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Gromov–Witten theory and Noether–Lefschetz theory for holomorphic-symplectic varieties

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Abstract

We use Noether–Lefschetz theory to study the reduced Gromov–Witten invariants of a holomorphic-symplectic variety of $K3^{[n]}$-type. This yields strong evidence for a new conjectural formula that expresses Gromov–Witten invariants of this geometry for arbitrary classes in terms of primitive classes. The formula generalizes an earlier conjecture by Pandharipande and the author for K3 surfaces. Using Gromov–Witten techniques, we also determine the generating series of Noether–Lefschetz numbers of a general pencil of Debarre–Voisin varieties. This reproves and extends a result of Debarre, Han, O'Grady and Voisin on Hassett–Looijenga–Shah (HLS) divisors on the moduli space of Debarre–Voisin fourfolds.

Contents

0. Introduction

0.1. K3 surfaces

Gromov–Witten theory is the intersection theory of the moduli space $\overline{M}_{g,n}(X, \beta)$ of stable maps to a target X in degree $\beta \in H_2(X, \mathbb{Z})$. If X carries a holomorphic symplectic form, the virtual fundamental class of the moduli space vanishes. Instead Gromov–Witten theory is defined through the reduced virtual fundamental class

$$[\overline{M}_{g,n}(X, \beta)]^{\mathrm{red}} \in A_*(\overline{M}_{g,n}(X, \beta)).$$

When working with reduced Gromov–Witten invariants, one observes in many examples the following dichotomy:

1. The invariants are notoriously difficult to compute, in particular if the class β is not primitive.
2. The structure of the invariants is simpler than for general target X. That is, the invariants have additional nongeometric symmetries such as the independence (understood correctly) from the divisibility of the curve class.

Physicists would say that X has additional super-symmetry, which should explain this phenomenon. A mathematical explanation is unfortunately very much missing so far.

As an example, let us consider a K3 surface S and the Hodge integral

$$R_{g,\beta} = \int_{[\overline{M}_{g,n}(S,\beta)]^{\mathrm{red}}} (-1)^g \lambda_g.$$

We can formally subtract "multiple cover contributions" from β by[1]

$$\widetilde{r}_{g,\beta} = \sum_{k|\beta} k^{2g-3} \mu(k) R_{g,\beta/k}$$

where we have used the Möbius function

$$\mu(k) = \begin{cases} (-1)^\ell & \text{if } k = p_1 \cdots p_\ell \text{ for distinct primes } p_i \\ 0 & \text{else.} \end{cases}$$

By deformation invariance, $\widetilde{r}_{g,\beta}$ depends on (S,β) only through the divisibility $m = \mathrm{div}(\beta)$ and the square $s = \beta \cdot \beta$. One writes

$$\widetilde{r}_{g,\beta} = \widetilde{r}_{g,m,s}.$$

The following remarkable result by Pandharipande and Thomas shows that the invariants $\widetilde{r}_{g,\beta}$ do not depend on the divisibility.

Theorem 1 ([49]). *For all g, m, s we have: $\widetilde{r}_{g,m,s} = \widetilde{r}_{g,1,s}$.*

The calculation of the primitive invariants $\widetilde{r}_{g,1,s}$ is much easier compared to the imprimitive case and was performed first in [38]. Several other proofs are available in the literature by now. Together with the theorem, this yields a complete determination of $\widetilde{r}_{g,\beta}$, called the Katz–Klemm–Vafa formula [25].

In [45], it was conjectured that more generally *any* Gromov–Witten invariant of a K3 surface is independent of the divisibility after subtracting multiple covers. The divisibility 2 case was solved recently [1], but the general case remains a challenge.

0.2. Holomorphic-symplectic varieties

A smooth projective variety X is (irreducible) holomorphic-symplectic if it is simply connected and the space of holomorphic 2-forms $H^0(X, \Omega_X^2)$ is spanned by a symplectic form. These varieties can be viewed as higher-dimensional analogues of K3 surfaces. For example, the cohomology $H^2(X, \mathbb{Z})$ carries a canonical nondegenerate integer-valued quadratic form. The prime example of a holomorphic-symplectic variety is the Hilbert scheme of n points of a K3 surface and its deformations, which we call varieties of $K3^{[n]}$ type.[2]

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[1]The notation $\widetilde{r}_{g,\beta}$ in choosen to clearly distinguish from the Gopakumar–Vafa BPS invariants $r_{g,\beta}$ which appear in [49]. The relationship between these two sets of invariants is discussed in B.

[2]We also allow the case $n = 1$ below, that is, our formulas apply also to the case of K3 surfaces, where they reduce to [45].

We conjecture in this paper that the Gromov–Witten theory of $K3^{[n]}$-type varieties is independent of the divisibility of the curve class, made precise in the following sense: Let $\beta \in H_2(X, \mathbb{Z})$ be an effective (hence, nonzero) curve class, and consider the Gromov–Witten class

$$\mathcal{C}_{g,N,\beta}(\alpha) = \mathrm{ev}_* \left(\tau^*(\alpha) \cap [\overline{M}_{g,N}(X, \beta)]^{\mathrm{red}} \right) \in H^*(X^N)$$

where $\tau : \overline{M}_{g,N}(X, \beta) \to \overline{M}_{g,N}$ is the forgetful morphism to the moduli space of curves, and $\alpha \in H^*(\overline{M}_{g,N})$ is a tautological class [16]. The classes $\mathcal{C}_{g,N,\beta}(\alpha)$ encode the full numerical Gromov–Witten theory of X.

We formally subtract the multiple cover contributions from this class:

$$\mathsf{c}_{g,N,\beta}(\alpha) = \sum_{k|\beta} \mu(k) k^{3g-3+N-\deg(\alpha)} (-1)^{[\beta]+[\beta/k]} \mathcal{C}_{g,N,\beta/k}(\alpha) \tag{1}$$

where we use that the residue of β with respect to the quadratic form

$$[\beta] \in H_2(X, \mathbb{Z})/H^2(X, \mathbb{Z})$$

can be canonically identified up to multiplication by ± 1 with an element of $\mathbb{Z}/(2n-2)\mathbb{Z}$; see Section 1.5.

Let X' be any variety of $K3^{[n]}$-type, and let

$$\varphi : H^2(X, \mathbb{R}) \longrightarrow H^2(X', \mathbb{R})$$

be any *real isometry* such that $\varphi(\beta) \in H_2(X', \mathbb{Z})$ is a *primitive* effective curve class satisfying

$$\pm[\varphi(\beta)] = \pm[\beta] \ \text{ in } \ \mathbb{Z}/(2n-2)\mathbb{Z}.$$

Extend φ to the full cohomology as a parallel transport lift (Section 1.6)

$$\varphi : H^*(X, \mathbb{R}) \longrightarrow H^*(X', \mathbb{R}).$$

The following is the main conjecture.

Conjecture A.

$$\mathsf{c}_{g,N,\beta}(\alpha) = \varphi^{-1} \left(\mathcal{C}_{g,N,\varphi(\beta)}(\alpha) \right)$$

The right-hand side of the conjecture is given by the Gromov–Witten theory for a primitive class. Hence, the conjecture reduces calculations in imprimitive classes (which are hard) to those for primitive ones (which are easier). A different but equivalent version of the conjecture is formulated in Section 2.3. The equivalent version shows that the above reduces for K3 surfaces to the conjecture [45, Conj.C2].

The moduli space $\overline{M}_{g,0}(X, \beta)$ is of reduced virtual dimension

$$(\dim(X) - 3)(1 - g) + 1. \tag{2}$$

Hence, the Gromov–Witten theory of X of $K3^{[n]}$-type vanishes for $g > 1$ if $n \geq 3$, and for $g > 2$ if $n = 2$. Moreover, for $g = 1$ the virtual dimension (2) is always one-dimensional, so relatively small. Hence, in dimension > 2, Conjecture A mainly concerns the genus zero theory. Our main evidence in genus $g > 0$ comes from the case of K3 surfaces [49, 1, 45] and the remarkable fact that there is a single formula which governs all $K3^{[n]}$ at the same time.

In genus 0, up to the sign $(-1)^{[\beta]+[\beta/k]}$, equation (22) is precisely the formula that defines the BPS numbers of a Calabi–Yau manifold in terms of Gromov–Witten invariants. However, the appearance of the sign is a new feature, particular to the holomorphic-symplectic case. For example, it does not appear

in the definition of genus 0 BPS numbers of Calabi–Yau fourfolds as given by Klemm–Pandharipande [27]. We expect a similar multiple cover formula to hold for all holomorphic-symplectic varieties. What stops us from formulating it is that the precise term that generalizes the sign is not clear (aside from that there would be no evidence available at all).

In the appendix, we also formulate a multiple cover rule for abelian surfaces, extending a proposal for abelian varieties in [7].

0.3. Noether–Lefschetz theory

There are three types of invariants associated to a one-parameter family $\pi : \mathcal{X} \to C$ of quasi-polarized holomorphic-symplectic varieties:

  (i) the Noether–Lefschetz numbers of π,
 (ii) the Gromov–Witten invariants of $\mathcal{X}$ in fiber classes,
(iii) the reduced Gromov–Witten invariants of a holomorphic-symplectic fiber of the family.

We refer to Section 3 for the definition of a one-parameter family of quasi-polarized holomorphic-symplectic varieties and its Noether–Lefschetz numbers. By a result of Maulik and Pandharipande [37], there is a geometric relation intertwining these three invariants. This relation (for a carefully selected family π) was used in [26] to prove Theorem 1 in genus 0 and then later in [49] in the general case. Roughly, for a nice family, the relation becomes invertible and reduces the problem to considering ordinary Gromov–Witten invariants of a K3-fibered threefold which then can be attacked by more standard methods.

