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ON THE p-LENGTH AND THE WIELANDT LENGTH OF A FINITE p-SOLUBLE GROUP

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Abstract

The p-length of a finite p-soluble group is an important invariant parameter. The well-known Hall–Higman p-length theorem states that the p-length of a p-soluble group is bounded above by the nilpotent class of its Sylow p-subgroups. In this paper, we improve this result by giving a better estimation on the p-length of a p-soluble group in terms of other invariant parameters of its Sylow p-subgroups.

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1. Introduction

All groups considered in this paper are finite and the terminology and notation are standard; see [8]. For a finite group G, we use $|G|$ and $\pi(G)$ to denote the order of G and the set of all primes dividing $|G|$, respectively; for a prime $p \in \pi(G)$, let G_p be a Sylow p-subgroup of G.

The celebrated Hall–Higman p-length theorem [7] establishes a connection between the p-length of a p-soluble group G and the nilpotent class of its Sylow p-subgroup G_p, showing that the p-length of G is bounded above by the nilpotent class of G_p.

For a finite group G, the *Wielandt subgroup* $\omega(G)$, introduced by Wielandt [12] in 1958, is the intersection of the normalisers of all subnormal subgroups of G. In that paper, Wielandt defined a series of normal subgroups

$$\omega_0(G), \omega_1(G), \dots, \omega_\ell(G) = G$$

for a group G as follows:

First, set $\omega_0(G) = 1$, and then if $\omega_i(G)$ is defined, set $\omega_{i+1}(G)/\omega_i(G) = \omega(G/\omega_i(G))$.

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He showed that $\omega(G)$ contains all minimal normal subgroups of G. Obviously, for a finite group G, $\omega_n(G) = G$ for some positive integer n. The smallest such value of n is called the *Wielandt length* of G, and is denoted by $w^*(G)$ in this paper.

Several authors have investigated relations between the Wielandt length and other invariant parameters of G; see [3, 4, 9] for instances. Let p be a prime and P a p-group. It is easy to see that $\omega(P)$ contains the centre of P, and the Wielandt length of P is not greater than the nilpotent class of P. For a p-group P, the Wielandt length may be less than the nilpotent class; for example, the quaternion group of order eight has nilpotent class 2 and Wielandt length 1. Furthermore, it is shown in [9] that a metabelian p-group of odd order has Wielandt length at most its nilpotent class minus one. An example in [9] of a 5-group has nilpotent class 6 and Wielandt length 4.

Let G be a p-soluble group and let $c(G_p)$ denote the *nilpotent class* of G_p. As mentioned above, the p-length of G is bounded above by $c(G_p)$, and $w^*(G_p) \leq c(G_p)$. This motivates the following question.

Question 1.1. For a p-soluble group, is the p-length bounded above by the Wielandt length of its Sylow p-subgroups?

The main purpose of this paper is to give an affirmative answer to this question. We will prove a more general result. To state our main result, we need to introduce a few more definitions.

A subgroup H of a group G is said to be *permutable* if for any subgroup K of G, we have $HK = KH$. The *normaliser* of a subgroup H in G consists of the elements x such that $xH = Hx$, and the *permutiser* of a subgroup H in G consists of the elements x such that $\langle x\rangle H = H\langle x\rangle$; see [10]. A normal subgroup series is called a *central series* if every member is in the centre of the corresponding quotient group. The *nilpotent class* of a nilpotent group is the shortest length of its central series. We introduce the following definition.

Definition 1.2. Let G be a nilpotent group. A normal series

$$1 = H_0 \leq H_1 \leq \cdots \leq H_n = G$$

is called a *permutable series* of G if, for any $1 \leq i \leq n$ and any element x of H_i, $\langle x\rangle H_{i-1}/H_{i-1}$ is permutable in G/H_{i-1}. In this case, the integer n is called the *length* of this series.

Let G be a nilpotent group. Then a central series of G is a permutable series of G. It follows that a nilpotent group has a permutable series.

Definition 1.3. Let G be a nilpotent group, and let

$$\mathfrak{p}(G) = \min\{n \mid n \text{ is the length of some permutable series of } G\}.$$

Then $\mathfrak{p}(G)$ is called the *permutable length* of G.

The main result of this paper is stated as follows.

