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RECURSIVE EQUIVALENCE TYPES AND OCTAHEDRA

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Abstract

Let the word "graph" be used in the sense of a countable, connected, simple graph with at least one vertex. We write Q_n and Oc_n for the graphs associated with the n-cube Q^n and the n-octahedron Oc^n respectively. In a previous paper (Dekker, 1981) we generalized Q_n and Q^n to a graph Q_N and a cube Q^N, for any nonzero recursive equivalence type N. In the present paper we do the same for Oc_n and Oc^n. We also examine the nature of the duality between Q^N and Oc^N, in case N is an infinite isol. There are c RETs, c denoting the cardinality of the continuum.

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1. Preliminaries

This paper is closely related to 'RETs and cubes', Dekker (1981), to which the reader is referred for notations and terminology not explained below. Propositions of that paper are referred to as P1.1, P1.2, ..., P2.1, P2.2, ..., and so on. We write ε for the set $(0, 1, \ldots)$, o for the empty set, Λ for the collection of all isols and Ω for the collection of all RETs. Also, $\varepsilon_0 = \varepsilon - (0)$, $\Lambda_0 = \Lambda - (0)$ and $\Omega_0 = \Omega - (0)$. We again use the canonical enumeration $\langle \rho_n \rangle$ of the class of all finite sets, that is, finite subsets of ε. For a finite set σ the unique number i such that $\sigma = \rho_i$ is the *canonical index* of σ, written $\mathrm{can}(\sigma)$. Put $r_n = \mathrm{card}\, \rho_n$; for $\nu \subset \varepsilon$, $i \in \varepsilon$ we define $2^\nu = \{x \in \varepsilon \mid \rho_x \subset \nu\}$ and $[\nu; i] = \{x \in \varepsilon \mid \rho_x \subset \nu \,\&\, r_x = i\}$. These recursive, combinatorial operators enable us to extend the functions 2^n and $[n; i]$ from ε into ε to functions 2^N and $[N; i]$ from Ω into Ω. The ordinary, vertical notations for the binomial functions $[n; i]$ and $[N; i]$ are only used in displayed

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formulas. If f is a function from a subset of ε into ε, we denote its domain by δf, its range by ρf and the image of n under f by $f(n)$ or f_n, sometimes in the same context. Throughout this paper the symbols ν, ν_0, ν_1, ν_2 denote nonempty sets, while μ, μ_0, μ_1, μ_2 stand for sets of cardinality $\geqslant 2$.

We define the (*undirected*) *ω-cube* Q^ν on the set ν as the ordered pair $\langle 2^\nu, F_\nu \rangle$, where F_ν is the class of all *faces* of Q^ν, that is, the class of all finite subsets σ of 2^ν for which there exist disjoint finite subsets ρ_p and ρ_q of ν such that $\sigma = \{x \in 2^\nu \mid \rho_p \subset \rho_x \subset \rho_p \cup \rho_q\}$. The number $j(p, q)$ is the *G-number* $G(\sigma)$ of σ and the number $k = r_q$ the *dimension* of σ; we refer to σ as a *k-face* of Q^ν and write $F_{\nu k}$ for the class of all k-faces of Q^ν. All faces of Q^ν are therefore finite-dimensional, even if the RET $N = \text{Req}\,\nu$, the so-called *ω-dimension* of Q^ν, is infinite, that is, belongs to $\Omega - \varepsilon$. We use the term "*N-cube*" for an ω-cube of ω-dimension N. For $k \in \varepsilon$, $N \subset \Omega_0$, $\nu \in N$ we define

$$(1.1) \qquad \alpha_{\nu k} = \{G(\sigma) \in \varepsilon \mid \sigma \in F_{\nu k}\}, \qquad \alpha_\nu = \{G(\sigma) \in \varepsilon \mid \sigma \in F_\nu\},$$

$$(1.2) \qquad A_{Nk} = \text{Req}\,\alpha_{\nu k}, \qquad A_n = \text{Req}\,\alpha_\nu.$$

The functions A_{Nk} and A_N are well-defined, that is, independent of the representative ν of N. According to P4.1 and P4.2 we have for $N \in \Omega_0$ and $0 \leqslant k \leqslant N$,

$$(1.3) \qquad A_{Nk} = \binom{N}{k} 2^{N-k}, \qquad A_n = 3^N.$$

2. Octahedral graphs

The six vertices and twelve edges of a regular octahedron in E^3 form a graph; its vertices can be denoted by $1, \ldots, 6$ so that the pairs of opposite vertices are $(1,2)$, $(3,4)$, $(5,6)$; it is therefore isomorphic to the complete tripartite graph $O_3 = K(2,2,2)$ with two vertices in each partite set. For $n \geqslant 1$ the *n-octahedral graph* O_n is defined as the n-partite graph $K(2, \ldots, 2)$ with two vertices in each of its n partite sets, Jungerman and Ringel (1978). Let O_n have $\mu = (1, \ldots, 2n)$ as set of vertices and $((1,2), \ldots, (2n-1, 2n))$ as class of its partite sets. Define f as the permutation of μ which interchanges $2k-1$ and $2k$, for $1 \leqslant k \leqslant n$. We now drop the condition that $\mu = (1, \ldots, 2n)$ and define an *involution without fixed points* (iwfp) of μ as a permutation f of μ such that $f^2 = i_\mu$ and $f(x) \neq x$, for $x \in \mu$. Denote the family of all iwfps of μ by $\text{Inv}(\mu)$, associate with every $f \in \text{Inv}(\mu)$ the graph $G_f = \langle \mu, \theta \rangle$, where $\theta = \{\text{can}(x, y) \in [\mu; 2] \mid f(x) \neq y\}$, and call a (countable) graph $G = \langle \mu, \theta \rangle$ *octahedral*, if $G = G_f$, for some $f \in \text{Inv}(\mu)$. Note that $f \to G_f$ maps $\text{Inv}(\mu)$ one-to-one onto the family of all octahedral graphs with μ as set of vertices. The vertices p and q of G_f are *opposite*, if $f(p) = q$ or equivalently, $f(q) = p$; thus p and q are adjacent iff they are not opposite. The iwfp f of μ is an *ω-iwfp* of μ, if it has a partial recursive one-to-one extension; here "one-to-one" can be deleted without changing the concept defined, since $f = f^{-1}$. Denote the family of all ω-iwfps of μ by $\mathrm{Inv}_\omega(\mu)$. If $G = \langle \mu, \theta \rangle$ we call the function f such that $G = G_f$ the *octahedral* function of G; the graph G is *ω-octahedral*, if its octahedral function belongs to $\mathrm{Inv}_\omega(\mu)$. We claim that in case $f \in \mathrm{Inv}(\mu)$, we have: $f \in \mathrm{Inv}_\omega(\mu)$ if and only if f is the restriction to μ of some partial recursive iwfp of some r.e. superset of μ. For if $\bar{f}$ is a partial recursive extension of f we put $\bar{\mu} = \{x \in \delta\bar{f} \mid \bar{f}(x) \neq x \,\&\, \bar{f}(x) \in \delta\bar{f} \,\&\, \bar{f}^2(x) = x\}$; then $\bar{\mu}$ is r.e., $\mu \subset \bar{\mu}$ and $\bar{f} \mid \bar{\mu}$ is a partial recursive extension of f. Note that $\mathrm{Inv}_\omega \mu = \mathrm{Inv}\, \mu$, if μ is finite. Thus a finite graph is ω-octahedral if and only if it is octahedral.