In this paper, we follow the same strategy for holomorphic symplectic varieties of $K3^{[n]}$-type. We first discuss the Maulik–Pandharipande relation in this case (Section 4) and then apply it in two cases:

 (i) A generic pencil of Fano varieties of a cubic fourfold [2]:

$$\mathcal{X} \subset \mathrm{Gr}(2,6) \times \mathbb{P}^1, \quad \pi : \mathcal{X} \longrightarrow \mathbb{P}^1.$$

(ii) A generic pencil of Debarre–Voisin varieties [15]:

$$\mathcal{X} \subset \mathrm{Gr}(6,10) \times \mathbb{P}^1, \quad \pi : \mathcal{X} \longrightarrow \mathbb{P}^1.$$

The examples are choosen such that the Gromov–Witten invariants of $\mathcal{X}$ can be computed using mirror symmetry [11]. For the family of Fano varieties, the Noether–Lefschetz numbers have already been computed by Li and Zhang [31]. The Gromov–Witten/Noether–Lefschetz relation for the Fano family then yields an explicit infinite family of relations that need to hold for Conjecture A to be true. Using a computer, we checked that these relations are satisfied up to degree $d \leq 38$ (with respect to the Plücker polarization). The relations involve curve classes of both high self-intersection and high divisibility. Together with previously known cases, we also obtain the following.

Proposition 1. *In genus* 0 *and* $K3^{[2]}$-*type Conjecture* A *holds for* $\beta = m \cdot \alpha$, *where* α *is primitive whenever:*

 ○ $(\alpha, \alpha) < 0$, *or*
 ○ $(\alpha, \alpha) = 0$ *and* $(m = 2$ *or* $N = 1)$, *or*
 ○ $(\alpha, \alpha) = 3/2$ *and* $m \in \{2, 3, 5\}$.

Ideally, we would like to apply our methods to other families. However, it is quite difficult to find appropriate one-parameter families. They must be (a) a zero section of a homogeneous vector bundle

on the geometric invariant theory (GIT) quotient of a vector space by a reductive group, and (b) their singular fibers must have mild singularieties. A promising candidate seemed to be a generic pencil of Iliev–Manivel fourfolds [24],

$$\mathcal{X} \subset \mathrm{Gr}(2,4) \times \mathrm{Gr}(2,4) \times \mathrm{Gr}(2,4) \times \mathbb{P}^1, \quad \pi : \mathcal{X} \longrightarrow \mathbb{P}^1,$$

but unfortunately, the singular fibers appear to be too singular.[3]

0.4. Debarre–Voisin fourfolds

For the generic pencil $\pi : \mathcal{X} \to \mathbb{P}^1$ of Debarre–Voisin fourfolds, the Noether–Lefschetz numbers have not yet been determined. Instead, we use the known cases of Conjecture A and the mirror symmetry calculations for the total space $\mathcal{X}$ to obtain constraints for the Noether–Lefschetz numbers. By using the modularity of the generating series of Noether–Lefschetz numbers due to Borcherds and McGraw [5, 39], we can then determine the full series.

Consider the generating series of Noether–Lefschetz numbers of π as defined in Section 3.8.1,

$$\varphi(q) = \sum_{D \geq 0} q^{D/11} \mathrm{NL}^\pi(D),$$

where D runs over all squares modulo 11.

Define the weight 1, 2, 3 modular forms

$$E_1(\tau) = 1 + 2 \sum_{n \geq 1} q^n \sum_{d|n} \chi_P\left(\frac{n}{d}\right), \quad \Delta_{11}(\tau) = \eta(\tau)^2 \eta(11\tau)^2$$

$$E_3(\tau) = \sum_{n \geq 1} q^n \sum_{d|n} d^2 \chi_P\left(\frac{n}{d}\right),$$

where $\eta(\tau) = q^{1/24} \prod_{n \geq 1}(1 - q^n)$ is the Dirichlet eta function, $q = e^{2\pi i \tau}$ and χ_{11} is the Dirichlet character given by the Legendre symbol $\left(\frac{\cdot}{11}\right)$. Consider the following weight 11 modular forms for $\Gamma_0(11)$ and character χ_{11}:

$$\begin{aligned}
\varphi_0(q) = &-5\,E_1^{11} + 430\,E_1^8 E_3 + \frac{5,199,920}{9}\Delta_{11}^3 E_1^5 - \frac{35,407,490}{27}\Delta_{11}^4 E_1^3 \\
&+ \frac{49,194,440}{9}\Delta_{11}^2 E_1^4 E_3 + 248,350\,E_1^5 E_3^2 - \frac{596,661,440}{27}\Delta_{11}^3 E_1^2 E_3 \\
&- \frac{306,631,760}{9}\Delta_{11} E_1^3 E_3^2 + \frac{51,243,500}{3}\Delta_{11}^4 E_3 + \frac{1,331,452,540}{27}\Delta_{11}^2 E_1 E_3^2 \\
&+ \frac{349,019,440}{9}E_1^2 E_3^3 \\
= &-5 + 320q + 255,420q^2 + 14,793,440q^3 + 262,345,260q^4 + \dots \\
\varphi_1(q) = &-5\,E_1^{11} + 110\,E_1^8 E_3 + \frac{722,740}{3,993}\Delta_{11}^3 E_1^5 - \frac{1,805,750}{3993}\Delta_{11}^4 E_1^3 \\
&- \frac{12,660,620}{11979}\Delta_{11}^2 E_1^4 E_3 - 990 E_1^5 E_3^2 + \frac{118,940}{363}\Delta_{11}^5 E_1 + \frac{5,609,180}{3,993}\Delta_{11}^3 E_1^2 E_3 \\
&+ \frac{29,208,460}{11,979}\Delta_{11} E_1^3 E_3^2 + \frac{3,500}{33}\Delta_{11}^4 E_3 + \frac{2,610,980}{1,089}E_1^2 E_3^3 \\
= &-5 + 320q^{11} + 990q^{12} + 5,500q^{14} + 11,440q^{15} + \dots
\end{aligned}$$

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[3]Another candidate is the family of Fano varieties of Pfaffian cubics, $X \subset \mathrm{Gr}(4,6) \times \mathrm{Gr}(2,6)$, found by Fatighenti and Mongardi [17].

Theorem 2. *Let* $\pi : \mathcal{X} \to \mathbb{P}^1$ *be a generic pencil of Debarre–Voisin fourfolds. Then the generating series of its Noether–Lefschetz numbers is*

$$\varphi(q^{11}) = \varphi_0(q^{11}) + \varphi_1(q) = -10 + 640q^{11} + 990q^{12} + 5,500q^{14} + 11,440q^{15}$$
$$+ 21,450q^{16} + 198,770q^{20} + 510,840q^{22} + \dots$$

Debarre–Voisin varieties are parametrized by a 20-dimensional projective irreducible GIT quotient

$$\mathcal{M}_{\mathrm{DV}} = \mathbb{P}(\wedge^3 V_{10}^\vee)//\mathrm{SL}(V_{10}),$$

where V_{10} is a vector space of dimension 10. The period map from this moduli space to the moduli space of holomorphic-symplectic varieties

$$p : \mathcal{M}_{\mathrm{DV}} \dashrightarrow \mathcal{M}_H$$

is birational [41] and regular on the open locus corresponding to smooth Debarre–Voisin varieties of dimension four. When passing to the Baily–Borel compactification $\overline{\mathcal{M}_H}$, this birational map can be resolved. An Hassett–Looijenga–Shah (HLS) divisor in $\mathcal{M}_H$ is the image of an exceptional divisor under this resolved map. These divisors reflect a difference between the GIT and the Baily–Borel compactification as they parametrize holomorphic-symplectic fourfolds of the same polarization type as a Debarre–Voisin fourfold, but for which the generic member is not a Debarre–Voisin fourfold.

Let e be a square modulo 11, and let

$$\mathcal{C}_{2e} \subset \mathcal{M}_H$$

be the Noether–Lefschetz divisor of the first type of discriminant e; see Section 3.8.2 for the precise definition. Observe that there is a natural gap[4] in the modular form $\varphi(q)$:

$$\varphi(q^{11}) = -10 + \underbrace{0 \cdot q^1 + 0 \cdot q^3 + \dots + 0 \cdot q^9}_{\text{gap}} + 640q^{11} + 990q^{12} + 5500q^{14} + \dots$$

Translating from Heegner divisors to the irreducible divisors $\mathcal{C}_{2e}$, this gap yields the following.

Corollary 1. *The divisors* $\mathcal{C}_2, \mathcal{C}_6, \mathcal{C}_8, \mathcal{C}_{10}, \mathcal{C}_{18}$ *are HLS divisors of the moduli space of Debarre–Voisin fourfolds. The divisor* $\mathcal{C}_{30}$ *is not HLS.*

The statement that $\mathcal{C}_2, \mathcal{C}_6, \mathcal{C}_{10}, \mathcal{C}_{18}$ are HLS is the main result of [13]. The argument here gives an independent and mostly formal proof of the main result of [13]. The only geometric input lies in describing in understanding the geometry of the singular fibers (a result of J. Song, see C). The result that $\mathcal{C}_{30}$ is not a HLS divisor answers a question of [13]. The fact that $\mathcal{C}_8$ is HLS seems to be new, and it would be interesting to understand the geometry of these loci, as is done in [13] for the other cases. In principle, Theorem 2 can be used to show that the divisors listed in Corollary 1 are the only Noether–Lefschetz divisors $\mathcal{C}_{2e}$ which are HLS.

Remark 1. After the first version of this paper appeared online, I learned that Theorem 2 and Corollary 1 was independently obtained by Calla Tschanz based on the results of [13].