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THEOREM 1.4. *Let G be a p-soluble group. Then the p-length of G is no larger than the permutable length of G_p.*

2. Observations and examples

Let p be a prime and P a p-group. From the definition, one can see that the permutable length of P is determined only by the structure of P. Since the upper central series of P is a permutable series of P and $c(P)$ is equal to the length of the upper central series of P, we have $\mathfrak{p}(P) \le c(P)$. Now let us consider the relationship between the permutable length of P and the Wielandt length of P.

PROPOSITION 2.1. *Let p be a prime and P a p-group. We have $\mathfrak{p}(P) \le w^*(P)$.*

PROOF. We only need to show that $1 = \omega_0(P) \le \omega_1(P) \le \cdots \le \omega_n(P) = P$ is a permutable series of P. Let i be an integer such that $1 \le i \le n$ and let x be an element of $\omega_i(P)$. Let $K/\omega_{i-1}(P)$ be a subgroup of $P/\omega_{i-1}(P)$. Since every subgroup of $P/\omega_{i-1}(P)$ is subnormal in $P/\omega_{i-1}(P)$, $\omega_i(P)/\omega_{i-1}(P) = \omega(P/\omega_{i-1}(P))$ is the intersection of the normalisers of all subgroups of $P/\omega_{i-1}(P)$. In particular, $\langle x\rangle\omega_{i-1}(P)/\omega_{i-1}(P) \le N_{P/\omega_i(P)}(K/\omega_{i-1}(P))$. Hence $K\langle x\rangle/\omega_{i-1}(P)$ is a subgroup of $P/\omega_{i-1}(P)$. Because $K/\omega_{i-1}(P)$ is chosen arbitrarily, one can see that $\langle x\rangle\omega_{i-1}(P)/\omega_{i-1}(P)$ is permutable in $P/\omega_{i-1}(P)$. It follows that $1 = \omega_0(P) \le \omega_1(P) \le \cdots \le \omega_n(P) = P$ is a permutable series of P. □

Let p be a prime. The following example indicates that the permutable length of a p-group can be less than its Wielandt length.

EXAMPLE 2.2 [11, p. 65, Example 2.3.19]. Let $p > 2$ and $P = \langle a, x \mid a^{p^3} = 1,\ x^{p^3} = a^{p^2},\ a^x = a^{1+p}\rangle$. Since a^{p^2} is centralised by the automorphism σ of $\langle a\rangle$ with $a^\sigma = a^{1+p}$, P is an extension of $\langle a\rangle$ by a cyclic group of order p^3. By Iwasawa's theorem [11, p. 55, Theorem 2.3.1] and [11, p. 55, Lemma 2.3.2], every subgroup of P is permutable in P. Thus, we have $\mathfrak{p}(P) = 1$. On the other hand, since P is a nonabelian p-group of odd order, it follows from [11, p. 60, Theorem 2.3.12] that not every subgroup of P is normal in P. Hence $\omega(P) < P$ and $w^*(P) > 1$. Therefore, $\mathfrak{p}(P) < w^*(P)$.

As observed above, $\mathfrak{p}(P) \le w^*(G_p) \le c(G_p)$, and we also see that there are examples with the strict relations $w^*(P) < c(P)$ and $\mathfrak{p}(P) < w^*(P)$. Let $l_p(G)$ denote the p-length of a p-soluble group G. Hall–Higman's p-length theorem states that the p-length of G is bounded by the nilpotent class of G_p. The main result of this paper is to improve Hall–Higman's p-length theorem by a better bound. Actually, we will prove that $l_p(G) \le \mathfrak{p}(G_p)$.

3. The proof of the main theorem

We first present some basic facts about the p-length and the permutable length.

LEMMA 3.1 [8, p. 689, Hilfssatz 6.4]. *Let G be a p-soluble group.*

(1) *If $N \trianglelefteq G$, then $l_p(G/N) \le l_p(G)$.*
(2) *If $U \le G$, then $l_p(U) \le l_p(G)$.*

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(3) *Let N_1 and N_2 be two normal subgroups of G. Then*

$$l_p(G/(N_1 \cap N_2)) = \max\{l_p(G/N_1), l_p(G/N_2)\}.$$

(4) $l_p(G/\Phi(G)) = l_p(G)$.