PROPOSITION B2.1. *Let $M = \mathrm{Req}\, \mu$. Then there is an ω-octahedral graph with μ as set of vertices if and only if M is even.*

PROOF. Let $G = \langle \mu, \theta \rangle$ and $M = \mathrm{Req}\, \mu$. Suppose G is ω-octahedral, say $G = G_f$, for $f \in \mathrm{Inv}_\omega(\mu)$. Then $f = \bar{f} \mid \mu$, for some partial recursive iwfp $\bar{f}$ of some r.e. superset of μ, say $\bar{\alpha}$. Define

$$(2.1)\quad \begin{cases} \bar{\alpha}_0 = \{x \in \bar{\alpha} \mid x < \bar{f}(x)\}, & \bar{\alpha}_1 = \{x \in \bar{\alpha} \mid x > \bar{f}(x)\}, \\ \alpha_0 = \bar{\alpha}_0 \cap \mu, \quad \alpha_1 = \bar{\alpha}_1 \cap \mu, & \delta\bar{g} = \bar{\alpha}_0, \quad \bar{g} = \bar{f} \mid \bar{\alpha}_0, \quad g = f \mid \alpha_0, \end{cases}$$

then $\bar{g}$ is a partial recursive one-to-one function from $\bar{\alpha}_0$ onto $\bar{\alpha}_1$ which maps α_0 onto α_1, hence $\alpha_0 \simeq \alpha_1$. Clearly, $\mu = \alpha_0 \cup \alpha_1$, where $\alpha_0 \mid \alpha_1$, hence $M = 2\,\mathrm{Req}\, \alpha_0$ is even. Now assume that $M = \mathrm{Req}\, \mu$ is even, say $\mu = \alpha_0 \cup \alpha_1$, $\alpha_0 \simeq \alpha_1$ and $\alpha_0 \mid \alpha_1$. Let g be a one-to-one function from α_0 onto α_1 with a partial recursive one-to-one extension. Put $\delta f = \mu$, $f(x) = g(x)$, for $x \in \alpha_0$ and $f(x) = g^{-1}(x)$, for $x \in \alpha_1$. Then f is an ω-iwfp of μ and G_f is an ω-octahedral graph with μ as set of vertices.

REMARK. Since $\Lambda - \varepsilon$ has cardinality c and the mappings $X \to 2X$ and $X \to 2X + 1$ are one-to-one, there are exactly c infinite even isols and exactly c infinite odd isols. Thus we see by B2.1 that there are exactly c ω-octahedral graphs. With every infinite odd isol M we can associate a set $\mu \in M$ and (since $\aleph_0 = 2\aleph_0$) an iwfp f of μ and the octahedral graph G_f. In view of B2.1 the graph G_f is not ω-octahedral. It follows that there also are exactly c octahedral graphs which are not ω-octahedral.

The graph $G = \langle \nu, \eta \rangle$ is *r.e.*, if the sets ν and η are r.e. We call a connected graph G an *ω-graph*, if it has an MPA (*minimal path algorithm*), that is, if there is an effective procedure Π which associates with every two distinct vertices of G a path of minimal length between them. An ω-graph $G = \langle \nu, \eta \rangle$ is *uniform*, if G is an induced subgraph of some r.e. ω-graph $\bar{G}$ which has an MPA $\bar{\Pi}$ which when applied to distinct vertices p and q of G yields a minimal path between p and q in G, that is, a minimal path with vertices in ν and edges in η. It was proved by Remmel (1981) that an ω-graph $G = \langle \nu, \eta \rangle$ need not be uniform, even if ν is immune.

PROPOSITION B2.2. *Let $G = \langle \mu, \theta \rangle$ be an ω-octahedral graph. Then G is a uniform ω-graph and there is a nonzero* RET N *such that* $\operatorname{Req} \mu = 2N$ *and* $\operatorname{Req} \theta = 2N(N-1)$.

PROOF. Let $G = \langle \mu, \theta \rangle$, say $G = G_f$, for $f \in \operatorname{Inv}_\omega(\mu)$. Define α_0 and α_1 as in (2.1), then $M = \operatorname{Req} \mu = 2N$, where $N = \operatorname{Req} \alpha_0$. Put

$$\lambda = \{\operatorname{can}(x, y) \in [\mu; 2] | x \in \alpha_0 \,\&\, y \in \alpha_1 \,\&\, f(x) \neq y\},$$

then $\operatorname{Req} \lambda = N(N-1)$ and $\theta = [\mu_0; 2] \cup [\mu_1; 2] \cup \lambda$, where the three sets on the right are separable. Hence $\operatorname{Req} \theta$ equals $2[N; 2] + \operatorname{Req} \lambda$, that is, $2N(N-1)$. Note that this proof is valid both in case $N \in \Lambda_0$, that is, if $N - 1 < N$ and in case $N \in \Omega_0 - \Lambda$, that is, if $N - 1 = N$. In the latter case $\operatorname{Req} \theta = 2N^2$. We now prove that $G_f = \langle \mu, \theta \rangle$ is a uniform ω-graph. Let $f = \bar{f} | \mu$, where $\bar{f}$ is a partial recursive iwfp of a r.e. superset of μ, say $\bar{\alpha}$. Define $\bar{\alpha}_0$, $\bar{\alpha}_1$, α_0, α_1 as in (2.1) and $\bar{\theta} = \{\operatorname{can}(x, y) \in [\bar{\alpha}; 2] \,|\, \bar{f}(x) \neq y\}$, then G_f is an induced subgraph of the r.e. graph $\bar{G} = \langle \bar{\alpha}, \bar{\theta} \rangle$. We may assume without loss of generality that card $\alpha_0 \geqslant 2$. Let $p, u \in \alpha_0$, $p \neq u$, $q = f(p)$, then p, q, u are distinct vertices of G and p, q are opposite. Put for $x, y \in \bar{\alpha}$, $x \neq y$,

$$\bar{\pi}(x, y) = \begin{cases} \langle x, u, y \rangle, & \text{if } x = p \text{ and } y = q, \text{ or } x = q \text{ and } y = p, \\ \langle x, p, y \rangle, & \text{if } x \notin (p, q), \text{ but } \bar{f}(x) = y, \\ \langle x, y \rangle, & \text{if } \bar{f}(x) \neq y. \end{cases}$$

