ON THE p-LENGTH AND THE WIELANDT LENGTH OF
A FINITE p-SOLUBLE GROUP

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Abstract

The p-length of a finite p-soluble group is an important invariant parameter. The well-known Hall–
Higman p-length theorem states that the p-length of a p-soluble group is bounded above by the nilpotent
class of its Sylow p-subgroups. In this paper, we improve this result by giving a better estimation on the
p-length of a p-soluble group in terms of other invariant parameters of its Sylow p-subgroups.


Keywords and phrases: p-length, Wielandt length, nilpotent class, permutable.

1. Introduction

All groups considered in this paper are finite and the terminology and notation are
standard; see [8]. For a finite group G, we use |G| and π(G) to denote the order of G
and the set of all primes dividing |G|, respectively; for a prime p ∈ π(G), let Gp be a
Sylow p-subgroup of G.

The celebrated Hall–Higman p-length theorem [7] establishes a connection
between the p-length of a p-soluble group G and the nilpotent class of its Sylow p-
subgroup Gp, showing that the p-length of G is bounded above by the nilpotent class
of Gp.

For a finite group G, the Wielandt subgroup ω(G), introduced by Wielandt [12] in
1958, is the intersection of the normalisers of all subnormal subgroups of G. In that
paper, Wielandt defined a series of normal subgroups

ω0(G), ω1(G), . . . , ωℓ(G) = G

for a group G as follows:

First, set ω0(G) = 1, and then if ωi(G) is defined, set ωi+1(G)/ωi(G) =

ω(G/ωi(G)).
He showed that $\omega(G)$ contains all minimal normal subgroups of $G$. Obviously, for a finite group $G$, $\omega_n(G) = G$ for some positive integer $n$. The smallest such value of $n$ is called the Wielandt length of $G$, and is denoted by $w^*(G)$ in this paper.

Several authors have investigated relations between the Wielandt length and other invariant parameters of $G$; see [3, 4, 9] for instances. Let $p$ be a prime and $P$ a $p$-group. It is easy to see that $\omega(P)$ contains the centre of $P$, and the Wielandt length of $P$ is not greater than the nilpotent class of $P$. For a $p$-group $P$, the Wielandt length may be less than the nilpotent class; for example, the quaternion group of order eight has nilpotent class 2 and Wielandt length 1. Furthermore, it is shown in [9] that a metabelian $p$-group of odd order has Wielandt length at most its nilpotent class minus one. An example in [9] of a 5-group has nilpotent class 6 and Wielandt length 4.

Let $G$ be a $p$-soluble group and let $c(G_p)$ denote the nilpotent class of $G_p$. As mentioned above, the $p$-length of $G$ is bounded above by $c(G_p)$, and $w^*(G_p) \leq c(G_p)$. This motivates the following question.

**Question 1.1.** For a $p$-soluble group, is the $p$-length bounded above by the Wielandt length of its Sylow $p$-subgroups?

The main purpose of this paper is to give an affirmative answer to this question. We will prove a more general result. To state our main result, we need to introduce a few more definitions.

A subgroup $H$ of a group $G$ is said to be permutable if for any subgroup $K$ of $G$, we have $HK = KH$. The normaliser of a subgroup $H$ in $G$ consists of the elements $x$ such that $xH = Hx$, and the permutiser of a subgroup $H$ in $G$ consists of the elements $x$ such that $\langle x \rangle H = H \langle x \rangle$; see [10]. A normal subgroup series is called a central series if every member is in the centre of the corresponding quotient group. The nilpotent class of a nilpotent group is the shortest length of its central series. We introduce the following definition.

**Definition 1.2.** Let $G$ be a nilpotent group. A normal series

$$1 = H_0 \leq H_1 \leq \cdots \leq H_n = G$$

is called a permutable series of $G$ if, for any $1 \leq i \leq n$ and any element $x$ of $H_i$, $\langle x \rangle H_{i-1}/H_{i-1}$ is permutable in $G/H_{i-1}$. In this case, the integer $n$ is called the length of this series.