0.5. Convention

If $\gamma \in H^i(X, \mathbb{Q})$ is a cohomology class, we write $\deg(\gamma) = i/2$ for the complex cohomological degree of γ. For X holomorphic-symplectic, we identify $\mathrm{Pic}(X)$ with its image in $H^2(X, \mathbb{Z})$ under the map taking

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[4]In fact, the gap determines the modular form (viewed as a vector-valued modular form) up to a constant.

the first Chern class. Let $\mathsf{Hilb}_n(S)$ be the Hilbert scheme of points of a K3 surface. Given $\alpha \in H^*(S, \mathbb{Q})$ and $i > 0$, we let

$$\mathfrak{q}_i(\alpha) : H^*(\mathsf{Hilb}_n(S)) \longrightarrow H^*(\mathsf{Hilb}_{n+i}(S))$$

be the i-th Nakajima operator [40] given by adding a i-fat point on a cycle with class α; we use the convention of [46]. We write $A \in H_2(\mathsf{Hilb}_n(S))$ for the class of a generic fiber of the singular locus of the Hilbert Chow morphism, and we let -2δ be the class of the diagonal. We identify

$$H^2(\mathsf{Hilb}_n(S)) \equiv H^2(S, \mathbb{Z}) \oplus \mathbb{Z}\delta, \quad H_2(\mathsf{Hilb}_n(S)) \equiv H_2(S, \mathbb{Z}) \oplus \mathbb{Z}A$$

using the Nakajima operators [42].

0.6. Subsequent work

In [43], the main conjecture of this paper (Conjecture A) is proven for all $K3^{[n]}$ in genus 0 and for $N \leq 3$ markings.

1. The monodromy in $K3^{[n]}$-type

1.1. Overview

Let X be a (irreducible) holomorphic-symplectic variety. The lattice $H^2(X, \mathbb{Z})$ is equipped with the integral and nondegenerate Beauville–Bogomolov–Fujiki quadratic form. We will also equip $H^*(X, \mathbb{Z})$ with the usual Poincaré pairing. Both pairings are extended to the $\mathbb{C}$-valued cohomology groups by linearity. Let $\mathsf{Mon}(X)$ be the subgroup of $O(H^*(X, \mathbb{Z}))$ generated by all monodromy operators, and let $\mathsf{Mon}^2(X)$ be its image in $O(H^2(X, \mathbb{Z}))$.

The goal of this section is to describe the monodromy group in the case that X is of $K3^{[n]}$-type, and we will assume so from now on. The main references for the sections are Markman's papers [33, 34].

1.2. Monodromy

Let X be of $K3^{[n]}$-type. By work of Markman [34, Thm.1.3], [35, Lemma 2.1], we have that

$$\mathsf{Mon}(X) \cong \mathsf{Mon}^2(X) = \widetilde{O}^+(H^2(X, \mathbb{Z})), \tag{3}$$

where the first isomorphism is the restriction map and $\widetilde{O}^+(H^2(X, \mathbb{Z}))$ is the subgroup of $O(H^2(X, \mathbb{Z}))$ of orientation preserving lattice automorphisms which act by ± 1 on the discriminant.[5] The first isomorpism implies that any parallel transport operator $H^*(X_1, \mathbb{Z}) \to H^*(X_2, \mathbb{Z})$ between two $K3^{[n]}$-type varieties is uniquely determined by its restriction to $H^2(X_1, \mathbb{Z})$.

If $g \in \mathsf{Mon}^2(X)$, we let $\tau(g) \in \{\pm 1\}$ be the sign by which g acts on the discriminant lattice. This defines a character

$$\tau : \mathsf{Mon}^2(X) \longrightarrow \mathbb{Z}_2.$$

1.3. Zariski closure

By [33, Lemma 4.11] if $n \geq 3$, the Zariski closure of $\mathsf{Mon}(X)$ in $O(H^*(X, \mathbb{C}))$ is $O(H^2(X, \mathbb{C})) \times \mathbb{Z}_2$. The inclusion yields the representation

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[5]Let $\mathcal{C} = \{x \in H^2(X, \mathbb{R}) | \langle x, x \rangle > 0\}$ be the positive cone. Then $\mathcal{C}$ is homotopy equivalent to S^2. An automorphism is orientation preserving if it acts by $+1$ on $H^2(\mathcal{C}) = \mathbb{Z}$.

$$\rho : O(H^2(X, \mathbb{C})) \times \mathbb{Z}_2 \longrightarrow O(H^*(X, \mathbb{C})) \tag{4}$$

which acts by degree-preserving orthogonal ring isomorphism. There is a natural embedding

$$\widetilde{O}^+(H^2(X, \mathbb{Z})) \longrightarrow O(H^2(X, \mathbb{C})) \times \mathbb{Z}_2, \, g \longmapsto (g, \tau(g))$$

under which ρ restricts to the monodromy representation. In case $n \in \{1, 2\}$, the Zariski closure is $O(H^2(X, \mathbb{C}))$. In this case, we define the representation (4) by letting it act through $O(H^2(X, \mathbb{C}))$.

The representation ρ is determined by the following properties.

Property 0. For any $(g, \tau) \in O(H^2(X, \mathbb{C})) \times \mathbb{Z}_2$, we have

$$\rho(g, \tau)|_{H^2(X, \mathbb{C})} = g.$$

Property 1. The restriction of ρ to $\mathrm{SO}(H^2(X, \mathbb{C})) \times \{0\}$ is the integrated action of the Looijenga–Lunts–Verbitsky algebra [32, 51].

Property 2. We have

$$\rho(1, -1) = D \circ \rho(-\mathrm{id}_{H^2(X, \mathbb{C})}, 1),$$

where D acts on $H^{2i}(X, \mathbb{C})$ by multiplication by $(-1)^i$.

Property 3. Assume that $X = \mathsf{Hilb}_n(S)$, and identify $H^2(X, \mathbb{Z})$ with $H^2(S, \mathbb{Z}) \oplus \mathbb{Z}\delta$. Then the restriction of ρ to $O(H^2(X, \mathbb{C}))_\delta \times 1$ (identified naturally with $O(H^2(S, \mathbb{C}))$) is the Zariski closure of the induced action on the Hilbert scheme by the monodromy representation of S.

In particular, the action is equivariant with respect to the Nakajima operators: For $g \in O(H^2(X, \mathbb{C}))_\delta$ let $\tilde{g} = g|_{H^2(S, \mathbb{C})} \oplus \mathrm{id}_{H^0(S, \mathbb{Z}) \oplus H^4(S, \mathbb{Z})}$. Then

$$\rho(g, 1)\left(\prod_i \mathfrak{q}_{k_i}(\alpha_i)1\right) = \prod_i \mathfrak{q}_{k_i}(\tilde{g}\alpha_i)1.$$

Property 4. Let $P_\psi : H^*(X_1, \mathbb{Z}) \to H^*(X_2, \mathbb{Z})$ be a parallel transport operator with $\psi = P_\psi|_{H^2(X_1, \mathbb{Z})}$. Then

$$P_\psi^{-1} \circ \rho(g, \tau) \circ P_\psi = \rho(\psi^{-1}g\psi, \tau).$$

Property 1 follows by [33, Lemma 4.13]. The other properties also follow from the results of [33]. Properties 1–3 determine the action ρ completely in the Hilbert scheme case. Moreover, by [46], this description is explicit in the Nakajima basis. The last condition extends this presentation then to arbitrary X. The parallel transport operator between different moduli spaces of stable sheaves can also be described more explicitly [33].

1.4. Parallel transport

Let

$$\Lambda = E_8(-1)^{\oplus 2} \oplus U^4$$

be the Mukai lattice. For $n \geq 2$, any holomorphic-symplectic variety of $K3[n]$-type is equipped with a canonical choice of a primitive embedding

$$\iota_X : H^2(X, \mathbb{Z}) \longrightarrow \Lambda$$

unique up to composition by an element by $O(\Lambda)$; see [34, Cor.9.4].

Theorem 3 ([34, Thm.9.8]). *An isometry $\psi : H^2(X_1, \mathbb{Z}) \to H^2(X_1, \mathbb{Z})$ is the restriction of a parallel transport operator if and only if it is orientation preserving and there exists an $\eta \in O(\Lambda)$ such that*

$$\eta \circ \iota_{X_1} = \iota_{X_2} \circ \psi.$$

Orientation preserving is here defined with respect to the canonical choice of orientation of the positive cone of X_i given by the real and imaginary part of the symplectic form and a Kähler class. If $X_1 = X_2$, the theorem reduces to the second isomorphism in equation (3).

1.5. Curve classes

By Eichler's criterion [19, Lemma 3.5], Theorem 3 yields a complete set of deformation invariants of curve classes in $K3^{[n]}$-type. To state the result we need the following constriction:

The orthogonal complement

$$L = \iota_X(H^2(X, \mathbb{Z}))^\perp \subset \Lambda$$

is isomorphic to the lattice $\mathbb{Z}$ with intersection form $(2n - 2)$. Let $v \in L$ be a generator, and consider the isomorphism of abelian groups

$$L^\vee/L \xrightarrow{\cong} \mathbb{Z}/(2n - 2)\mathbb{Z} \tag{5}$$

determined by sending $v/(2n - 2)$ to the residue class of 1. Since the generators of L are $\pm v$, equation (5) is canonical up to multiplication by 1.

Since Λ is unimodular, there exists a natural isomorphism ([22, Sec.14])

$$H^2(X, \mathbb{Z})^\vee/H^2(X, \mathbb{Z}) \xrightarrow{\cong} L^\vee/L.$$

If we use Poincaré duality to identify $H_2(X, \mathbb{Z})$ with $H^2(X, \mathbb{Z})^\vee$, this yields the *residue map*

$$r_X : H_2(X, \mathbb{Z})/H^2(X, \mathbb{Z}) \xrightarrow{\cong} L^\vee/L \xrightarrow{\cong} \mathbb{Z}/(2n - 2)\mathbb{Z}.$$

The map depends on the choice of the generator v and, hence, is unique up to multiplication by ± 1.

Definition 1. The *residue set* of a class $\beta \in H_2(X, \mathbb{Z})$ is defined by

$$\pm[\beta] = \{\pm r_X([\beta])\} \subset \mathbb{Z}/(2n - 2)\mathbb{Z}$$

if $n \geq 2$ and by $\pm[\beta] = 0$ otherwise.

Note that, since r_X is canonical up to sign, the residue set is independent of the choice of map r_X.