Then $\bar{\pi}_{xy}$ is a minimal path between x and y in $\bar{G}$ which is a minimal path in G in case $x, y \in \mu$. Since $\bar{\pi}_{xy}$ can be effectively obtained from x and y, the ω-graph G is uniform.

Define the functions d_0, d_1 by $\delta d_0 = \delta d_1 = \varepsilon$, $d_0(x) = 2x$, $d_1(x) = 2x + 1$ and associate with every set ν the sets $\nu_0 = d_0(\nu)$ and $\nu_1 = d_1(\nu)$. The *standard* ω-iwfp associated with ν is the function $f \in \operatorname{Inv}_\omega(\mu)$, where $\mu = \nu_0 \cup \nu_1$ and $f(2x) = 2x + 1$, $f(2x + 1) = 2x$, for $x \in \nu$. The *standard* ω-octahedral graph Oc_ν associated with ν is the graph $G_f = \langle \mu, \theta \rangle$, where f is the standard ω-iwfp associated with ν. Thus the vertices p and q of Oc_ν are opposite, if p is even and $p + 1 = q$, or q is even and $q + 1 = p$. An *ω-isomorphism* from the graph G_1 onto the graph G_2 is an isomorphism from G_1 onto G_2 which has a partial recursive one-to-one extension. G_1 is *isomorphic* (*ω-isomorphic*) to G_2, if there is at least one isomorphism (ω-isomorphism) from G_1 onto G_2. These two equivalence relations are denoted by $\cong$ and $\cong_\omega$. Two finite graphs are ω-isomorphic if and only if they are isomorphic. Under an ω-isomorphism from G_1 onto G_2 minimal paths in G_1 correspond to minimal paths in G_2. This implies

(2.2) *if* $G_1 \cong_\omega G_2$ *and* G_1 *is an* ω*-graph, so is* G_2.

Proposition B2.3. *A graph is* ω*-octahedral if and only if it is* ω*-isomorphic to some standard* ω*-octahedral graph.*

Proof. Let $G = \langle \mu, \theta \rangle$. (a) Suppose g is an ω-isomorphism from $\mathrm{Oc}_\nu = \langle \nu_0 \cup \nu_1, \eta \rangle$ onto G; put $\delta t = \nu_0 \cup \nu_1$, $t(x) = x + 1$, for $x \in \nu_0$, $t(x) = x - 1$, for $x \in \nu_1$; then $\mathrm{Oc}_\nu = G_t$. Define $\delta f = \mu$, and $f = gtg^{-1}$, then $f \in \mathrm{Inv}_\omega(\mu)$. Moreover, $\mathrm{can}(x, y) \in \theta$ if and only if $\mathrm{can}(g^{-1}(x), g^{-1}(y)) \in \eta$ if and only if $tg^{-1}(x) = g^{-1}(y)$ if and only if $f(x) \neq y$, so that $\theta = \{\mathrm{can}(x, y) \in [\mu; 2] \mid f(x) \neq y\}$ and $G = G_f$. The function t has a recursive one-to-one extension, namely $\bar{t}$, where $\delta \bar{t} = \varepsilon$, $\bar{t}(x) = x + 1$, for $x \in \delta_0$, while $\bar{t}(x) = x - 1$, for $x \in \delta_1$. Let $\bar{g}$ be a partial recursive one-to-one extension of g; put $\delta \bar{f} = \{x \in \rho \bar{g} \mid \bar{t}(x) \in \delta \bar{g}\}$ and $\bar{f}(x) = \bar{g}\bar{t}\bar{g}^{-1}(x)$, then $\bar{f}$ is a partial recursive one-to-one extension of f, hence $G = G_f$ is ω-octahedral.

(b) Assume that $G = \langle \mu, \theta \rangle$ is ω-octahedral, say $G = G_f$, where $f \in \mathrm{Inv}_\omega(\mu)$. If μ is finite, put $m = \frac{1}{2}\,\mathrm{card}\,\mu$, $\nu = (0, \ldots, m-1)$; then $\mathrm{Oc}_\nu \cong G$, hence $\mathrm{Oc}_\nu \cong_\omega G$. Now assume that μ is infinite. Let $\bar{f}$ be a partial recursive iwfp of some r.e. superset of μ, say $\bar{\mu}$, then there is a one-to-one recursive function $\bar{u}_n$ ranging over $\bar{\mu}$ such that $\bar{f}(\bar{u}_{2n}) = \bar{u}_{2n+1}$ and $\bar{u}_{2n} < \bar{u}_{2n+1}$. Then $\bar{u}_{2n} \in \mu$ if and only if $\bar{u}_{2n+1} \in \mu$, for $n \in \varepsilon$. Let $\nu = \{n \in \varepsilon \mid \bar{u}_{2n} \in \mu\}$, then $\bar{u}(\nu_0 \cup \nu_1) = \mu$. Define $\delta t = \nu_0 \cup \nu_1$, $t(x) = x + 1$, for $x \in \nu_0$, $t(x) = x - 1$, for $x \in \nu_1$, then $\mathrm{Oc}_\nu = G_t$. For $\mathrm{Oc}_\nu = \langle \nu_0 \cup \nu_1, \eta \rangle$ and $x, y \in \nu_0 \cup \nu_1$,

$$\begin{aligned}\mathrm{can}(x, y) \in \eta &\Leftrightarrow t(x) \neq y \Leftrightarrow \bar{u}_{t(x)} \neq \bar{u}_{t(y)} \\ &\Leftrightarrow f(\bar{u}_x) \neq \bar{u}_y \Leftrightarrow \mathrm{can}(\bar{u}_x, \bar{u}_y) \in \theta.\end{aligned}$$

Thus $\bar{u} \mid \nu_0 \cup \nu_1$ is an isomorphism from Oc_ν onto G with the recursive one-to-one extension $\bar{u}$, hence $\mathrm{Oc}_\nu \cong_\omega G$.