Let $G$ be a nilpotent group. Then a central series of $G$ is a permutable series of $G$. It follows that a nilpotent group has a permutable series.

**Definition 1.3.** Let $G$ be a nilpotent group, and let

$$p(G) = \min\{n \mid n \text{ is the length of some permutable series of } G\}.$$ 

Then $p(G)$ is called the permutable length of $G$.

The main result of this paper is stated as follows.
**Theorem 1.4.** Let $G$ be a $p$-soluble group. Then the $p$-length of $G$ is no larger than the permutable length of $G_p$.

### 2. Observations and Examples

Let $p$ be a prime and $P$ a $p$-group. From the definition, one can see that the permutable length of $P$ is determined only by the structure of $P$. Since the upper central series of $P$ is a permutable series of $P$ and $c(P)$ is equal to the length of the upper central series of $P$, we have $\nu(P) \leq c(P)$. Now let us consider the relationship between the permutable length of $P$ and the Wielandt length of $P$.

**Proposition 2.1.** Let $p$ be a prime and $P$ a $p$-group. We have $\nu(P) \leq w^*(P)$.

**Proof.** We only need to show that $1 = \omega_0(P) \leq \omega_1(P) \leq \cdots \leq \omega_n(P) = P$ is a permutable series of $P$. Let $i$ be an integer such that $1 \leq i \leq n$ and let $x$ be an element of $\omega_i(P)$. Let $K/\omega_{i-1}(P)$ be a subgroup of $P/\omega_{i-1}(P)$. Since every subgroup of $P/\omega_{i-1}(P)$ is normal in $P/\omega_{i-1}(P)$, $\omega_i(P)/\omega_{i-1}(P)$ is the intersection of the normalisers of all subgroups of $P/\omega_{i-1}(P)$. In particular, $\langle x \rangle \omega_{i-1}(P)/\omega_{i-1}(P) \leq N_{P/\omega_{i-1}(P)}(K/\omega_{i-1}(P))$. Hence $\langle x \rangle \omega_{i-1}(P)$ is a subgroup of $P/\omega_{i-1}(P)$. Because $K/\omega_{i-1}(P)$ is chosen arbitrarily, one can see that $\langle x \rangle \omega_{i-1}(P)/\omega_{i-1}(P)$ is permutable in $P/\omega_{i-1}(P)$. It follows that $1 = \omega_0(P) \leq \omega_1(P) \leq \cdots \leq \omega_n(P) = P$ is a permutable series of $P$. 

Let $p$ be a prime. The following example indicates that the permutable length of a $p$-group can be less than its Wielandt length.

**Example 2.2** [11, p. 65, Example 2.3.19]. Let $p > 2$ and $P = \langle a, x \mid a^{p^3} = 1, \ x^{p^3} = a^{p^2}, \ a^x = a^{1+p} \rangle$. Since $a^{p^2}$ is centralised by the automorphism $\sigma$ of $\langle a \rangle$ with $a^\sigma = a^{1+p}$, $P$ is an extension of $\langle a \rangle$ by a cyclic group of order $p^3$. By Iwasawa’s theorem [11, p. 55, Theorem 2.3.1] and [11, p. 55, Lemma 2.3.2], every subgroup of $P$ is permutable in $P$. Thus, we have $\nu(P) = 1$. On the other hand, since $P$ is a nonabelian $p$-group of odd order, it follows from [11, p. 60, Theorem 2.3.12] that not every subgroup of $P$ is normal in $P$. Hence $\omega(P) < P$ and $w^*(P) > 1$. Therefore, $\nu(P) < w^*(P)$.