Remark 2. (i) Since parallel transport operators respect the embedding i_X up to composing with an isomorphism of Λ, the residue set $[\beta]$ is preserved under deformation. (This is also reflected in the fact that the monodromy acts by ± 1 on the discriminant.) (ii) In the case of the Hilbert scheme $X = \mathsf{Hilb}_n(S)$ of a K3 surface, let $A \in H_2(X)$ be the class of an exceptional curve, that is, the class of a fiber of the Hilbert–Chow morphism $\mathsf{Hilb}_n(S) \to \mathrm{Sym}^n(S)$ over a generic point in the singular locus. We have a natural identification

$$H_2(X, \mathbb{Z}) = H_2(S, \mathbb{Z}) \oplus \mathbb{Z}A.$$

The morphism r_X then sends (up to sign) the class $[A]$ to $1 \in \mathbb{Z}_{2n-2}$. (iii) In other words, we could have defined the residue class also by first deforming to the Hilbert scheme and then taking the coefficient of A modulo $2n - 2$. This is usually the practical way to compute the residue class. □

Let $\beta \in H_2(X, \mathbb{Z})$ be a class. The class β has then the following deformation invariants:

(i) the divisibility $\text{div}(\beta)$ in $H_2(X, \mathbb{Z})$,
(ii) the Beauville–Bogomolov norm $(\beta, \beta) \in \mathbb{Q}$, and
(iii) the residue set $\pm \left[\frac{\beta}{\text{div}(\beta)} \right] \in \mathbb{Z}/(2n - 2)\mathbb{Z}$.

The global Torelli theorem and Eichler's criterion yields the following.

Corollary 2. *Two pairs (X, β) and (X', β') of a $K3^{[n]}$-type variety and a class in $H_2(X, \mathbb{Z})$ which pairs positively with a Kähler class are deformation equivalent if and only if the invariants (i–iii) agree.*

Remark 3. By the global Torelli theorem, if β and β' are both of Hodge type, the deformation in the corollary can be chosen such that the curve class stays of Hodge type.

1.6. Lifts of isometries of H^2

Let X_1, X_2 be of $K3^{[n]}$-type, and let

$$g : H^2(X_1, \mathbb{C}) \longrightarrow H^2(X_2, \mathbb{C})$$

be an isometry. An operator $\widetilde{g} : H^*(X_1, \mathbb{C}) \to H^*(X_2, \mathbb{C})$ is a *parallel transport lift* of g if it is of the form

$$\widetilde{g} = \rho(g \circ \psi^{-1}, 1) \circ P_\psi$$

for a parallel transport operator $P_\psi : H^*(X_1, \mathbb{Z}) \to H^*(X_2, \mathbb{Z})$ with restriction $\psi = P|_{H^2(X_1, \mathbb{Z})}$. In particular, any parallel transport lift is a degree-preserving orthogonal ring isomorphism.

Recall from Section 1.3, Property 2, the operator

$$\widetilde{D} = \rho(\text{id}, -1) = D \circ \rho(-\text{id}_{H^2(X_2, \mathbb{Z})}, 1).$$

Lemma 1. *A parallel transport lift of g is unique up to composition by $\widetilde{D}$.*

Proof. Consider two parallel transport lifts of g,

$$\widetilde{g}_i = \gamma(g \circ \psi_i^{-1}) \circ P_{\psi_i}, \quad i = 1, 2$$

for parallel transport operators P_{ψ_1}, P_{ψ_2}. We will show that

$$\widetilde{g}_1 = \widetilde{g}_2 \quad \text{or} \quad \widetilde{g}_1 = \widetilde{D} \circ \widetilde{g}_2.$$

Let $\gamma(h) = \rho(h, 1)$. If $\tau(\psi_1 \circ \psi_2^{-1}) = 1$, then

$$\gamma(g \circ \psi_1^{-1}) \circ P_{\psi_1} \circ P_{\psi_2}^{-1} = \gamma(g\psi_1^{-1}) \circ \gamma(\psi_1 \circ \psi_2^{-1}) = \gamma(g\psi_2^{-1}).$$

If $\tau(\psi_1 \circ \psi_2^{-1}) = -1$, then

$$\gamma(g \circ \psi_1^{-1}) \circ P_{\psi_1} \circ P_{\psi_2}^{-1} = \gamma(g\psi_1^{-1}) \circ D \circ \gamma(-\psi_1 \circ \psi_2^{-1}) = \widetilde{D} \circ \gamma(g\psi_2^{-1})$$

$\square$

2. The multiple cover conjecture

2.1. Overview

Let X be a variety of $K3^{[n]}$-type, and let $\beta \in H_2(X, \mathbb{Z})$ be an effective curve class. The moduli space $\overline{M}_{g,N}(X, \beta)$ of N-marked genus g stable maps to X in class β carries a reduced virtual fundamental

class $[\overline{M}_{g,N}(X,\beta)]^{\text{red}}$ of dimension $(2n-3)(1-g)+N+1$. Gromov–Witten invariants of X are defined by pairing with this class:

$$\langle\alpha;\gamma_1,\ldots,\gamma_n\rangle^X_{g,\beta} := \int_{[\overline{M}_{g,n}(X,\beta)]^{\text{red}}} \pi^*(\alpha) \cup \prod_i \text{ev}_i^*(\gamma_i), \tag{6}$$

where $\text{ev}_i : \overline{M}_{g,n}(X,\beta) \to X$ are the evaluation maps, $\tau : \overline{M}_{g,n}(X,\beta) \to \overline{M}_{g,n}$ is the forgetful map and $\alpha \in H^*(\overline{M}_{g,n})$ is a *tautological class* [16].

In this section, we state a conjecture to express the invariants (6) for β an arbitrary class in terms of invariants where β is primitive.

2.2. Invariance

Let $\beta \in H_2(X,\mathbb{Z})$ be an effective curve class, and let

$$\widetilde{O}^+(H^2(X,\mathbb{Z}))_\beta \subset \widetilde{O}^+(H^2(X,\mathbb{Z}))$$

be the subgroup fixing β (either via the monodromy representation or equivalently, via the dual action on $H^2(X,\mathbb{Z})^\vee \cong H_2(X,\mathbb{Z})$ under the Beauville–Bogomolov form). Applying Remark 3 and the deformation invariance of the reduced Gromov–Witten invariants, we find that

$$\left\langle\alpha\,;\,\gamma_1,\ldots,\gamma_n\right\rangle^X_{g,\beta} = \left\langle\alpha\,;\,\mu(\varphi)\gamma_1,\,\ldots,\,\mu(\varphi)\gamma_n\right\rangle^X_{g,\beta}$$

for all $\varphi \in \widetilde{O}^+(H^2(X,\mathbb{Z}))_\beta$, where we have used

$$\mu : \widetilde{O}^+(H^2(X,\mathbb{Z})) \longrightarrow O(H^*(X,\mathbb{Z}))$$

to denote the monodromy representation (defined by the isomorphism (3) composed with the inclusion $\text{Mon}(X) \subset O(H^*(X,\mathbb{Z}))$).

The image of $\widetilde{O}^+(H^2(X,\mathbb{Z}))_\beta$ is Zariski dense in

$$G_\beta = (O(H^2(X,\mathbb{C})) \times \mathbb{Z}/2)_\beta$$
$$:= \{g \in O(H^2(X,\mathbb{C})) \times \mathbb{Z}/2\mathbb{Z} \mid \rho(g)\beta = \beta\}.$$

It follows that for all $g \in G_\beta$ we have

$$\left\langle\alpha\,;\,\gamma_1,\ldots,\gamma_n\right\rangle^X_{g,\beta} = \left\langle\alpha\,;\,\rho(g)\gamma_1,\,\ldots,\,\rho(g)\gamma_n\right\rangle^X_{g,\beta}. \tag{7}$$

Equivalently, the pushforward of the reduced virtual class lies in the invariant part of the diagonal G_β action:

$$\text{ev}_*\left(\tau^*(\alpha) \cap [\overline{M}_{g,n}(X,\beta)]^{\text{red}}\right) \in H^*(X^n,\mathbb{Q})^{G_\beta}.$$

The representation ρ restricted to $\{\text{id}\} \times \mathbb{Z}_2$ acts trivially on $H^2(X,\mathbb{C})$. Hence, for X a K3 surface, we obtain the invariance of the Gromov–Witten class under the group $O(H^2(X,\mathbb{C}))_\beta$. This matches what was conjectured in [45, Conj.C1] and then proven along the above lines in [9].

2.3. Multiple-cover conjecture

The main conjecture is the following: Let $\beta \in H_2(X, \mathbb{Z})$ be any effective curve class. For every divisor $k|\beta$, let X_k be a variety of $K3^{[n]}$ type and let

$$\varphi_k : H^2(X, \mathbb{R}) \longrightarrow H^2(X_k, \mathbb{R})$$

be a real isometry such that:

○ $\varphi_k(\beta/k)$ is a primitive curve class
○ $\pm[\varphi_k(\beta/k)] = \pm[\beta/k]$.

We extend φ_k as a *parallel transport lift* (Section 1.6) to the full cohomology:

$$\varphi_k : H^*(X, \mathbb{R}) \longrightarrow H^*(X_k, \mathbb{R}).$$

By Section 2.6 below, pairs (X_k, φ_k) satisfying these properties can always be found.

Conjecture B. *For any effective curve class $\beta \in H_2(X, \mathbb{Z})$, we have*

$$\left\langle \alpha; \gamma_1, \ldots, \gamma_N \right\rangle_{g,\beta}^X$$
$$= \sum_{k|\beta} k^{3g-3+N-\deg(\alpha)} (-1)^{[\beta]+[\beta/k]} \left\langle \alpha; \varphi_k(\gamma_1), \ldots, \varphi_k(\gamma_N) \right\rangle_{g,\varphi_k(\beta/k)}^{X_k}.$$

The invariance property discussed in Section 7 and Property 4 of Section 1.3 imply that the right-hand side of the conjecture is independent of the choice of (X_k, φ_k). Using that $\sum_{k|a} \mu(k) = \delta_{a1}$, Conjecture B is also seen to be equivalent to Conjecture A of the introduction.

