\ &= \frac{X \pi^{1/2}}{(\log X)^{1/2}} + O\left( \frac{X}{\log X} \right). \end{aligned}$$

Combining the above results establishes the following lemma.

LEMMA 4. *Let $\chi_5$ denote the quadratic character modulo 5 and let*

$$A = \frac{1}{2\pi^{1/2}} \frac{L(2, \chi_5)}{(L(1, \chi_5))^{1/2}} \prod_{p \equiv \pm 2 (\mathrm{mod}\ 5)} \left(1 - \frac{1}{p^2}\right)^{1/2}. \tag{28}$$

*Then*

$$F\left(\rho e\left(\frac{a}{5}\right)\right) = \chi_5(a) A \frac{X}{(\log X)^{1/2}} + O\left(\frac{X}{\log X}\right) \qquad (5 \nmid a). \tag{29}$$

Suppose further that $1 \le a \le q \le (\log X)^3$, $(a, q) = 1$, $5 \mid q$, $X$ is large, and $z \in M(q, a)$. Then

$$|F(z)| \le AB(q/5)\frac{X}{(\log X)^{1/2}} + O\left(\frac{X}{\log X}\right), \tag{30}$$

where

$$B(1) = 1 \quad \text{and} \quad B(r) \le 4/(3r) \qquad (r > 1).$$

**7. Completion of the proof of Theorem 4.** Let $M$ be large and let

$$n = \prod_{\substack{p \le M \\ p \equiv \pm 2 (\mathrm{mod}\ 5)}} p,$$

so that $n \in \mathcal{N}$. We shall show for some $n'$ that $|a(M, n')| > \exp(cM^{1/2}(\log M)^{-1/4})$ for a suitable positive constant $c$. By (13) and (14) we have

$$\sum_{m=1}^{\infty} c_m z^m = \sum_{m=1}^{\infty} c(m) z^m + O\left(\sum_{m>M} |z|^m\right). \tag{32}$$

Suppose $|z| = \rho = e^{-1/X}$ with $X$ large. Then, by Lemmas 2, 3 and 4

$$\left|\sum_{m=1}^{\infty} c_m z^m\right| \le \frac{AX}{(\log X)^{1/2}} + O\left(\frac{X}{\log X} + X\exp\left(-\frac{M}{X}\right)\right)$$
$$\le \frac{AX}{(\log X)^{1/2}} + O\left(\frac{X}{\log X}\right)$$

provided that

$$M \ge X \log\log X. \tag{33}$$

By (12) and Cauchy's inequalities for the coefficients of power series we have

$$|a(m, n)| \le \rho^{-m} \exp\left(\frac{AX}{(\log X)^{1/2}} + O\left(\frac{X}{\log X}\right)\right)$$
$$= \exp\left(\frac{m}{X} + \frac{AX}{(\log X)^{1/2}} + O\left(\frac{X}{\log X}\right)\right).$$

Now we may choose

$$X = \left(\frac{m}{A}\right)^{1/2} (\tfrac{1}{2}\log m)^{1/4}$$

provided that (33) is satisfied, and it certainly will be when

$$2 \le m \le M^2/\log M. \tag{34}$$

Thus

$$|a(m, n)| \le \exp\left(\frac{2^{5/4}(Am)^{1/2}}{(\log m)^{1/4}} + O\left(\frac{m^{1/2}}{(\log m)^{3/4}}\right)\right). \tag{35}$$

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Now instead choose $X$ so that

$$\frac{AX^2}{(\log X)^{1/2}}=\frac{M}{100}. \tag{36}$$

Again (33) is satisfied. Thus, by (32) and Lemma 4,

$$\sum_{m=1}^{\infty} c_m\rho^m e(am/5)=\chi(a)A\frac{X}{(\log X)^{1/2}}+O\left(\frac{X}{\log X}\right). \tag{37}$$

The maximum of the function of $x$ given by

$$\frac{2^{5/4}(Ax)^{1/2}}{(\log X)^{1/4}}-\frac{x}{X}$$

occurs with

$$x\sim\frac{AX^2}{(\log X)^{1/2}}.$$

Hence, by (36) and (35),

$$\sum_{m\le\frac{M}{200}}|a(m,n)|\rho^m\le\exp\left(\frac{2^{5/4}(AM/200)^{1/2}}{(\log(M/200))^{1/4}}-\frac{M}{200X}+O\left(\frac{X}{\log X}\right)\right).$$