As observed above, $\nu(P) \leq w^*(G_p) \leq c(G_p)$, and we also see that there are examples with the strict relations $w^*(P) < c(P)$ and $\nu(P) < w^*(P)$. Let $l_p(G)$ denote the $p$-length of a $p$-soluble group $G$. Hall–Higman’s $p$-length theorem states that the $p$-length of $G$ is bounded by the nilpotent class of $G_p$. The main result of this paper is to improve Hall–Higman’s $p$-length theorem by a better bound. Actually, we will prove that $l_p(G) \leq \nu(G_p)$.

### 3. The proof of the main theorem

We first present some basic facts about the $p$-length and the permutable length.

**Lemma 3.1** [8, p. 689, Hilfssatz 6.4]. Let $G$ be a $p$-soluble group.

1. If $N \unlhd G$, then $l_p(G/N) \leq l_p(G)$.
2. If $U \leq G$, then $l_p(U) \leq l_p(G)$.
Let $N_1$ and $N_2$ be two normal subgroups of $G$. Then
\[ l_p(G/(N_1 \cap N_2)) = \max\{l_p(G/N_1), l_p(G/N_2)\}. \]

(4) \quad l_p(G/\Phi(G)) = l_p(G).

**Lemma 3.2** [5, 1.3 and 1.4]. Let $N$ be a normal subgroup of $G$ and $K$ a subgroup of $G$ containing $N$. Then $K/N$ is permutable in $G/N$ if and only if $K$ is permutable in $G$.

**Lemma 3.3.** Let $G$ be a nilpotent group.

1. If $N \leq G$, then $\varphi(G/N) \leq \varphi(G)$.
2. If $U \leq G$, then $\varphi(U) \leq \varphi(G)$.

**Proof.** Let $1 = H_0 \leq H_1 \leq \cdots \leq H_n = G$ be a permutable series of $G$ with length $n = \varphi(G)$. To prove (1), we only need to show that $1 = H_0N/N \leq H_1N/N \leq \cdots \leq H_nN/N = G/N$ is a permutable series of $G/N$. Since all $H_i$ are normal in $G$, all $H_iN/N$ are also normal in $G/N$. Let $i$ be an integer such that $1 \leq i \leq n$. Let $x$ be an element of $H_iN/N$. It is easy to see that there exists an element $y$ of $H_i$ such that $\langle x \rangle = \langle y \rangle N/N$. Let $K/N$ be a subgroup of $G/N$. By definition, $\langle y \rangle H_{i-1}/H_{i-1}$ is permutable in $G/H_{i-1}$. It follows that $\langle y \rangle H_{i-1}$ is permutable in $G$ by Lemma 3.2. Hence $\langle y \rangle H_{i-1}K$ is a subgroup of $G$. Therefore, $(\langle y \rangle N/N)(H_{i-1}N/N)K/N = (\langle y \rangle H_{i-1}KN)/N$ is a subgroup of $G/N$. The arbitrary choice of $K/N$ implies that $(\langle y \rangle N/N)(H_{i-1}N/N)$ is permutable in $G/N$. Again by Lemma 3.2, we know that $\prod (\langle y \rangle (H_{i-1}N/N)/N) = \prod (\langle y \rangle N/N)(H_{i-1}N/N)$ is permutable in $(G/N)/(H_{i-1}N/N)$. By definition, $1 = H_0N/N \leq H_1N/N \leq \cdots \leq H_nN/N = G/N$ is a permutable series of $G/N$.