If $G = \langle \mu, \theta \rangle$ is an ω-graph, $o(G) = \mathrm{Req}\,\mu$ is the *order* of G. Thus $o(G)$ has the usual meaning if and only if G is finite. If the graph $G = \langle \mu, \theta \rangle$ is ω-octahedral, its order M is even by B2.1, that is, $M = 2A$, for some $A \in \Omega$. We call A the ω*-dimension* of G, written $\dim_\omega G$. Since $2A = 2B$ implies $A = B$ by Friedberg (1961), $\dim_\omega G$ is well-defined for an ω-octahedral graph G. Hence two ω-octahedral graphs have the same order if and only if they have the same ω-dimension.

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PROPOSITION B2.4. *Two ω-octahedral graphs are ω-isomorphic if and only if they have the same ω-dimension.*

PROOF. Since $\dim_\omega \mathrm{Oc}_\nu = \mathrm{Req}\,\nu$, it suffices to show in view of B2.3 that $\alpha \simeq \beta \Leftrightarrow \mathrm{Oc}_\alpha \cong_\omega \mathrm{Oc}_\beta$, for nonempty sets α and β. The conditional from the right to the left is trivial, since ω-isomorphic graphs have the same order. Now assume $\alpha \simeq \beta$, say $\alpha \subset \delta p$, $p(\alpha) = \beta$, p partial recursive and one-to-one, $\mathrm{Oc}_\alpha = \langle \alpha_0 \cup \alpha_1, \theta_\alpha \rangle$, $\mathrm{Oc}_\beta = \langle \beta_0 \cup \beta_1, \theta_\beta \rangle$. Put $\delta q = d_0(\delta p) \cup d_1(\delta p)$, $q(x) = 2p(x/2)$, for $x \in \delta q \cap \delta_0$, while $q(x) = 2p(x - 1/2)$, for $x \in \delta q \cap \delta_1$. Then $\alpha_0 \cup \alpha_1 \subset \delta q$, $q(\alpha_0 \cup \alpha_1) = \beta_0 \cup \beta_1$, where q is partial recursive and one-to-one. Moreover, $\mathrm{can}(x, y) \in \theta_\alpha$ implies $\mathrm{can}(q_x, q_y) \in \theta_\beta$, for $x, y \in \alpha_0 \cup \alpha_1$, so that $q \mid \alpha_0 \cup \alpha_1$ is an ω-isomorphism from Oc_α onto Oc_β.

For $N \in \Omega_0$ we define Oc_N as any ω-octahedral graph of ω-dimension N, or equivalently, of order $2N$. Thus Oc_N is unique up to ω-isomorphism. For the definitions of an ω-regular graph and its ω-degree, see page 546 of Dekker (1981a). It can be shown that *every ω-octahedral graph of order $2N$ is ω-regular of ω-degree* $2(N - 1)$; the proof is routine.

3. Octahedra

With every $f \in \mathrm{Inv}(\mu)$ we associate the ordered pair $\mathrm{Oc}_f = \langle \mu, C_\mu \rangle$, where $\mu = \delta f$ and C_μ is the class of all finite subsets σ of μ such that no two elements of σ correspond to each other under f. An *octahedron* is an ordered pair $\mathrm{Oc} = \langle \mu, C_\mu \rangle$ such that $\mathrm{Oc} = \mathrm{Oc}_f$, for some $f \in \mathrm{Inv}(\mu)$; Oc_f is an *ω-octahedron*, if $f \in \mathrm{Inv}_\omega(\mu)$. The mapping $f \to \mathrm{Oc}_f$ maps $\mathrm{Inv}(\mu)$ one-to-one onto the family of all octahedra on μ and $\mathrm{Inv}_\omega(\mu)$ one-to-one onto the family of all ω-octahedra on μ. If $\mathrm{Oc}_f = \langle \mu, C_\mu \rangle$, then $G_f = \langle \mu, \theta \rangle$ where $\theta = \{\mathrm{can}(x, y) \in [\mu; 2] \mid (x, y) \in C_\mu\}$, hence Oc_f and G_f uniquely determine each other; we say that Oc_f and G_f are *associated*. The members of μ are the *vertices* of $\mathrm{Oc}_f = \langle \mu, C_\mu \rangle$, while the members of C_μ are the *faces* of Oc_f. If σ is a face we define $\dim \sigma$ as the number $k = \mathrm{card}\,\sigma - 1$ and refer to σ as a *k-face* of Oc_f; we write $C_{\mu k}$ for the class of all k-faces of Oc_f. Note that (i) every face of $\langle \mu, C_\mu \rangle$ is finite-dimensional, even if μ is infinite, (ii) every subset of a face is again a face, (iii) there is only one (-1)-face, namely the empty set, (iv) $\mu \notin C_\mu$, since our agreement that $\mathrm{card}\,\mu \geqslant 2$ implies that μ contains two opposite vertices. For the ω-octahedron $\mathrm{Oc} = \langle \mu, C_\mu \rangle$ with $M = \mathrm{Req}\,\mu$, we define the *order* $o(\mathrm{Oc})$ as M and the *ω-dimension* $\dim_\omega(\mathrm{Oc})$ as $M/2$. Hence $o(\mathrm{Oc}) = o(G)$ and $\dim_\omega(\mathrm{Oc}) = \dim_\omega(G)$, where G is associated with Oc. An *isomorphism* (*ω-isomorphism*) from $\mathrm{Oc}_1 = \langle \mu_1, C_{\mu(1)} \rangle$ onto $\mathrm{Oc}_2 = \langle \mu_2, C_{\mu(2)} \rangle$ is a one-to-one function (with a partial recursive one-to-one extension) from μ_1 onto μ_2 which preserves faces and their dimensions. We write $\cong$ for "isomorphic to" and $\cong_\omega$ for "ω-isomorphic to." If G_1 and G_2 are associated with Oc_1 and Oc_2 respectively, the isomorphisms (ω-isomorphisms) from G_1 onto G_2 are the same as the isomorphisms (ω-isomorphisms) from Oc_1 onto Oc_2. We have therefore by B2.4 for ω-octahedra $\mathrm{Oc}_1 = \langle \mu_1, C_{\mu(1)}\rangle$ and $\mathrm{Oc}_2 = \langle \mu_2, C_{\mu(2)}\rangle$,

$$(3.1) \qquad \mu_1 \simeq \mu_2 \Leftrightarrow \mathrm{Oc}_1 \cong_\omega \mathrm{Oc}_2 \Leftrightarrow \dim_\omega \mathrm{Oc}_1 = \dim_\omega \mathrm{Oc}_2.$$