The reduced Gromov–Witten invariants of X can only be nonzero if the dimension constraint

$$(\dim X - 3)(1 - g) + N + 1 = \deg(\alpha) + \sum_i \deg(\gamma_i)$$

is satisfied. Hence, the conjecture can also be rewritten as

$$\left\langle \alpha; \gamma_1, \ldots, \gamma_N \right\rangle_{g,\beta}^X = \sum_{k|\beta} k^{\dim(X)(g-1)-1+\sum_i \deg(\gamma_i)} (-1)^{[\beta]+[\beta/k]} \left\langle \alpha; \varphi_k(\gamma_1), \ldots, \varphi_k(\gamma_N) \right\rangle_{g,\varphi_k(\beta/k)}^X.$$

Since for K3 surfaces the residue always vanishes, Conjecture B specializes for K3 surfaces to the conjecture made in [45, Conj C2].

Remark 4. The condition that we ask of the residue, i.e., $\pm[\varphi_k(\beta/k)] = \pm[\beta/k]$, is necessary for the conjecture to hold. For example, consider X of $K3^{[5]}$-type and two primitive classes β_1, β_2 with $(\beta_1 \cdot \beta_1) = (\beta_2 \cdot \beta_2) = 16$ but $[\beta_1] = 0$ and $[\beta_2] = 4$ in $\mathbb{Z}/8\mathbb{Z}$. Then by [42] one has

$$\mathrm{ev}_*[\overline{M}_{0,1}(X, \beta_1)]^{\mathrm{red}} = 1464\beta_1^\vee, \quad \mathrm{ev}_*[\overline{M}_{0,1}(X, \beta_2)]^{\mathrm{red}} = 480\beta_2^\vee,$$

so an isometry taking β_1 to β_2 does not preserve Gromov–Witten invariants.

2.4. Uniruled divisors

An uniruled divisor $D \subset X$ which is swept out by a rational curve in class β is a component of the image of ev : $\overline{M}_{0,1}(X, \beta) \to X$. The virtual class of these uniruled divisors is given by the pushforward $\mathrm{ev}_*[\overline{M}_{0,1}(X, \beta)]^{\mathrm{red}}$. For β primitive and (X, β) very generally, the virtual class is closely related to the actual class [47].

By monodromy invariance (e.g., [47, Section 2.6]), there exists $N_\beta \in \mathbb{Q}$ such that

$$\mathrm{ev}_*[\overline{M}_{0,1}(X,\beta)]^{\mathrm{red}} = N_\beta \cdot h, \tag{8}$$

where $h = (\beta, -) \in H_2(X, \mathbb{Q})^\vee \cong H^2(X, \mathbb{Q})$ is the dual of β with respect to the Beauville–Bogomolov–Fujiki form. Since by deformation invariance N_β only depends on the divisibility $m = \mathrm{div}(\beta)$, the square $s = (\beta, \beta)$ and the residue $r = [\beta/\mathrm{div}(\beta)]$ we write

$$N_\beta = N_{m,s,r}.$$

Conjecture B then says that

$$N_{m,s,r} = \sum_{k \mid \beta} \frac{1}{k^3} (-1)^{mr + \frac{m}{k}r} N_{1, \frac{s}{k^2}, \frac{mr}{k}}. \tag{9}$$

The primitive numbers $N_{1,s,r}$ have been determined in [42].

Example 1. Let $X = \mathrm{Hilb}_2(S)$, and let $A \in H_2(X, \mathbb{Z})$ be the class of the exceptional curve. We have

$$\mathrm{ev}_*[\overline{M}_{0,1}(X,\beta)]^{\mathrm{red}} = \Delta_{\mathrm{Hilb}_2(S)} = -2\delta.$$

Since $A^\vee = -\frac{1}{2}\delta$, we see $N_{1,-\frac{1}{2},1} = 4$. The multiple cover formula then predicts that for even $\ell \in \mathbb{Z}_{\geq 1}$ we have

$$N_{\ell A} = -\frac{1}{\ell^3} N_{1,-\frac{1}{2},1} + \frac{1}{(\ell/2)^3} N_{1,-2,0} = \frac{-4}{\ell^3} + \frac{8}{\ell^3} = \frac{4}{\ell^3},$$

where we have used $N_{1,-2,0} = 1$. This matches the degree-scaling property discussed in [36, 48].

2.5. Fourfolds

We consider genus 0 Gromov–Witten invariants of a variety X of $K3^{[2]}$-type. By dimension considerations, all genus 0 Gromov–Witten invariants are determined by the two-point class:

$$\mathrm{ev}_*[\overline{M}_{0,2}(X,\beta)]^{\mathrm{red}} \in H^*(X \times X).$$

Following the arguments of [47, Sec.2][6], for every effective $\beta \in H_2(X, \mathbb{Z})$ there exist constants[7] $F_\beta, G_\beta \in \mathbb{Q}$ such that

$$\mathrm{ev}_*[\overline{M}_{0,2}(X,\beta)]^{\mathrm{red}} = F_\beta(h^2 \otimes h^2) + G_\beta\big(h \otimes \beta + \beta \otimes h + (h \otimes h) \cdot c_{BB}\big)$$
$$+ \left(-\frac{1}{30}(h^2 \otimes c_2 + c_2 \otimes h^2) + \frac{1}{900}(\beta, \beta)c_2 \otimes c_2\right)(G_\beta + (\beta, \beta)F_\beta),$$

where

○ $h = (\beta, -) \in H^2(X, \mathbb{Q})$ is the dual of the curve class,
○ $c_2 = c_2(X)$ is the second Chern class, and
○ $c_{BB} \in H^2(X) \otimes H^2(X)$ is the inverse of the Beauville–Bogomolov–Fujiki form.

---

[6]One uses that the class is monodromy invariant and a Lagrangian correspondence and that for very general (X, β) the image of the Hodge classes under this correspondence is annihilated by the symplectic form; see [47, Sec.1.3]. The reference treats only the case of primitive β, but the imprimitive case follows likewise with minor modifications.

[7]The constant N_β appearing in equation (8) is equal to G_β in the $K3^{[2]}$-case.

In $K3^{[2]}$-type, the residue $r = [\beta]$ of a curve class is determined by $s = (\beta, \beta)$ via $r = 2s \mod 2$. So we can write $F_\beta = F_{m,s}$ and $G_\beta = G_{m,s}$. The multiple cover conjecture for $K3^{[2]}$-type in genus 0 is then equivalent to

$$F_{m,s} = \sum_{k|m} \frac{1}{k^5} (-1)^{2(s+s/k^2)} F_{1,\frac{s}{k^2}}$$

$$G_{m,s} = \sum_{k|m} \frac{1}{k^3} (-1)^{2(s+s/k^2)} G_{1,\frac{s}{k^2}}.$$

We can also define

$$f_{m,s} = \sum_{k|m} \frac{\mu(k)}{k^5} (-1)^{2(s+s/k^2)} F_{\frac{m}{k},\frac{s}{k^2}}$$

$$g_{m,s} = \sum_{k|m} \frac{\mu(k)}{k^3} (-1)^{2(s+s/k^2)} G_{\frac{m}{k},\frac{s}{k^2}}.$$

and arrive at the following.

Lemma 2. *Conjecture* B *in* $K3^{[2]}$-type and genus 0 is equivalent to

$$\forall m, s: \ f_{m,s} = f_{1,s}, \quad g_{m,s} = g_{1,s}.$$

The first few cases are known.

Proposition 2. *Conjecture* B *in* $K3^{[2]}$-type and genus 0 holds for all classes β such that (i) $(\beta, \beta) < 0$ or (ii) $(\beta, \beta) = 0$ and $N = 1$.

Proof. The case $(\beta, \beta) < 0$ follows from [36, 48]. In case $(\beta, \beta) = 0$, the series $g_{m,s}$ is determined by intersecting the one-pointed class with a curve, and then using the methods of [42] to reduce to $\mathrm{Hilb}_2(\mathbb{P}^1 \times E)$. The resulting series is evaluated by T in [42, Thm.9]. □

For later use, we will also need to following expression for one-pointed descendence invariants:

$$\mathrm{ev}_* [\overline{M}_{0,1}(X, \beta)] = G_\beta \beta^\vee$$

$$\mathrm{ev}_* \left(\psi_1 \cdot [\overline{M}_{0,1}(X, \beta)] \right) = 2F_\beta h^2 - \frac{1}{15} \left(G_\beta + (\beta, \beta) F_\beta \right) c_2(X)$$

$$\mathrm{ev}_* \left(\psi_1^2 \cdot [\overline{M}_{0,1}(X, \beta)] \right) = -12 F_\beta$$

$$\mathrm{ev}_* \left(\psi_1^3 \cdot [\overline{M}_{0,1}(X, \beta)] \right) = 24 F_\beta.$$

This follows by monodromy invariance and topological recursions. In particular, to check the multiple cover formula in $K3^{[2]}$-type and genus 0 it is enough to consider one-point descendent invariants.