To prove (2), we need to show that $1 = (H_0 \cap U)/(H_1 \cap U) \leq \cdots \leq (H_n \cap U) = U$ is a permutable series of $U$. It is evident that $H_i \cap U$ is normal in $U$ for any $i$. Let $i$ be an integer such that $1 \leq i \leq n$. Let $x$ be an element of $H_i \cap U$ and $K$ be a subgroup of $U$. By definition, $\langle x \rangle H_{i-1}/H_{i-1}$ is permutable in $G/H_{i-1}$. It follows that $\langle x \rangle H_{i-1}$ is permutable in $G$, by Lemma 3.2. Hence $\langle x \rangle H_{i-1}K$ is a subgroup of $G$. Since $K \leq U$ and $\langle x \rangle \leq U$, we have $\langle x \rangle (H_{i-1} \cap U)K = \langle x \rangle (H_{i-1}K \cap U) = \langle x \rangle H_{i-1}K \cap U$ and thus $\langle x \rangle (H_{i-1} \cap U)K$ is a subgroup of $U$. The arbitrary choice of $K$ implies that $\langle x \rangle (H_{i-1} \cap U)$ is permutable in $U$. Again by Lemma 3.2, $\langle x \rangle (H_{i-1} \cap U)/(H_{i-1} \cap U)$ is permutable in $U/(H_{i-1} \cap U)$. Therefore, $1 = (H_0 \cap U) \leq (H_1 \cap U) \leq \cdots \leq (H_n \cap U) = U$ is a permutable series of $U$.

**Proof of Theorem 1.4.** Assume that this theorem is not true and let $G$ be a counterexample of minimal order. Then we have the following steps to the proof.

1. Assume that $O_{p'}(G) \neq \Phi(G) \neq 1$. Then, by the minimal choice of $G$, $l_p(G/O_{p'}(G)) \leq \varphi(G/O_{p'}(G))/\varphi(G)$ or $l_p(G/\Phi(G)) \leq \varphi(G/\Phi(G))/\varphi(G)$. By the definition of $p$-length, $l_p(G/O_{p'}(G)) = l_p(G)$. By Lemma 3.1(4), $l_p(G/\Phi(G)) = l_p(G)/\varphi(G)$ and $\varphi(G/O_{p'}(G))/\varphi(G) = \varphi(G/\Phi(G))/\varphi(G)$ or $\varphi(G/O_{p'}(G))/\varphi(G) = \varphi(G/\Phi(G))/\varphi(G)$, a contradiction.
2. $G$ has a unique minimal normal subgroup $N$. 

\[ \text{Suppose that} \quad O_{p'}(G) \neq \Phi(G) \neq 1. \quad \text{Then, by the minimal choice of} \quad G, \quad l_p(G/O_{p'}(G)) \leq \varphi(G/O_{p'}(G))/\varphi(G) \quad \text{or} \quad l_p(G/\Phi(G)) \leq \varphi(G/\Phi(G))/\varphi(G). \quad \text{By the definition of} \quad p\text{-length,} \quad l_p(G/O_{p'}(G)) = l_p(G). \quad \text{By Lemma 3.1(4),} \quad l_p(G/\Phi(G)) = l_p(G)/\varphi(G). \quad \text{On the other hand,} \quad \varphi(G/O_{p'}(G))/\varphi(G) = \varphi(G/(G \cap O_{p'}(G))) \leq \varphi(G) \quad \text{and} \quad \varphi(G/\Phi(G))/\varphi(G) = \varphi(G/(G \cap \Phi(G))) \leq \varphi(G)$ from Lemma 3.3(1). Hence $l_p(G) \leq \varphi(G)$, a contradiction.

\[ \text{Therefore,} \quad G \text{ has a unique minimal normal subgroup} \quad N. \]
Suppose that $G$ has two different minimal normal subgroups $N_1$ and $N_2$. From the minimal choice of $G$, $l_p(G/N_1) \leq \nu(G_pN_1/N_1)$ and $l_p(G/N_2) \leq \nu(G_pN_2/N_2)$. Without loss of generality, we may assume that $\nu(G_pN_1/N_1) \geq \nu(G_pN_2/N_2)$. Obviously, $N_1 \cap N_2 = 1$. From Lemmas 3.1(3) and 3.3(1),

$$l_p(G) = l_p(G/(N_1 \cap N_2)) \leq \max\{l_p(G/N_1), l_p(G/N_2)\} = l_p(G/N_1) \leq \nu(G_pN_1/N_1) = \nu(G_p/(G_p \cap N_1)) \leq \nu(G_p),$$

a contradiction.