Therefore, by (36),

$$\sum_{m\le\frac{M}{200}}|a(m,n)|\rho^m\le\exp\left((2^{1/2}-\tfrac{1}{2})\frac{AX}{(\log X)^{1/2}}+O\left(\frac{X}{\log X}\right)\right). \tag{38}$$

A similar argument shows that

$$\sum_{\frac{M}{50}<m\le\frac{M^2}{\log M}}|a(m,n)|\rho^m\le\exp\left((2^{3/2}-2)\frac{AX}{(\log X)^{1/2}}+O\left(\frac{X}{\log X}\right)\right). \tag{39}$$

Also it follows easily from Theorem 3 that

$$\sum_{m>\frac{M^2}{\log M}}|a(m,n)|\rho^m<1.$$

Hence, by (12), (37), (38) and (39) there is an $a$ such that

$$\left|\sum_{\frac{M}{200}<m\le\frac{M}{50}}a(m,n)e^{-m/X}e\left(\frac{a}{5}m\right)\right|=\exp\left(\frac{AX}{(\log X)^{1/2}}+O\left(\frac{X}{\log X}\right)\right).$$

Therefore there is an $M_0$ and a positive constant $C$ such that

$$\frac{M}{200}<M_0\le\frac{M}{50}\quad\text{and}\quad|a(M_0,n)|>\exp(CM^{1/2}(\log M)^{-1/4}). \tag{40}$$

Let

$$\Lambda=\exp\left(\frac{c}{2}M^{1/2}(\log M)^{-1/4}\right). \tag{41}$$

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If $|a(M, n)| > \Lambda$, then we are finished. Hence we may assume that

$$|a(M, n)| \le \Lambda. \tag{42}$$

By (10)

$$\Phi_{mn}(z) = \prod_{d\mid m} \Phi_n(z^{m/d})^{\mu(d)} \quad ((m, n) = 1). \tag{43}$$

Let $p, q$ denote distinct prime numbers with $p, q \equiv \pm 1 (\mathrm{mod}\ 5)$, $q > M$. Further let $a_-(m, n)$ denote the coefficient of $z^m$ in the power series expansion of $\Phi_n(z)^{-1}$, valid for $|z| < 1$. Note that

$$a_-(0, n) = 1, \qquad a_-(1, n) = \mu(n). \tag{44}$$

Now, since $q > M$, it follows from (43) with $m = pq$ that

$$\begin{aligned} a(M, npq) &= \sum_{\substack{u \ge 0,\, v \ge 0 \\ u + pv = M}} a(u, n) a_-(v, n) \\ &= a(M, n) + b_1(M, p), \end{aligned}$$

where

$$b_1(M, p) = \sum_{1 \le v \le M/p} a(M - pv, n) a_-(v, n).$$

If $|b_1(M, p)| > 2\Lambda$, then we are finished. Hence we may suppose that

$$|b_1(M, p)| \le 2\Lambda \quad \text{for each prime} \quad p \equiv \pm 1 (\mathrm{mod}\ 5). \tag{45}$$

Now let $p_1, p_2$ denote distinct prime numbers in the residue classes $\pm 1 (\mathrm{mod}\ 5)$ with $p_1^2 > M$, $p_2^2 > M$. Then, by (43) with $m = p_1 p_2$, we have

$$\begin{aligned} a(M, np_1p_2) &= \sum_{\substack{u \ge 0,\, v_1 \ge 0,\, v_2 \ge 0 \\ u + p_1 v_1 + p_2 v_2 = M}} a(u, n) a_-(v_1, n) a_-(v_2, n) \\ &= a(M, n) + b_1(M, p_1) + b_1(M, p_2) + b_2(M, p_1, p_2), \end{aligned}$$

where

$$b_2(M, p_1, p_2) = \sum_{\substack{v_1 \ge 1,\, v_2 \ge 1 \\ p_1 v_1 + p_2 v_2 \le M}} a(M - p_1 v_1 - p_2 v_2, n) a_-(v_1, n) a_-(v_2, n).$$

If $|b_2(M, p_1, p_2)| > 6\Lambda$, then the desired conclusion follows from (42) and (45). Hence we may suppose that

$$|b_2(M, p_1, p_2)| \le 6\Lambda \tag{46}$$

for all distinct primes $p_1, p_2 \equiv \pm 1 (\mathrm{mod}\ 5)$ with $p_1^2 > M$, $p_2^2 > M$.