The *standard* ω-octahedron Oc^ν associated with the set ν is the ω-octahedron Oc_f, where f is the standard ω-iwfp associated with ν. In view of B2.3 and (3.1) we conclude that (i) an octahedron is an ω-octahedron if and only if it is ω-isomorphic to a standard ω-octahedron, (ii) $\alpha \simeq \beta \Leftrightarrow \mathrm{Oc}^\alpha \cong_\omega \mathrm{Oc}^\beta$, for $\alpha, \beta \neq o$. If $N \in \Omega_0$ we define Oc^N as any ω-octahedron of ω-dimension N; it is unique up to ω-isomorphism. Let for $n \geqslant 1$, $-1 \leqslant k \leqslant n-1$, Oc^n have c_{nk} k-faces and c_n faces. It is well-known and readily seen that $c_{nk} = 2^{n+1}[n; k+1]$ and $c_n = 3^n$. In order to generalize these formulas to Oc^N, for $N \in \Omega_0$, we define for $\mathrm{Oc}^\nu = \langle \mu, C_\mu \rangle$, $-1 \leqslant k \leqslant N-1$,

$$(3.2) \qquad \tilde{\gamma}_{\nu k} = \{x \in \varepsilon \mid \rho_x \in C_{\mu k}\}, \qquad \tilde{\gamma}_\nu = \{x \in \varepsilon \mid \rho_x \in C_\mu\},$$

$$(3.3) \qquad C_{Nk} = \mathrm{Req}\, \tilde{\gamma}_{\nu k}, \quad \text{for } \nu \in N, \qquad C_N = \mathrm{Req}\, \tilde{\gamma}_\nu, \quad \text{for } \nu \in N,$$

$$(3.4) \qquad \gamma_{\nu k} = \{j(p,q) \in \varepsilon | \rho_p, \rho_q \subset \nu \,\&\, \rho_p \cap \rho_q = o \,\&\, r_{p+q} = k+1\},$$

$$(3.5) \qquad \gamma_\nu = \{j(p,q) \in \varepsilon | \rho_p, \rho_q \subset \nu \,\&\, \rho_p \cap \rho_q = o\}.$$

The functions C_{Nk} and C_N are well-defined, that is, independent of the choice of ν. Note that $x \in \tilde{\gamma}_{\nu k}$ if and only if $d_0^{-1}(\rho_x \cap \delta_0)$ and $d_1^{-1}(\rho_x \cap \delta_1)$ are disjoint subsets of ν and $r_x = k+1$. The *G-number* $G(\rho_x)$ of the face ρ_x of $\mathrm{Oc}^\nu = \langle \mu, C_\mu \rangle$ is defined as $j(p,q)$, where $\rho_p = d_0^{-1}(\rho_x \cap \delta_0)$, $\rho_q = d_1^{-1}(\rho_x \cap \delta_1)$. It can now be proved that

$$(3.6) \qquad \tilde{\gamma}_{\nu k} \simeq \gamma_{\nu k} \quad \textit{and} \quad \tilde{\gamma}_\nu \simeq \gamma_\nu, \qquad \textit{for } -1 \leqslant k \leqslant N-1, N \geqslant 1, \nu \in N.$$

PROPOSITION B3.1. *For $N \in \Omega_0$ and $-1 \leqslant k \leqslant N-1$,*

$$C_{Nk} = 2^{k+1}\binom{N}{k+1} \quad \textit{and} \quad C_N = 3^N.$$

PROOF. Let $\nu \in N$. Then we have $\gamma_\nu \simeq \alpha_\nu$ and $C_N = A_N = 3^N$ by (1.1), (1.3), and (3.6). The formula for C_{Nk} is trivial for $k = -1$. Now assume $k \geqslant 0$. Put $\alpha = (0, \ldots, k)$ and $\beta = j[2^\alpha \times [\nu; k+1]]$ then $\mathrm{Req}\, \beta = 2^{k+1}[N; k+1]$ and it suffices to prove $\beta \simeq \gamma_{\nu k}$. Let $\delta g = \beta$ and for $j(x,y) \in \beta$, $gj(x,y) = j(p,q)$, where p and q are computed as follows: find the enumeration according to size of ρ_y, say $z_0, \ldots, z_k$; then $\rho_p = \{z_i \mid 0 \leqslant i \leqslant k \,\&\, i \in \rho_x\}$ and $\rho_q = \rho_y - \rho_p$. Then ρ_p and ρ_q are disjoint subsets of ν, while $r_{p+q} = r_y = k+1$, hence $j(p,q) \in \gamma_{\nu k}$, so that $g(\beta) \subset \gamma_{\nu k}$. It can now be proved that $g(\beta) = \gamma_{\nu k}$, where g is one-to-one and both g and g^{-1} have partial recursive extensions. Thus $\beta \simeq \gamma_{\nu k}$.

4. Some sequences of isols

This section deals with the sequences defined by the functions $[n; k]$, for $0 \leqslant k \leqslant n$, $a_{nk} = 2^{n-k}[n; k]$ and $c_{nk} = 2^{k+1}[n; k+1]$, for $0 \leqslant k \leqslant n-1$, namely

$$\begin{array}{lll} \text{I} & \left\langle \binom{n}{k} \right\rangle_{k=0}^{n} & : \binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n-1}, \binom{n}{n}, \\ \text{II} & \langle a_{nk} \rangle_{k=0}^{n-1} & : 2^n, 2^{n-1}\binom{n}{1}, \ldots, 2^2\binom{n}{n-2}, 2\binom{n}{n-1} = 2n, \\ \text{III} & \langle c_{nk} \rangle_{k=0}^{n-1} & : 2n, 2^2\binom{n}{2}, \ldots, 2^{n-1}\binom{n}{n-1}, 2^n\binom{n}{n} = 2^n. \end{array}$$