(3) $N = C_G(N) = O_p(G)$.

Since $\Phi(G) = 1$, $O_p(G)$ is the direct product of some minimal normal subgroups of $G$. But $N$ is the unique minimal normal subgroup of $G$, so $N = O_p(G)$. Because $G$ is $p$-soluble and $O_p'(G) = 1$, we have $C_G(O_p'(G)) \leq O_p(G)$ by [6, p. 228, Theorem 3.2]. Since $O_p(G) = N$ is abelian, $N = C_G(N)$.

(4) There exists a maximal subgroup $M$ of $G$ such that $G = [N]M$.

This follows directly from the fact that $\Phi(G) = 1$ and $N$ is an abelian minimal normal subgroup of $G$.

(5) Suppose that $1 = H_0 \leq H_1 \leq \cdots \leq H_n = G_p$ is a permutable series of $G_p$ with length $n = \nu(G_p)$. Then $H_1 \cap M = 1$.

Assume that $H_1 \cap M \neq 1$. Let $x$ be an element of $H_1 \cap M$ of order $p$, and let $y$ be an element of $N$. Clearly $y$ is also of order $p$. By definition, $\langle x \rangle$ is permutable in $G_p$. Since $\langle y \rangle \leq O_p(G) \leq G_p$, $\langle x \rangle \langle y \rangle$ is a subgroup of $G_p$. Note that $|x| = |y| = p$ and $\langle x \rangle \cap N \leq M \cap N = 1$, and $\langle x \rangle \langle y \rangle$ is a group of order $p^2$. Therefore, $\langle x \rangle \langle y \rangle$ is an abelian group and $x \in C_G(\langle y \rangle)$. Since $y$ is chosen arbitrarily, we have $x \in C_G(N)$. But then $x \in C_G(N) \cap M = N \cap M = 1$, a contradiction.

(6) Final contradiction.

Let $M_p$ be a Sylow $p$-subgroup of $M$ such that $M_p \leq G_p$. By (4), $G_p = NM_p$ and $N \cap M_p = 1$. By (5), $M_p = M_p/(M_p \cap H_1) \cong N_H/H_1$. Hence $G_p/N = NM_p/N \cong M_p/(N \cap M_p) \cong M_pH_1/H_1$ and $\nu(G_p/N) = \nu(M_pH_1/H_1)$. But $M_pH_1/H_1$ is a subgroup of $G_p/H_1$ and thus $\nu(M_pH_1/H_1) \leq \nu(G_p/H_1)$ by Lemma 3.3(2). As a result, $\nu(G_p/N) = \nu(M_pH_1/H_1) \leq \nu(G_p/H_1)$.

From the proof of Lemma 3.3(1), we know that the normal series $1 = H_1 \leq H_2/H_1 \leq \cdots \leq H_n/H_1 = G_p/H_1$, whose length is $n - 1 = \nu(G_p) - 1$, is a permutable series of $G_p/H_1$. As a result, $\nu(G_p/H_1) \leq \nu(G_p) - 1$. From (1), (3) and the definition of the $p$-length, we know that $l_p(G/N) = l_p(G) - 1$. From the minimal choice of $G$, $l_p(G/N) \leq \nu(G_p/N)$. Hence $l_p(G) - 1 = l_p(G/N) \leq \nu(G_p/N) \leq \nu(G_p/H_1) \leq \nu(G_p) - 1$ and it follows that $l_p(G) \leq \nu(G_p)$, a final contradiction. □

4. Some applications

The following corollary is an immediate consequence of Proposition 2.1 and Theorem 1.4. It gives an affirmative answer to Question 1.1.