Now let $p_1, p_2, p_3, q$ denote distinct primes in the residue classes $\pm 1 (\mathrm{mod}\ 5)$ with $p_i^2 > M$, $q > M$. Then, by (43) with $m = p_1 p_2 p_3 q$, we have

$$\begin{aligned} a(M, np_1p_2p_3q) &= \sum_{\substack{u \ge 0,\, v_1 \ge 0,\, v_2 \ge 0,\, v_3 \ge 0 \\ u + p_1 v_1 + p_2 v_2 + p_3 v_3 = M}} a(u, n) a_-(v_1, n) a_-(v_2, n) a_-(v_3, n) \\ &= a(M, n) + \sum_i b_1(M, p_i) + \sum_{i<j} b_2(M, p_i, p_j) + b_3(M, p_1, p_2, p_3) \end{aligned}$$

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with

$$b_3(M, p_1, p_2, p_3) = \sum_{\substack{v_1 \geq 1, v_2 \geq 1, v_3 \geq 1 \\ p_1v_1 + p_2v_2 + p_3v_3 \leq M}} a(M - p_1v_1 - p_2v_2 - p_3v_3)a_-(v_1, n)a_-(v_2, n)a_-(v_3, n).$$

When $M - M_0$ is odd a straightforward application of the Hardy–Littlewood–Vinogradov method as expounded in Chapter 3 of Vaughan [**9**] shows that there are distinct primes $p_1, p_2, p_3$ with $p_i > \frac{1}{4}M$, $p_i \equiv \pm 1 \pmod 5$ and $p_1 + p_2 + p_3 = M - M_0$. For such a choice of $p_1, p_2, p_3$ we have

$$b_3(M, p_1, p_2, p_3) = a(M_0, n)a(1, n)^3.$$

Hence, by (40), (41) and (44), we have

$$|b_3(M, p_1, p_2, p_3)| > 26\Lambda.$$

Thus, when $M - M_0$ is odd, it follows from (42), (45) and (46) that

$$|a(M, np_1p_2p_3q)| > \Lambda,$$

as required.

Now suppose that $M - M_0$ is even. If $|b_3(M, p_1, p_2, p_3)| > 26\Lambda$, then we are finished. Hence we may suppose that

$$|b_3(M, p_1, p_2, p_3)| \leq 26\Lambda, \tag{47}$$

for all distinct primes $p_1, p_2, p_3 \equiv \pm 1 \pmod 5$ with $p_i^2 > M$. By applying the Hardy–Littlewood–Vinogradov method as above one can readily show that there are primes $p_1, p_2, p_3, p_4$ with $p_i > \frac{1}{5}M$, $p_i \equiv \pm 1 \pmod 5$ and $p_1 + p_2 + p_3 + p_4 = M - M_0$. By (43) with $m = p_1p_2p_3p_4$ we obtain

$$a(M, np_1p_2p_3p_4) = a(M, n) + \sum_i b_1(M, p_i) + \sum_{i<j} b_2(M, p_i, p_j)$$
$$+ \sum_{i<j<k} b_3(M, p_i, p_j, p_k) + a(M_0, n)a_-(1, n)^4.$$

The argument can now be completed much as in the previous case. Thus we have shown that

$$\max_n |a(M, n)| > \Lambda.$$

This completes the proof of Theorem 4.

## REFERENCES

**1.** H. Davenport, On some infinite series involving arithmetical functions II, *Quart. J. Math Oxford*, **8** (1937), 313–320.

**2.** H. Davenport, *Multiplicative number theory*, Second Edition, revised by H. L. Montgomery, (Springer-Verlag 1980).

**3.** P. Erdös, and R. C. Vaughan, Bounds for the $r$-th coefficients of cyclotomic polynomials, *J. London Math. Soc.* (2) **8** (1974), 393–400.

https://doi.org/10.1017/S0017089500006145 Published online by Cambridge University Press

**4.** H. Halberstam, and H.-E. Richert, *Sieve methods*, (Academic Press, London, 1974).
**5.** Edmund Landau, *Vorlesungen über Zahlentheorie*, (Chelsea, New York, 1955).
**6.** H. L. Montgomery and R. C. Vaughan, Exponential sums with multiplicative coefficients, *Invent. Math.* **43** (1977), 69–82.
**7.** E. C. Titchmarsh, *The theory of the Riemann zeta function*, (Clarendon Press, Oxford, 1951).
**8.** R. C. Vaughan, Bounds for the coefficients of cyclotomic polynomials, *Michigan Math J.* **21** (1974), 289–295.
**9.** R. C. Vaughan, *The Hardy–Littlewood method*, (Cambridge University Press, 1981).

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