Remark 5. For convenience, we recall the evaluation of $f_{1,m,s}$ and $g_{1,m,s}$. Let

$$\vartheta(q) = \sum_{n \in \mathbb{Z}} q^{n^2}, \quad \alpha(q) = \sum_{\substack{\text{odd } n > 0 \\ d|n}} d q^n,$$

$$G_2(q) = -\frac{1}{24} + \sum_{n \geq 1} \sum_{d|n} d q^n, \quad \Delta(q) = q \prod_{n \geq 1} (1 - q^n)^{24}.$$

Then by [42] one has

$$\sum_s f_{1,s}(-q)^{4s} = \frac{1}{4} \frac{-1}{\vartheta(q)\alpha(q)\Delta(q^4)}$$

$$\sum_s g_{1,s}(-q)^{4s} = \frac{1}{12} \frac{\vartheta^4(q) + 4\alpha(q) + 24G_2(q^4)}{\vartheta(q)\alpha(q)\Delta(q^4)}.$$

2.6. Hilbert schemes of elliptic K3 surfaces

Let

$$\pi : S \longrightarrow \mathbb{P}^1$$

be an elliptic K3 surface with a section. Let $B, F \in H^2(S, \mathbb{Z})$ be the class of the section and a fiber, respectively, and let

$$W = B + F.$$

We consider the Hilbert scheme $X = \mathsf{Hilb}_n S$ and the generating series of Gromov–Witten invariants

$$F_{g,m}(\alpha; \gamma_1, \ldots, \gamma_N) = \sum_{d=-m}^{\infty} \sum_{r \in \mathbb{Z}} \langle \alpha; \gamma_1, \ldots, \gamma_N \rangle_{g,mW+dF+rA}^{\mathsf{Hilb}_n S} q^d (-p)^r.$$

We state a characterization of the multiple cover formula:

Consider the basis of $H^*(S, \mathbb{R})$ defined by

$$\mathcal{B} = \{1, \mathsf{p}, W, F, e_3, \ldots, e_{22}\},$$

where p is the class of a point and $e_3, \ldots, e_{22}$ is a basis of $\mathbb{Q}\langle W, F \rangle^{\perp}$ in $H^2(S, \mathbb{R})$. An element $\gamma \in H^*(X, \mathbb{R})$ is in the Nakajima basis with respect to $\mathcal{B}$ if it is of the form

$$\gamma = \prod_i \mathfrak{q}_{k_i}(\alpha_i) 1, \quad \alpha_i \in \mathcal{B}.$$

Let $w(\gamma)$ and $f(\gamma)$ be the number of classes α_i which are equal to W and F, respectively, and define a modified degree grading $\underline{\deg}$ by:

$$\underline{\deg}(\gamma) = \deg(\gamma) + w(\gamma) - f(\gamma).$$

For a series $f = \sum_{d,r} c(d,r) q^d p^r$, define the formal Hecke operator by

$$T_{m,\ell} f = \sum_{d,r} \left(\sum_{k | (m,d,r)} k^{\ell-1} c\left(\frac{md}{k^2}, \frac{r}{k} \right) \right) q^d p^r.$$

Lemma 3. *Conjecture B holds if and only if for all $m > 0$, all g, N, α and all $(\deg, \underline{\deg})$-bihomogeneous classes γ_i we have*

$$F_{g,m}(\alpha; \gamma_1, \ldots, \gamma_N) = m^{\sum_i \deg(\gamma_i) - \underline{\deg}(\gamma_i)} \cdot T_{m,\ell} F_{g,1}(\alpha; \gamma_1, \ldots, \gamma_N) \tag{10}$$

where $\ell = 2n(g-1) + \sum_i \underline{\deg}(\gamma_i)$.

Proof. Given the class $\beta = mW + dF + rA$ and a divisor $k \mid \beta$, consider the real isometry $\varphi_k : H^2(X, \mathbb{R}) \to H^2(X, \mathbb{R})$ defined by

$$
W \longmapsto \frac{k}{m} W
$$
$$
F \longmapsto \frac{m}{k} F
$$
$$
\gamma \longmapsto \gamma \text{ for all } \gamma \perp W, F.
$$

We extend this map to the full cohomology by $\varphi_k = \rho(\phi_k, 0) : H^*(X, \mathbb{R}) \to H^*(X, \mathbb{R})$. We then have

$$
\varphi_k \left(\frac{\beta}{k} \right) = W + \frac{dm}{k^2} F + \frac{r}{k} A.
$$

The Nakajima operators are equivariant with respect to the action of φ_k, and the isometry of $H^*(S, \mathbb{R})$ given by

$$
\widetilde{\varphi}_k = \varphi_k |_{H^2(S, \mathbb{R})} \oplus \mathrm{id}_{H^0(S, \mathbb{R}) \oplus H^4(S, \mathbb{R})};
$$

see Property 3 of Section 1.3.

Let $\gamma_i \in H^*(X, \mathbb{Q})$ be elements in the Nakajima basis with respect to $\mathcal{B}$. If Conjecture B holds, then its application with respect to φ_k yields:

$$
F_{g,m}(\alpha; \gamma_1, \ldots, \gamma_N) = \sum_{d,r} q^d p^r \sum_{k \mid (m,d,r)} (-1)^{r/k} k^{3g-3+N-\deg(\alpha)}
$$
$$
\times \left(\frac{m}{k} \right)^{\sum_i f(\gamma_i) - w(\gamma_i)} \langle \alpha; \gamma_1, \ldots, \gamma_N \rangle_{g, W + \frac{md}{k^2} F + \frac{r}{k} A}^{\mathsf{Hilb}_n}. \tag{11}
$$

Using the dimension constraint, our modified degree function and the formal Hecke operators this becomes: Define the weight of a cohomology class in the Nakajima basis with respect to $\mathcal{B}$ by

$$
\mathsf{wt} \left(\prod_i \mathfrak{q}_{a_i}(\alpha_i) 1 \right) = \sum_i \mathsf{wt}(\alpha_i),
$$

where

$$
\mathsf{wt}(\alpha) = \begin{cases} 1 & \text{if } \alpha \in \{\mathsf{p}, W\} \\ -1 & \text{if } \alpha \in \{1, F\} \\ 0 & \text{if } \alpha \in \{1, F, W, \mathsf{p}\}^{\perp}. \end{cases}
$$

For a homogeneous $\gamma \in H^*(\mathsf{Hilb}_n(S))$, we then set $\underline{\deg}(\gamma) = \mathsf{wt}(\gamma) + n$.

Since the Nakajima operator $\mathfrak{q}_i(\alpha)$ has degree $\overline{i - 1} + \deg(\alpha)$, we have for $\gamma = \prod_{i=1}^{\ell} \mathfrak{q}_i(\alpha_i) 1 \in H^*(\mathsf{Hilb}_n S)$ that

$$
\underline{\deg}(\gamma) = n - \ell + \sum_i \underline{\deg}(\alpha_i).
$$

Hence, we can rewrite

$$\sum_{i=1}^{n} \deg(\gamma_i) + w - f = nN - \sum_i \ell(\gamma_i) + \sum_{i,j} \deg(\alpha_{ij}) + w - f$$

$$= nM + \sum_i \mathsf{wt}(\gamma_i)$$

$$= \sum_i \underline{\deg}(\gamma_i)$$

$$F_{g,m}(\alpha; \gamma_1, \ldots, \gamma_N) = m^{\sum_i \deg(\gamma_i) - \underline{\deg}(\gamma_i)}$$
$$\times \sum_{d,r} q^d p^r \sum_{k \mid (m,d,r)} (-1)^{r/k} k^{2n(g-1)-1+\sum_i \underline{\deg}(\gamma_i)} \langle \alpha; \gamma_1, \ldots, \gamma_N \rangle_{g, W + \frac{md}{k^2} F + \frac{r}{k} A}^{\mathsf{Hilb}_n}$$

(12)

which is equation (10).

Conversely, equality (10) implies Conjecture B since any pair (X, β) is deformation equivalent to some $(\mathsf{Hilb}_n(S), \tilde{\beta} = mW + dF + kA)$ with $m > 0$, and the right-hand side of Conjecture B is independent of choices. □

We will reinterpret the lemma in terms of Jacobi forms in [44]. See also [1] for a parallel discussion in the case of K3 surfaces.

3. Noether–Lefschetz theory

3.1. Lattice polarized holomorphic-symplectic varieties

Let V be the abstract lattice, and let $L \subset V$ be a primitive nondegenerate sublattice[8].

An L-polarization of a holomorphic-symplectic variety X is a primitive embedding

$$j : L \hookrightarrow \mathrm{Pic}(X)$$

such that

○ there exists an isometry $\varphi : V \xrightarrow{\cong} H^2(X, \mathbb{Z})$ with $\varphi|_L = j$, and
○ the image $j(L)$ contains an ample class.

We call the isometry φ as above a marking of (X, j). If the image $j(L)$ only contains a big and nef line bundle, we say that X is L-quasipolarized. Let $\mathcal{M}_L$ be the moduli space of L-quasipolarized holomorphic-symplectic varieties of a given fixed deformation type.

The period domain associated to $L \subset V$ is

$$\mathcal{D}_L = \{x \in \mathbb{P}(L^\perp \otimes \mathbb{C}) \mid \langle x, x \rangle = 0, \langle x, \bar{x} \rangle > 0\}$$

and has two connected components. Let $\mathcal{D}_L^+$ be one of these components. Consider the subgroup

$$\mathrm{Mon}(V) \subset O(V)$$

which, for some choice of marking $\varphi : V \to H^2(X, \mathbb{Z})$ for some (X, j) defining a point in $\mathcal{M}_L$, can be identified with the monodromy group $\mathrm{Mon}^2(X)$ of X.[9] Let $\mathrm{Mon}(V)_L$ be the subgroup of $\mathrm{Mon}(V)$ that

---

[8]i.e., the quotient is torsion free.
[9]If the monodromy group $\mathrm{Mon}(V)$ is normal in $O(V)$, then it does not depend on the choice of (X, j); this is known for all known examples of holomorphic-symplectic varieties.

acts trivially on L. Then the global Torelli theorem says that the period mapping

$$\text{Per} : \mathcal{M}_L \longrightarrow \mathcal{D}_L^+/\text{Mon}(V)_L$$

is surjective, restricts to an open embedding on the open locus of L-polarized holomorphic symplectic varieties and any fiber consists of birational holomorphic symplectic varieties; see [34] for a survey and references.

3.2. One-parameter families

Let $L_i, i = 1, \ldots, \ell$ be an integral basis of L.

Given a compact complex manifold $\mathcal{X}$ of dimension $2n + 1$, line bundles

$$\mathcal{L}_1, \ldots, \mathcal{L}_\ell \in \text{Pic}(\mathcal{X})$$

and a morphism $\pi : \mathcal{X} \to C$ to a smooth proper curve, following [26, 0.2.2], we call the tuple $(\mathcal{X}, \mathcal{L}_1, \ldots, \mathcal{L}_\ell, \pi)$ a *one-parameter family of L-quasipolarized holomorphic-symplectic varieties* if the following holds:

 (i) For every $t \in C$, the fiber $(X_t, L_i \mapsto \mathcal{L}_i|_{X_t})$ is a L-quasipolarized holomorphic-symplectic variety.
 (ii) There exists an vector $h \in L$ which yields a quasi-polarization on all fibers of π simultaneously.