Lemma 3.2 [5, 1.3 and 1.4]. *Let N be a normal subgroup of G and K a subgroup of G containing N. Then K/N is permutable in G/N if and only if K is permutable in G.*

Lemma 3.3. *Let G be a nilpotent group.*

(1) *If $N \trianglelefteq G$, then $\mathfrak{p}(G/N) \le \mathfrak{p}(G)$.*
(2) *If $U \le G$, then $\mathfrak{p}(U) \le \mathfrak{p}(G)$.*

Proof. Let $1 = H_0 \le H_1 \le \cdots \le H_n = G$ be a permutable series of G with length $n = \mathfrak{p}(G)$. To prove (1), we only need to show that $1 = H_0N/N \le H_1N/N \le \cdots \le H_nN/N = G/N$ is a permutable series of G/N. Since all H_i are normal in G, all H_iN/N are also normal in G/N. Let i be an integer such that $1 \le i \le n$. Let x be an element of H_iN/N. It is easy to see that there exists an element y of H_i such that $\langle x\rangle = \langle y\rangle N/N$. Let K/N be a subgroup of G/N. By definition, $\langle y\rangle H_{i-1}/H_{i-1}$ is permutable in G/H_{i-1}. It follows that $\langle y\rangle H_{i-1}$ is permutable in G by Lemma 3.2. Hence $\langle y\rangle H_{i-1}K$ is a subgroup of G. Therefore, $(\langle y\rangle N/N)(H_{i-1}N/N)(K/N) = (\langle y\rangle H_{i-1}KN)/N$ is a subgroup of G/N. The arbitrary choice of K/N implies that $(\langle y\rangle N/N)(H_{i-1}N/N)$ is permutable in G/N. Again by Lemma 3.2, we know that $[\langle x\rangle(H_{i-1}N/N)]/(H_{i-1}N/N) = [(\langle y\rangle N/N)(H_{i-1}N/N)]/(H_{i-1}N/N)$ is permutable in $(G/N)/(H_{i-1}N/N)$. By definition, $1 = H_0N/N \le H_1N/N \le \cdots \le H_nN/N = G/N$ is a permutable series of G/N.

To prove (2), we need to show that $1 = (H_0 \cap U) \le (H_1 \cap U) \le \cdots \le (H_n \cap U) = U$ is a permutable series of U. It is evident that $H_i \cap U$ is normal in U for any i. Let i be an integer such that $1 \le i \le n$. Let x be an element of $H_i \cap U$ and K be a subgroup of U. By definition, $\langle x\rangle H_{i-1}/H_{i-1}$ is permutable in G/H_{i-1}. It follows that $\langle x\rangle H_{i-1}$ is permutable in G, by Lemma 3.2. Hence $\langle x\rangle H_{i-1}K$ is a subgroup of G. Since $K \le U$ and $\langle x\rangle \le U$, we have $\langle x\rangle(H_{i-1} \cap U)K = \langle x\rangle(H_{i-1}K \cap U) = \langle x\rangle H_{i-1}K \cap U$ and thus $\langle x\rangle(H_{i-1} \cap U)K$ is a subgroup of U. The arbitrary choice of K implies that $\langle x\rangle(H_{i-1} \cap U)$ is permutable in U. Again by Lemma 3.2, $\langle x\rangle(H_{i-1} \cap U)/(H_{i-1} \cap U)$ is permutable in $U/(H_{i-1} \cap U)$. Therefore, $1 = (H_0 \cap U) \le (H_1 \cap U) \le \cdots \le (H_n \cap U) = U$ is a permutable series of U. □

Proof of Theorem 1.4. Assume that this theorem is not true and let G be a counterexample of minimal order. Then we have the following steps to the proof.

(1) $O_{p'}(G) = \Phi(G) = 1$.

Assume that $O_{p'}(G) \ne 1$ or $\Phi(G) \ne 1$. Then, by the minimal choice of G, $l_p(G/O_{p'}(G)) \le \mathfrak{p}(G_pO_{p'}(G)/O_{p'}(G))$ or $l_p(G/\Phi(G)) \le \mathfrak{p}(G_p\Phi(G)/\Phi(G))$. By the definition of p-length, $l_p(G/O_{p'}(G)) = l_p(G)$. By Lemma 3.1(4), $l_p(G/\Phi(G)) = l_p(G)$. On the other hand, $\mathfrak{p}(G_pO_{p'}(G)/O_{p'}(G)) = \mathfrak{p}(G_p/(G_p \cap O_{p'}(G))) \le \mathfrak{p}(G_p)$ and $\mathfrak{p}(G_p\Phi(G)/\Phi(G)) = \mathfrak{p}(G_p/(G_p \cap \Phi(G))) \le \mathfrak{p}(G_p)$ from Lemma 3.3(1). Hence $l_p(G) \le \mathfrak{p}(G_p)$, a contradiction.