Using the algebraic relations $[n; k] = [n; n-k]$, for $0 \leqslant k \leqslant n$ and $a_{nk} = c_{n,n-k-1}$, $c_{nk} = a_{n,n-k-1}$, for $0 \leqslant k \leqslant n-1$, we see that for $n \geqslant 3$ these sequences are first strictly increasing and then strictly decreasing; moreover, II and III have the same elements, but in reverse order. If we replace the number n by an infinite isol N, we may or may not permit k to assume values of the type $N - i$, where $i \in \varepsilon$. We obtain

$$\begin{array}{llll} \text{I}' & \binom{N}{0}, \binom{N}{1}, \ldots, & \text{I}'' & \binom{N}{0}, \binom{N}{1}, \ldots, \binom{N}{N-1}, \binom{N}{N}, \\ \text{II}' & 2^N, 2^{N-1}\binom{N}{1}, \ldots, & \text{II}'' & 2^N, 2^{N-1}\binom{N}{1}, \ldots, 2^2\binom{N}{N-2}, \\ & & & 2\binom{N}{N-1} = 2N, \\ \text{III}' & 2N, 2^2\binom{N}{2}, \ldots, & \text{III}'' & 2N, 2^2\binom{N}{2}, \ldots, 2^{N-1}\binom{N}{N-1}, \\ & & & 2^N\binom{N}{N} = 2^N. \end{array}$$

PROPOSITION B4.1. *For an infinite isol N, sequences* I′, II′, III′ *are strictly increasing. Moreover, sequences* II′ *and* III′ *have no elements in common.*

PROOF. Define for $A \in \Lambda$, $P(A, 0) = 1$ and

(4.1) $\quad P(A, k) = A \cdot (A-1) \cdot \cdots \cdot (A-k+1), \quad \text{for } 1 \leqslant k \leqslant A;$

then

(4.2) $\quad P(A, k+1) = (A-k)P(A, k), \quad \text{for } k+1 \leqslant A,$

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(4.3) $$P(A,k) = k!\binom{A}{k}, \quad \text{for } k \leqslant A,$$

(4.4) $$mP(A,k) < P(A,k+1), \quad \text{for } m,k \in \varepsilon, A \in \Lambda - \varepsilon.$$

Relation (4.3) holds by Theorem 113 of Dekker and Myhill (1960). For $k \in \varepsilon$, $N \in \Lambda - \varepsilon$ we have

(4.5) $$\binom{N}{k} < \binom{N}{k+1},$$
$$2^{N-k}\binom{N}{k} < 2^{N-k-1}\binom{N}{k+1}, \qquad 2^{k}\binom{N}{k} < 2^{k+1}\binom{N}{k+1}$$

and the desired statements can be proved from (4.1),...,(4.5).

We also have

PROPOSITION B4.2. *For an infinite isol N the tail of* I″ *is the reverse of the head of* I″. *Moreover, the tail of* II″ *is the reverse of the head of* III″, *while the head of* II″ *is the reverse of the tail of* III″. *Finally,* I″, II″, III″ *are first strictly increasing and then strictly decreasing.*

5. Duality

Let the word "*n-polytope*" be used the sense of a bounded, convex, n-dimensional polytope. For an n-polytope P we write $F(P)$ for the class of all its faces and $f_{nk}(P)$ for the number of its k-dimensional faces, for $0 \leqslant k \leqslant n-1$. If P and P^* are n-polytopes, a *duality-mapping* from $F(P)$ onto $F(P^*)$ is a one-to-one mapping D such that both D and D^{-1} are inclusion-reversing, that is, such that

(5.1) $$\alpha \subset \beta \Leftrightarrow D(\beta) \subset D(\alpha), \quad \text{for } \alpha, \beta \in F(P).$$

P and P^* are *dual*, if there exists a duality-mapping from $F(P)$ onto $F(P^*)$; see page 46 of Grünbaum (1967). Note that (5.1) implies: $\dim \alpha + \dim D(\alpha) = n-1$, for $\alpha \in F(P)$. Thus $\dim \alpha = k$ if and only if $\dim D(\alpha) = n-k-1$, hence $f_{n,k}(P) = f_{n,n-k-1}(P^*)$, for $0 \leqslant k \leqslant n-1$; the sequences $\langle f_{n0}(P),\ldots,f_{n,n-1}(P)\rangle$ and $\langle f_{n0}(P^*),\ldots,f_{n,n-1}(P^*)\rangle$ have therefore the same elements, but in reverse order.

Let $Q^N = \langle 2^\nu, F_\nu\rangle$ and $\mathrm{Oc}^N = \langle \mu, C_\mu\rangle$, where $N \geqslant 2$. It is well-known that Q^N and Oc^N are dual, if N is finite, say $N = n$. Then the relation $f_{nk}(P) = f_{n,n-k-1}(P^*)$ becomes $a_{nk} = c_{n,n-k-1}$, for $0 \leqslant k \leqslant n-1$, so that $\langle a_{n0},\ldots,a_{n,n-1}\rangle$ and $\langle c_{n0},\ldots,c_{n,n-1}\rangle$ have the same elements, but in reverse order. Now assume that

N is an infinite isol. Are Q^N and Oc^N effectively dual under a suitable generalization of the notion of duality? More specifically, if ν is immune, $\mu = \nu_0 \cup \nu_1$, $Q^\nu = \langle 2^\nu, F_\nu \rangle$ and $\mathrm{Oc}^\nu = \langle \mu, C_\mu \rangle$, is there a duality-mapping D from F_ν onto C_μ such that the corresponding one-to-one mapping d from α_ν onto γ_μ has a partial recursive one-to-one extension and maps the sets in $\langle \alpha_{\nu 0}, \alpha_{\nu 1}, \ldots \rangle$ onto the sets in $\langle \gamma_{\nu 0}, \gamma_{\nu 1}, \ldots \rangle$? The answer is clearly negative, since $\langle A_{N0}, A_{N1}, \ldots \rangle$ and $\langle C_{N0}, C_{N1}, \ldots \rangle$ have no elements in common for $N \in \Lambda - \varepsilon$ by B4.1. However, II″ and III″ have the same elements, but in reverse order by B4.2. This suggests that while Q^ν is not effectively dual to Oc^ν, if ν is immune, there is a system closely related to $Q^\nu = \langle 2^\nu, F_\nu \rangle$ which might be dual to Oc^ν, namely the system consisting of 2^ν and a class of subsets of ν which behave like faces, but have ω-dimensions of type $N - k$, for $k \geqslant 0$.