Any one-parameter family as above defines a morphism $\iota_\pi : C \to \mathcal{M}_L$ into the moduli space of L-quasipolarized holomorphic-symplectic varieties (of the deformation type specified by a fiber).

3.3. Noether–Lefschetz cycles

Given primitive sublattices $L \subset \widetilde{L} \subset V$, consider the open substack

$$\mathcal{M}'_{\widetilde{L}} \subset \mathcal{M}_{\widetilde{L}}$$

parametrizing pairs $(X, j : \widetilde{L} \hookrightarrow \text{Pic}(X))$ such that $j(L)$ contains a quasi-polarization. There exists a natural proper morphism $\iota : \mathcal{M}'_{\widetilde{L}} \to \mathcal{M}_L$ defined by restricting j to L. The Noether–Lefschetz cycle associated to $\widetilde{L}$ is the class of the reduced image of this map:

$$\text{NL}_{\widetilde{L}} = [\iota(\mathcal{M}'_{\widetilde{L}})] \in A^c(\mathcal{M}_L).$$

The codimension c of the cycle is given by $\text{rank}(\widetilde{L}) - \text{rank}(L)$. For $c = 1$, we call $\text{NL}_{\widetilde{L}}$ a Noether–Lefschetz divisor of the first type.

3.4. Heegner divisors

We review the construction of Heegner divisors. Their relation to Noether–Lefschetz divisors will yield modularity results for intersection numbers with Noether–Lefschetz divisors.

Consider the lattice

$$M = L^\perp \subset V$$

and the subgroup

$$\Gamma_M = \{g \in O^+(M) | g \text{ acts trivially on } M^\vee/M\},$$

where $O^+(M)$ stands for those automorphisms which preserve the orientation, or equivalently, the component $\mathcal{D}_L^+$. We consider the quotient

$$\mathcal{D}_L^+/\Gamma_M.$$

For every $n \in \mathbb{Q}_{<0}$ and $\gamma \in M^\vee/M$, the associated Heegner divisor is:

$$y_{n,\gamma} = \left(\sum_{\substack{v \in M^\vee \\ \frac{1}{2} v \cdot v = n, \, [v] = \gamma}} v^\perp \right) / \Gamma_M.$$

For $n = 0$, we define $y_{n,\gamma}$ by the descent $\mathcal{K}$ of the tautological line bundle $\mathcal{O}(-1)$ on $\mathcal{D}_L$ equipped with the natural Γ_M-action. Concretely, we set

$$y_{0,\gamma} = \begin{cases} c_1(\mathcal{K}^*) & \text{if } \gamma = 0 \\ 0 & \text{otherwise.} \end{cases}$$

In case $n > 0$, we set $y_{n,\gamma} = 0$ in all cases.

Define the formal power series of Heegner divisors

$$\Phi(q) = \sum_{n \in \mathbb{Q}_{\leq 0}} \sum_{\gamma \in M^\vee/M} y_{n,\gamma} q^{-n} e_\gamma$$

which is an element of $\mathrm{Pic}(\mathcal{D}_L^+/\Gamma_M)[[q^{1/N}]] \otimes \mathbb{C}[M^\vee/M]$, where e_γ are the elements of the group ring $\mathbb{C}[M^\vee/M]$ indexed by γ and N is the smallest integer for which $M^\vee(N)$ is an even lattice.

We recall the modularity result of Borcherds in the formulation of [37].

Theorem 4. *([5, 39]) The generating series $\Phi(q)$ is the Fourier-expansion of a modular form of weight* $\mathrm{rank}(M)/2$ *for the dual of the Weil representation $\rho_M^\vee$ of the metaplectic group $\mathrm{Mp}_2(\mathbb{Z})$:*

$$\Phi(q) \in Pic(\mathcal{D}_L^+/\Gamma_M) \otimes \mathsf{Mod}(\mathrm{Mp}_2(\mathbb{Z}), \mathrm{rank}(M)/2, \rho_M^\vee).$$

The modular forms for the dual of the Weil representations can be computed easily by a Sage program of Brandon Williams [53].

3.5. Noether–Lefschetz divisors of the second type

The precise relationship between Noether–Lefschetz and Heegner divisors for arbitrary holomorphic-symplectic varieties is somewhat painful to state. For once the monodromy group $\mathrm{Mon}^2(X)$ is not known in general, and even if it is known it usually does not contain Γ_M or is contained in Γ_M. To simplify the situation, we from now on restrict to the case of $K3[n]$-type for $n \geq 2$. Hence, we let

$$V = E_8(-1)^{\oplus 2} \oplus U^3 \oplus (2 - 2n),$$

and we fix an identification $V^\vee/V = \mathbb{Z}/(2n - 2)\mathbb{Z}$.

We define the Noether–Lefschetz divisors of second type:

$$\mathsf{NL}_{s,d,\pm r} \in A^1(\mathcal{M}_L),$$

where $d = (d_1, \ldots, d_\ell) \in \mathbb{Z}^\ell$, $s \in \mathbb{Q}$ and $r \in \mathbb{Z}/(2n-2)\mathbb{Z}$ are given. Consider the intersection matrix of the basis L_i:

$$\mathsf{a} = (a_{ij})_{i,j=1}^\ell = (L_i \cdot L_j)_{i,j=1}^\ell.$$

We set

$$\Delta(s,d) = \det\begin{pmatrix} a & d^t \\ d & s \end{pmatrix} = \det\begin{pmatrix} a_{11} & \cdots & a_{1\ell} & d_1 \\ \vdots & & \vdots & \vdots \\ a_{\ell 1} & \cdots & a_{\ell\ell} & d_\ell \\ d_1 & \cdots & d_\ell & s \end{pmatrix}.$$

Case: $\Delta(s,d) \neq 0$. We define

$$\mathsf{NL}_{s,d,\pm r} = \sum_{L \subset \tilde{L} \subset V} \mu(s,d,r | L \subset \tilde{L} \subset V) \cdot \mathsf{NL}_{\tilde{L}},$$

where the sum runs over all isomorphism classes of primitive embeddings $L \subset \tilde{L} \subset V$ with $\mathrm{rank}(\tilde{L}) = \ell + 1$. The multiplicity[10] $\mu(s,d,r | L \subset \tilde{L} \subset V)$ is the number of elements $\beta \in V^\vee$ which are contained in $\tilde{L} \otimes \mathbb{Q}$ and satisfy:

$$\beta \cdot L_i = d_i, \quad \beta \cdot \beta = s, \quad \pm[\beta] = \pm r \text{ in } \mathbb{Z}/(2n-2).$$

Here we have used the canonical embeddings $V^\vee \subset V \otimes \mathbb{Q}$ and $\tilde{L} \otimes \mathbb{Q} \subset V \otimes \mathbb{Q}$.

Case: $\Delta(s,d) = 0$. In this case, any curve class with these invariants has to lie in $L \otimes \mathbb{Q}$ and is uniquely determined by the degree d. Hence, we let $\beta \in L \otimes \mathbb{Q}$ be the unique class so that $\beta \cdot L_i = d_i$ for all i.[11] If β lies in $V^\vee$ and has residue $[\beta] = \pm r$, we define

$$\mathsf{NL}_{s,d,\pm r} = c_1(\mathcal{K}^\vee),$$

and we define $\mathsf{NL}_{s,d,\pm r} = 0$ otherwise.

Remark 6. Often the residue set $\pm[\beta]$ of a class $\beta \in H_2(X,\mathbb{Z})$ is determined by the degrees $d_i = \beta \cdot L_i$. For example, if L contains a class ℓ such that

$$\langle \ell, H^2(X,\mathbb{Z}) \rangle = (2n-2)\mathbb{Z}, \quad \langle \ell, H_2(X,\mathbb{Z}) \rangle = \mathbb{Z}$$

we may define a natural isomorphism by cupping with ℓ:

$$H_2(X,\mathbb{Z})/H^2(X,\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}/(2n-2)\mathbb{Z}, \gamma \longmapsto \gamma \cdot \ell.$$

(Not every polarization is of that form, for example the case of double covers of EPW sextics.) In other cases, the residue set is determined by the norm $\beta \cdot \beta$, for example in $K3^{[2]}$-type. When the residue is determined by s and d, we will drop it from the notation of Noether–Lefschetz divisors. □

---

[10]For this construction it would be more natural to work with pairs of a holomorphic-symplectic varieties and a primitive embedding $j : L \to N_1(X)$ into the group of effective 1-cycles $N_1(X) \subset H_2(X,\mathbb{Z})$. If we then consider a rank 1 overlattice $L \subset \tilde{L} \subset N_1(X)$, we define the multiplicity μ as the number of $\beta \in \tilde{L}$ such that $\beta \cdot L_i = d_i, \beta \cdot \beta = s$ and $[\beta] = \pm r$. The condition above is more cumbersome but equivalent to this definition.

[11]The class is given by $\sum_{i,j=1}^\ell d_i (a^{-1})_{ij} L_j$.

3.6. Heegner and Noether–Lefschetz divisors

By the result of Markman, $\mathsf{Mon}(V) \subset O(V)$ is the subgroup of orientation-preserving isometries which act by $\pm\mathrm{id}$ on the discriminant. Hence, we have the inclusion $\Gamma_M \subset \mathsf{Mon}(V)_L$ of index 1 or 2. This yields the diagram

$$
\begin{array}{c}
\mathcal{D}_L^+/\Gamma_M \\
\downarrow{\scriptstyle \pi} \\
\mathcal{M}_L \xrightarrow{\ \mathrm{Per}\ } \mathcal{D}_L^+/\mathsf{Mon}(V)_L,
\end{array}
$$

where π is either an isomorphism or of degree 2.

Let $C \subset \mathcal{M}_L$ be a complete curve, and define the modular form

$$
\Phi_C(q) = \langle \Phi(q), \pi^*[\mathrm{Per}(C)] \rangle.
$$

We write $\Phi_C[n, \gamma]$ for the coefficient of $q^n e^\gamma$ in the Fourier-expansion of Φ_C.