(2) G has a unique minimal normal subgroup N.

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Suppose that G has two different minimal normal subgroups N_1 and N_2. From the minimal choice of G, $l_p(G/N_1) \leq \mathfrak{p}(G_pN_1/N_1)$ and $l_p(G/N_2) \leq \mathfrak{p}(G_pN_2/N_2)$. Without loss of generality, we may assume that $\mathfrak{p}(G_pN_1/N_1) \geq \mathfrak{p}(G_pN_2/N_2)$. Obviously, $N_1 \cap N_2 = 1$. From Lemmas 3.1(3) and 3.3(1),

$$\begin{aligned} l_p(G) &= l_p(G/(N_1 \cap N_2)) \leq \max\{l_p(G/N_1), l_p(G/N_2)\} = l_p(G/N_1) \\ &\leq \mathfrak{p}(G_pN_1/N_1) = \mathfrak{p}(G_p/(G_p \cap N_1)) \leq \mathfrak{p}(G_p), \end{aligned}$$

a contradiction.

(3) $N = C_G(N) = O_p(G)$.

Since $\Phi(G) = 1$, $O_p(G)$ is the direct product of some minimal normal subgroups of G. But N is the unique minimal normal subgroup of G, so $N = O_p(G)$. Because G is p-soluble and $O_{p'}(G) = 1$, we have $C_G(O_p(G)) \leq O_p(G)$ by [6, p. 228, Theorem 3.2]. Since $O_p(G) = N$ is abelian, $N = C_G(N)$.

(4) There exists a maximal subgroup M of G such that $G = [N]M$.

This follows directly from the fact that $\Phi(G) = 1$ and N is an abelian minimal normal subgroup of G.

(5) Suppose that $1 = H_0 \leq H_1 \leq \cdots \leq H_n = G_p$ is a permutable series of G_p with length $n = \mathfrak{p}(G_p)$. Then $H_1 \cap M = 1$.

Assume that $H_1 \cap M \neq 1$. Let x be an element of $H_1 \cap M$ of order p, and let y be an element of N. Clearly y is also of order p. By definition, $\langle x\rangle$ is permutable in G_p. Since $\langle y\rangle \leq O_p(G) \leq G_p$, $\langle x\rangle\langle y\rangle$ is a subgroup of G_p. Note that $|x| = |y| = p$ and $\langle x\rangle \cap N \leq M \cap N = 1$, and $\langle x\rangle\langle y\rangle$ is a group of order p^2. Therefore, $\langle x\rangle\langle y\rangle$ is an abelian group and $x \in C_G(\langle y\rangle)$. Since y is chosen arbitrarily, we have $x \in C_G(N)$. But then $x \in C_G(N) \cap M = N \cap M = 1$, a contradiction.

(6) Final contradiction.

Let M_p be a Sylow p-subgroup of M such that $M_p \leq G_p$. By (4), $G_p = NM_p$ and $N \cap M_p = 1$. By (5), $M_p = M_p/(M_p \cap H_1) \cong M_pH_1/H_1$. Hence $G_p/N = NM_p/N \cong M_p/(N \cap M_p) \cong M_p \cong M_pH_1/H_1$ and $\mathfrak{p}(G_p/N) = \mathfrak{p}(M_pH_1/H_1)$. But M_pH_1/H_1 is a subgroup of G_p/H_1 and thus $\mathfrak{p}(M_pH_1/H_1) \leq \mathfrak{p}(G_p/H_1)$ by Lemma 3.3(2). As a result, $\mathfrak{p}(G_p/N) = \mathfrak{p}(M_pH_1/H_1) \leq \mathfrak{p}(G_p/H_1)$.