Let $N = \mathrm{Req}\,\nu$ and $N \geqslant 2$. We define a *coface* of Q^ν as a subset τ of 2^ν for which there exist disjoint subsets β and γ of ν such that β is infinite, γ is cofinite relative to ν and $\tau = \{x \in 2^\nu \mid \beta \subset \rho_x \subset \beta \cup \gamma\}$. We call $\mathrm{Req}\,\gamma$ the *ω-dimension* of τ, written $\dim_\omega \tau$. A coface of ω-dimension $N - k$ is called an *$(N-k)$-coface* of Q^ν. Clearly, $\tau \simeq \{x \in 2^\nu \mid o \subset \rho_x \subset \gamma\}$, that is, $\tau \simeq 2^\gamma$. Thus each $(N-k)$-coface of Q^ν has RET 2^{N-k}. Note that τ is a coface of Q^ν if and only if there exist finite subsets β and δ of ν such that $\beta \cap (\nu - \delta) = o$, that is, $\beta \subset \delta$ and $\tau = \{x \in 2^\nu \mid \beta \subset \rho_x \subset \beta \cup (\nu - \delta)\}$. The *G-number* of the coface τ is the number $j(p, s)$ such that

$$\tag{5.2} \tau = \{x \in 2^\nu \mid \rho_p \subset \rho_x \subset \rho_p \cup (\nu - \rho_s)\}, \quad \text{where } \rho_p \subset \rho_s \subset \nu.$$

Let $N = \mathrm{Req}\,\nu$ and $k \leqslant N$. Recall that we write $F_{\nu k}$ for the class of all k-faces of Q^ν and F_ν for the class of all faces of Q^ν. Similarly, we write $L_{\nu, N-k}$ for the class of all $(N-k)$-cofaces of Q^ν and L_ν for the class of all cofaces of Q^ν. It is readily seen that the classes F_ν and L_ν are equal if and only if ν is finite; they are disjoint if and only if ν is infinite, for then F_ν consists of finite sets and L_ν of infinite sets. Let $\nu \in N$, $k \leqslant N$, $N \in \Lambda_0$. Then $\lambda_{\nu, N-k}$ stands for the set of all G-numbers of $(N-k)$-cofaces of Q^ν and λ_ν for the set of all G-numbers of cofaces of Q^ν. In symbols,

$$\tag{5.3} \lambda_{\nu, N-k} = \{j(p, s) \in \varepsilon \mid \rho_p \subset \rho_s \subset \nu \;\&\; r_s = k\},$$

$$\tag{5.4} \lambda_\nu = \{j(p, s) \in \varepsilon \mid \rho_p \subset \rho_s \subset \nu\}.$$

Let $\alpha \in N$, $N \in \Lambda_0$, $k \leqslant N$. Then $\alpha \simeq \beta$ implies $\lambda_{\alpha, N-k} \simeq \lambda_{\beta, N-k}$ and $\lambda_\alpha \simeq \lambda_\beta$. This enables us to define $L_{N, N-k} = \mathrm{Req}\,\lambda_{\nu, N-k}$ and $L_N = \mathrm{Req}\,\lambda_\nu$, for any $\nu \in N$.

In the remainder of this section it is essential that we distinguish between the G-number $G(\tau)$ of some coface τ of Q^ν and the G-number $G(\sigma)$ of some face σ of Oc^ν. To stress this distinction we shall henceforth write $G'(\sigma)$ rather than $G(\sigma)$ for the G-number of a face σ of Oc^ν.

Definition. Let $N \in \Lambda$, $N \geqslant 2$, $Q^\nu = \langle 2^\nu, F_\nu \rangle$, $\mathrm{Oc}_f = \langle \mu, C_\mu \rangle$. Then an *effective duality-mapping* from L_ν onto C_μ is a one-to-one mapping D from L_ν onto C_μ such that (a) $\alpha \subset \beta$ if and only if $D(\beta) \subset D(\alpha)$, for $\alpha, \beta \in L_\nu$, (b) $D(L_{\nu,N-k}) = C_{\mu,k-1}$, for $0 \leqslant k \leqslant N$, (c) the one-to-one function d from λ_ν onto γ_μ such that $dG(\tau) = G'D(\tau)$ has a partial recursive one-to-one extension.

For an N-cube Q^ν with $N \in \Lambda$, $N \geqslant 2$ we define a *facet* of Q^ν as an $(N-1)$-coface of Q^ν [hence as an $(N-1)$-face of Q^ν if and only if N is finite]. Recall that in E^3 we can with a solid cube Q associate its dual, namely a solid octahedron Oc by defining the vertices of Oc as the midpoints of the faces of Q and the edges of Oc as the line segments which join the midpoints of nonparallel faces of Q. We generalize this procedure in the proof of the next theorem, but we replace the midpoints of the facets of the cube by the G-numbers of these facets, since we are working with discrete cubes and discrete octahedra.

Proposition B5.1. *Let $N \in \Lambda$ and $N \geqslant 2$. Then we can associate with every N-cube $Q^\nu = \langle 2^\nu, F_\nu \rangle$ an N-octahedron $\mathrm{Oc}_f = \langle \mu, C_\mu \rangle$ and an effective duality-mapping from L_ν onto C_μ.*

Proof. Assume the hypothesis, $k \geqslant 0$ and write ν_u for $\nu - (u)$, if $u \in \nu$. Recall that every $(N-k)$-coface of Q^ν has RET 2^{N-k}. The following three statements can now be proved:

(A) Let τ be a facet of Q^ν and $G(\tau) = j(p, s)$. Then τ is of one of the two types:

(I) $\tau = \{x \in 2^\nu \mid (u) \subset \rho_x \subset (u) \cup \nu_u\}$, for some $u \in \nu$.

(II) $\tau = \{x \in 2^\nu \mid o \subset \rho_x \subset o \cup \nu_u\}$, for some $u \in \nu$.

In fact, τ is of type (I) if and only if $r_p = 1 \,\&\, r_s = 1$, while τ is of type (II) if and only if $r_p = 0 \,\&\, r_s = 1$.

(B) For each facet τ of Q^ν there exists exactly one facet of Q^ν disjoint from it, namely $2^\nu - \tau$ [we call two facets of Q^ν *opposite*, if they are complementary subsets of 2^ν, otherwise *adjacent*].

(C) Let $k \geqslant 2$ and $\tau_1, \ldots, \tau_k$ be k mutually adjacent facets of Q^ν and $\tau = \tau_1 \cap \cdots \cap \tau_k$. Then τ is an $(N-k)$-coface of Q^ν and $G(\tau)$ can be computed from $G(\tau_1), \ldots, G(\tau_k)$. Moreover, for every $(N-k)$-coface of Q^ν there exists exactly one class $(\tau_1, \ldots, \tau_k)$ of k mutually adjacent facets of Q^ν such that $\tau = \tau_1 \cap \cdots \cap \tau_k$.