**Corollary 4.1.** Let $G$ be a $p$-soluble group. Then the $p$-length of $G$ is no larger than the Wielandt length of $G_p$. 
Let $G$ be a $p$-soluble group. As a special case of Theorem 1.4, we know that if the permutable length of $G_p$ is at most 1, then the $p$-length of $G$ is also at most 1. By definition, the permutable length of $G_p$ is at most 1 if and only if every subgroup of $G_p$ is permutable in $G_p$. By [11, p. 55, Lemma 2.3.2], $G_p$ satisfies such properties if and only if $G_p$ is a modular $p$-group. As a result, we have the following corollary.

**Corollary 4.2.** Let $G$ be a $p$-soluble group. If the Sylow $p$-subgroups of $G$ are modular $p$-subgroups, then $l_p(G) \leq 1$.

A Hamiltonian group is a group all of whose subgroups are normal. From this definition, one can see that a Hamiltonian $p$-group must be a modular $p$-group. (The converse is not true, see Example 2.2.) By Corollary 4.2, we have the following further corollary.

**Corollary 4.3** [2]. Let $G$ be a $p$-soluble group. If the Sylow $p$-subgroups of $G$ are Hamiltonian $p$-subgroups, then $l_p(G) \leq 1$.

The well-known Burnside’s theorem tells us that if $N_G(G_p) = C_G(G_p)$, then $G$ is $p$-nilpotent. In other words, if $G_p$ is an abelian $p$-group and $N_G(G_p)$ is $p$-nilpotent, then $G$ is $p$-nilpotent. In [1, Theorem 1], this result was extended to show that if $G_p$ is a modular $p$-group, then $G$ is $p$-nilpotent if and only if $N_G(G_p)$ is $p$-nilpotent. An interesting question is whether we can get an analogous result for the case of $p$-supersoluble. That is, suppose that $G_p$ is a modular $p$-group, can we obtain that $G$ is $p$-supersoluble provided $N_G(G_p)$ is $p$-supersoluble? The answer to this question is no. For instance, the alternating group $A_5$ has modular Sylow 5-subgroups and the normalisers of its Sylow 5-subgroups are also 5-supersoluble, but $A_5$ itself is not 5-supersoluble. However, the following theorem indicates that in the class of all $p$-soluble groups, the modularity of the Sylow $p$-subgroups and the $p$-supersolvability of $N_G(G_p)$ do yield the $p$-supersolvability of $G$.

**Theorem 4.4.** Let $G$ be a $p$-soluble group with modular Sylow $p$-subgroups. Let $\mathcal{F}$ be a formation satisfying $E_p^{\mathcal{F}} = \mathcal{F}$ (where $E_p^{\mathcal{F}}$ denotes the class of all groups with order coprime to $p$). If $N_G(G_p) \in \mathcal{F}$, then $G \in \mathcal{F}$. In particular, under the circumstances that $G$ is a $p$-soluble group and $G_p$ is a modular $p$-group, $G$ is $p$-supersoluble if and only if $N_G(G_p)$ is $p$-supersoluble.

**Proof.** By Corollary 4.2, we know that $l_p(G) \leq 1$. Hence $G_pO_p'(G)/O_p'(G)$ is normal in $G/O_p'(G)$. It follows that $G_pO_p'(G)$ is normal in $G$ and $N_G(G_pO_p'(G)) = G$. Since any two Sylow $p$-subgroups are conjugated, we have $N_G(G_pO_p'(G)) = N_G(G_pO_p'(G))$. Consequently, $G/O_p'(G) = N_G(G_pO_p'(G))/O_p'(G) = N_G(G_pO_p'(G))/O_p'(G) \cong N_G(G_p)/(N_G(G_p) \cap O_p'(G)) \in \mathcal{F}$. This implies that $G \in E_p^{\mathcal{F}}$ and the hypothesis that $E_p^{\mathcal{F}} = \mathcal{F}$ guarantees that $G \in \mathcal{F}$.

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