From the proof of Lemma 3.3(1), we know that the normal series $1 = H_1/H_1 \leq H_2/H_1 \leq \cdots \leq H_n/H_1 = G_p/H_1$, whose length is $n - 1 = \mathfrak{p}(G_p) - 1$, is a permutable series of G_p/H_1. As a result, $\mathfrak{p}(G_p/H_1) \leq \mathfrak{p}(G_p) - 1$. From (1), (3) and the definition of the p-length, we know that $l_p(G/N) = l_p(G) - 1$. From the minimal choice of G, $l_p(G/N) \leq \mathfrak{p}(G_p/N)$. Hence $l_p(G) - 1 = l_p(G/N) \leq \mathfrak{p}(G_p/N) \leq \mathfrak{p}(G_p/H_1) \leq \mathfrak{p}(G_p) - 1$ and it follows that $l_p(G) \leq \mathfrak{p}(G_p)$, a final contradiction. □

4. Some applications

The following corollary is an immediate consequence of Proposition 2.1 and Theorem 1.4. It gives an affirmative answer to Question 1.1.

COROLLARY 4.1. *Let G be a p-soluble group. Then the p-length of G is no larger than the Wielandt length of G_p.*

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Let G be a p-soluble group. As a special case of Theorem 1.4, we know that if the permutable length of G_p is at most 1, then the p-length of G is also at most 1. By definition, the permutable length of G_p is at most 1 if and only if every subgroup of G_p is permutable in G_p. By [11, p. 55, Lemma 2.3.2], G_p satisfies such properties if and only if G_p is a modular p-group. As a result, we have the following corollary.

Corollary 4.2. *Let G be a p-soluble group. If the Sylow p-subgroups of G are modular p-subgroups, then $l_p(G) \le 1$.*

A Hamiltonian group is a group all of whose subgroups are normal. From this definition, one can see that a Hamiltonian p-group must be a modular p-group. (The converse is not true, see Example 2.2.) By Corollary 4.2, we have the following further corollary.

Corollary 4.3 [2]. *Let G be a p-soluble group. If the Sylow p-subgroups of G are Hamiltonian p-subgroups, then $l_p(G) \le 1$.*

The well-known Burnside's theorem tells us that if $N_G(G_p) = C_G(G_p)$, then G is p-nilpotent. In other words, if G_p is an abelian p-group and $N_G(G_p)$ is p-nilpotent, then G is p-nilpotent. In [1, Theorem 1], this result was extended to show that if G_p is a modular p-group, then G is p-nilpotent if and only if $N_G(G_p)$ is p-nilpotent. An interesting question is whether we can get an analogous result for the case of p-supersoluble. That is, suppose that G_p is a modular p-group, can we obtain that G is p-supersoluble provided $N_G(G_p)$ is p-supersoluble? The answer to this question is no. For instance, the alternating group A_5 has modular Sylow 5-subgroups and the normalisers of its Sylow 5-subgroups are also 5-supersoluble, but A_5 itself is not 5-supersoluble. However, the following theorem indicates that in the class of all p-soluble groups, the modularity of the Sylow p-subgroups and the p-supersolvability of $N_G(G_p)$ do yield the p-supersolvability of G.

Theorem 4.4. *Let G be a p-soluble group with modular Sylow p-subgroups. Let $\mathcal{F}$ be a formation satisfying $\mathcal{E}_{p'}\mathcal{F} = \mathcal{F}$ (where $\mathcal{E}_{p'}$ denotes the class of all groups with order coprime to p). If $N_G(G_p) \in \mathcal{F}$, then $G \in \mathcal{F}$. In particular, under the circumstances that G is a p-soluble group and G_p is a modular p-group, G is p-supersoluble if and only if $N_G(G_p)$ is p-supersoluble.*

Proof. By Corollary 4.2, we know that $l_p(G) \le 1$. Hence $G_pO_{p'}(G)/O_{p'}(G)$ is normal in $G/O_{p'}(G)$. It follows that $G_pO_{p'}(G)$ is normal in G and $N_G(G_pO_{p'}(G)) = G$. Since any two Sylow p-subgroups are conjugated, we have $N_G(G_pO_{p'}(G)) = N_G(G_p)O_{p'}(G)$. Consequently, $G/O_{p'}(G) = N_G(G_pO_{p'}(G))/O_{p'}(G) = N_G(G_p)O_{p'}(G)/O_{p'}(G) \cong N_G(G_p)/(N_G(G_p) \cap O_{p'}(G)) \in \mathcal{F}$. This implies that $G \in \mathcal{E}_{p'}\mathcal{F}$ and the hypothesis that $\mathcal{E}_{p'}\mathcal{F} = \mathcal{F}$ guarantees that $G \in \mathcal{F}$. □

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