Using (A), (B) and (C) we now finish the proof. Write μ for the set of all G-numbers of the facets of Q^ν; call two elements of μ *opposite* (*adjacent*), if they are G-numbers of opposite (adjacent) facets of Q^ν. Let f be the iwfp of μ which maps each element of μ onto its opposite. Put $\mu_1 = \{G(\tau) \in \mu \mid \tau \text{ is of type (I)}\}$, $\mu_2 = \{G(\tau) \in \mu \mid \tau \text{ is of type II}\}$. According to (A) we have $\mu_1 = \{j(2^t, 2^t) \mid t \in \nu\}$

and $\mu_2 = \{j(0, 2^t) \mid t \in \nu\}$, so that $\mu = \mu_1 \cup \mu_2$, and $\mu_1 \mid \mu_2$. Then the function f which maps $f(2^t, 2^t)$ and $j(0, 2^t)$ onto each other, for $t \in \nu$, has a partial recursive one-to-one extension. Hence $f \in \mathrm{Inv}_\omega(\mu)$, $\mu_1 \simeq \mu_2 \simeq \nu$ and $\mathrm{Req}\, \mu = 2N$. We now show that $\mathrm{Oc}_f = \langle \mu, C_\mu \rangle$ satisfies the requirements. Note that $\dim_\omega \mathrm{Oc}_f = N$, since $\mathrm{Req}\, \mu = 2N$. Define for $k \geq 2$ the mappings D_k and d_k by:

$$(5.5) \qquad \mathrm{Dom}\, D_k = L_{\nu, N-k}, \quad D_k(\tau_1 \cap \cdots \cap \tau_k) = (G(\tau_1), \ldots, G(\tau_k)),$$

$$(5.6) \qquad \delta d_k = \lambda_{\nu, N-k}, \quad d_k G(\tau_1 \cap \cdots \cap \tau_k) = G'(G(\tau_1), \ldots, G(\tau_k)).$$

for any k mutually adjacent facets $\tau_1, \ldots, \tau_k$ of Q^ν. By (C) the mapping D_k maps $L_{\nu, N-k}$ one-to-one onto $C_{\mu, k-1}$ so that d_k maps $\lambda_{\nu, N-k}$ one-to-one onto $\gamma_{\mu, k-1}$. Moreover, d_k and d_k^{-1} have partial recursive extensions, hence d_k has a partial recursive one-to-one extension. Define the mappings D and d as follows: for $\tau \in L_\nu$, $x = G(\tau)$ and $\dim_\omega \tau = N - k$,

$$\begin{array}{ll} D(\tau) = o, & d(x) = G'(o), \text{ for } k = 0, \text{ that is, } \tau = 2^\nu, \\ D(\tau) = (G(\tau)), & d(x) = G' \text{ of } (x), \text{ for } k = 1, \text{ that is, } \tau \text{ is a facet}, \\ D(\tau) = (G(\tau_1), \ldots, G(\tau_k)), & d(x) = G'(G(\tau_1), \ldots, G(\tau_k)), \text{ for } k \geq 2, \end{array}$$

where in case $k \geq 2$, $\tau_1, \ldots, \tau_k$ are the k mutually adjacent facets of Q^ν such that $\tau = \tau_1 \cap \cdots \cap \tau_k$. Given an element $x = j(p, s) \in \lambda_\nu$, we can compute $k = r_s$ and $d(x) = d_k j(p, s)$. Thus the one-to-one function d from λ_ν onto γ_μ has a partial recursive one-to-one extension. Also, $D(L_{\nu, N-k}) = D_k(L_{\nu, N-k}) = C_{\mu, k-1}$, hence $d(\lambda_{\nu, N-k}) = \gamma_{\mu, k-1}$, for $k \geq 0$. Let $(\tau_1, \ldots, \tau_k)$ and $(\tau_1^*, \ldots, \tau_m^*)$ be classes of mutually adjacent facets of Q^ν with $\tau_1 \cap \cdots \cap \tau_k = \tau$ and $\tau_1^* \cap \cdots \cap \tau_m^* = \tau^*$. Then $\tau \subset \tau^*$ if and only if $(\tau_1^*, \ldots, \tau_m^*) \subset (\tau_1, \ldots, \tau_k)$ if and only if $(G(\tau_1^*), \ldots, G(\tau_m^*)) \subset (G(\tau_1), \ldots, G(\tau_k))$ if and only if $D(\tau^*) \subset D(\tau)$.

COROLLARY. $L_{N, N-k} = 2^k \binom{N}{k}$ *and* $L_N = 3^N$, *for* $N \in \Lambda_0$, $0 \leq k \leq N$.

REMARK (A). Let $Q^\nu = \langle 2^\nu, F_\nu \rangle$, $\mathrm{Oc}^\nu = \langle \mu, C_\mu \rangle$, $N = \mathrm{Req}\, \nu$, $N \in \Lambda$, $N \geq 2$. Write CQ^ν for $\langle 2^\nu, L_\nu \rangle$ and call CQ^ν *effectively dual* to Oc^ν, if there is an effective duality-mapping from L_ν onto C_μ. Then we have

(I) If N is infinite, Q^ν is not effectively dual to Oc^ν,

(II) CQ^ν is effectively dual to Oc^ν, for every N (finite or infinite); however, for a finite N this only yields the well-known fact that the n-cube is dual to the n-octahedron, for in that case $L_\nu = F_\nu$, hence $CQ^\nu = Q^\nu$.

REMARK (B). Under the hypothesis of the preceding remark it is also possible to define a class H_μ of cofaces of $\mathrm{Oc}^\nu = \langle \mu, C_\mu \rangle$ such that there exists an effective duality-mapping from F_ν onto H_μ. Thus the systems $\langle 2^\nu, F_\nu, L_\nu \rangle$ and $\langle \mu, C_\mu, H_\mu \rangle$ are effectively dual in the sense that there is not only an effective duality-mapping

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from L_ν onto C_μ, but also one from F_ν onto H_μ. If $N = \text{Req}\,\nu$ is finite, we have $F_\nu = C_\mu$ and $L_\nu = H_\mu$ and we can identify $\langle 2^\nu, F_\nu, L_\nu \rangle$ with $Q^\nu = \langle 2^\nu, F_\nu \rangle$ and $\langle \mu, C_\mu, H_\mu \rangle$ with $\text{Oc}^\nu = \langle \mu, C_\mu \rangle$.

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