We will need also

$$
\widetilde{\Delta}(s, d) := -\frac{1}{2} \cdot \frac{1}{\det(a)} \det\begin{pmatrix} a & d^t \\ d & s \end{pmatrix}.
$$

The following gives the main connection between the Noether–Lefschetz divisors of the second type and the Heegner divisors.

Proposition 3. *There exists a canonically defined class $\gamma(s, d, r) \in M^\vee/M$ (abbreviated also by $\gamma(r)$) such that we have the following:*

(a) *If π is an isomorphism,*

$$
C \cdot \mathsf{NL}_{s,d,\pm r} = \Phi_C\left[\widetilde{\Delta}(s, d), \gamma(r)\right].
$$

(b) *If π is of degree 2,*

$$
C \cdot \mathsf{NL}_{s,d,\pm r} = \begin{cases} \frac{1}{2}\Phi_C\left[\widetilde{\Delta}(s, d), \gamma(r)\right] & \text{if } r = -r \\ \frac{1}{2}\left(\Phi_C\left[\widetilde{\Delta}(s, d), \gamma(r)\right] + \Phi_C\left[\widetilde{\Delta}(s, d), \gamma(-r)\right]\right) & \text{otherwise.} \end{cases}
$$

In $K3^{[2]}$-type, we have $V^\vee/V = \mathbb{Z}_2$ so that π is an isomorphism. Moreover, the residue r of any $\beta \in V^\vee$ is determined by its norm $s = \beta \cdot \beta$. Hence, omitting r from the notation we find the following.

Corollary 3. *In $K3^{[2]}$ type, there exists a canonical class $\gamma = \gamma(d, s)$ with*

$$
C \cdot \mathsf{NL}_{s,d} = \Phi_C[\widetilde{\Delta}(s, d), \gamma].
$$

Remark 7. In fact, in $K3^{[2]}$ type, the proof below will imply the equality of divisors

$$
\mathsf{NL}_{s,d} = \Phi[\widetilde{\Delta}(s, d), \gamma] = y_{-\widetilde{\Delta}(s,d),\gamma}
$$

on $\mathcal{M}_L$, where we have omitted the pullback by the period map Per on the right-hand side.

For the proof of Proposition 3, we will repeatedly use the following basic linear algebra fact whose proof we skip.

Lemma 4. *Consider an $\mathbb{R}$-vector space Λ with inner product $\langle -, - \rangle$ and an orthogonal decomposition $L \oplus M = \Lambda$. Let L_i be a basis of L with intersection matrix $a_{ij} = L_i \cdot L_j$. For $\beta \in \Lambda$ with $d_i = \beta \cdot L_i$, let $v = \beta - \sum_{i,j} d_i a^{ij} L_j$ be the projection of β onto M, where a^{ij} are the entries of a^{-1}. Then we have*

$$\langle v, v \rangle = \frac{1}{\det(a)} \det \begin{pmatrix} a & d^t \\ d & \langle \beta, \beta \rangle \end{pmatrix},$$

where $d = (d_1, \ldots, d_\ell)$.

The main step in the proof of the proposition is given by the following lemma: For fixed $d = (d_1, \ldots, d_\ell), s$ and $r \in \mathbb{Z}_{2n-2}$ consider the divisor on $\mathcal{D}_L / \Gamma_M$ given by

$$\mathsf{NL}_{s,d,r} = \left(\sum_\beta \beta^\perp \right) / \Gamma_M,$$

where the sum is over all classes $\beta \in V^\vee$ such that

$$\beta \cdot \beta = s, \quad \beta \cdot L_i = d_i, i = 1, \ldots, \ell, \quad \text{and} \quad [\beta] = r \in V^\vee / V. \tag{13}$$

Moreover, $\beta^\perp$ stands for the hyperplane in $\mathbb{P}(V)$ orthogonal to β intersected with the period domain D_L.

Lemma 5. *There exists a canonically defined class $\gamma = \gamma(s, d, r) \in M^\vee / M$ such that*

$$\mathsf{NL}_{s,d,r} = y_{n,\gamma} \in A^1(\mathcal{D}_L / \Gamma_M),$$

where $n = \frac{1}{2} \frac{1}{\det(a)} \det \begin{pmatrix} a & d \\ d & s \end{pmatrix}$.

Proof. In view of the definition of both sides of the claimed equation, it is enough to establish a bijection between

(a) the set of classes $\beta \in V^\vee$ satisfying (13), and
(b) the set of classes $v \in M^\vee$ satisfying $v^2 = \frac{1}{\det(a)} \det \begin{pmatrix} a & d \\ d & s \end{pmatrix}$ and $[v] = \gamma$ for an appropriately defined γ.

Consider a primitive embedding $V \subset \Lambda$ into the Mukai lattice

$$\Lambda = E_8(-1)^{\oplus 2} \oplus U^{\oplus 4}.$$

Let e, f be a symplectic basis of one summand of U. We choose the embedding such that $V^\perp = \mathbb{Z}L_0$, where $L_0 = e + (n - 1)f$. Since Λ is unimodular, there exists a canonical isomorphism

$$V^\vee / V \cong (\mathbb{Z}L_0)^\vee / \mathbb{Z}L_0, \tag{14}$$

and we may assume that under this isomorphism the class $L_0 / (2n - 2) \mod \mathbb{Z}L_0$ corresponds to $1 \in \mathbb{Z}/(2n - 2)\mathbb{Z}$.

Step 1. Let $d_0 \in \mathbb{Z}$ be any integer such that $d_0 \equiv r$ modulo $2n - 2$, and let $\tilde{s} \in 2\mathbb{Z}$ such that

$$s = \frac{1}{2n - 2} \det \begin{pmatrix} 2n - 2 & d_0 \\ d_0 & \tilde{s} \end{pmatrix} \iff \tilde{s} = s + \frac{d_0^2}{2n - 2}.$$

(We may assume such $\tilde{s}$ exists: Otherwise, the set in (a) is empty and by the argument below also the set in (b)). Then we claim that there exists a bijection between the set in (a) and

(c) the set of classes $\tilde{\beta} \in \Lambda$ such that $\tilde{\beta} \cdot L_i = d_i$ for all $i = 0, \ldots, \ell$ and $\tilde{\beta} \cdot \tilde{\beta} = \tilde{s}$.

Proof of Step 1. Given $\widetilde{\beta}$ satisfying the conditions in (c), then

$$\beta = \widetilde{\beta} - \frac{d_0}{2n-2} L_0$$

lies in $V^\vee$. Moreover, $[\beta]$ is the class in $V^\vee/V$ corresponding to $d_0/(2n-2)$ in $(\mathbb{Z}L_0)^\vee/\mathbb{Z}L_0$ under (14); hence, $[\beta] = r$. Also, $\beta \cdot L_i = d_i$ for $i = 1, \ldots, \ell$. The equality $\beta \cdot \beta = s$ is by definition of $\widetilde{s}$ and Lemma 4.

Conversely, let $\delta = -e + (n-1)f$, and observe that $L_0 \cdot \delta = 0$ and $L_0/(2n-2) + \delta/(2n-2) = f$. Hence, if β satisfies (a) then β is an element of $d_0 \cdot \frac{\delta}{2n-2} + V$, and hence,

$$\beta \longmapsto \beta + \frac{d_0}{2n-2} L_0 \in \Lambda$$

defines the required inverse. $\qquad\square$

We consider now the embedding $M \subset \Lambda$ and the orthogonal complement

$$\widehat{L} = M^\perp.$$

Since Λ is unimodular, we have an isomorphism

$$\widehat{L}^\vee/\widehat{L} \cong M^\vee/M.$$

We specify the class γ via this isomorphism. Concretely, we set

$$\gamma := \left[\sum_{i,j=0}^{\ell} d_i a^{ij} L_j \right] \in \widehat{L}^\vee/\widehat{L},$$

where we let a^{ij} denote the entries of the inverse of the extended intersection matrix $\hat{a} = (L_i \cdot L_j)_{i,j=0,\ldots,\ell}$.

If we replace d_0 by $d_0 + (2n-2)$, then since $a^{00} = 1/(2n-2)$ and $a^{0j} = 0$ for $j \neq 0$, the expression $\sum d_i a^{ij} L_j$ gets replaced by the same expression plus L_0. Hence, the class γ only depends on $s, (d_1, \ldots, d_\ell), r$.

Step 2. There exists a bijection between the classes in (c) and (b).

Proof of Step 2. We have the bijection

$$\widetilde{\beta} \longmapsto \widetilde{\beta} - \sum_{i,j=0}^{r} d_i a^{ij} L_j \in M^\vee.$$

$\qquad\square$

Combining Step 1 and 2 finished the proof of the lemma. $\qquad\square$

Proof of Proposition 3. By definition we have:

$$\mathrm{NL}_{s,d_1,\ldots,d_r,\pm r} = \mathrm{Per}^* \left[\left(\sum_{\beta} \beta^\perp \right) / \mathrm{Mon}(V)_L \right],$$

where the sum is over all $\beta \in V^\vee$ such that

$$\beta \cdot \beta = s, \quad \beta \cdot L_i = d_i, \quad [\beta] = \pm r.$$

Hence, if π is an isomorphism, the result follows from this by Lemma 5.

Hence, assume now π is of degree 2, and let $\widetilde{C} = \mathrm{Per}(C)$.

If $-r = r$, then the morphism $\mathrm{NL}_{s,d,r} \to \mathrm{NL}_{s,d,\pm r}$ given by restriction of π is of degree 2. Therefore,

$$C \cdot \mathrm{NL}_{s,d,\pm r} = \frac{1}{2}\widetilde{C} \cdot \pi_* \mathrm{NL}_{s,d,r} = \frac{1}{2}\pi^*[\widetilde{C}] \cdot \mathrm{NL}_{s,d,r}$$

which then implies the claim by Lemma 5. If $r \neq -r$, then we have $\pi_* \mathrm{NL}_{s,d,r} = \mathrm{NL}_{s,d,\pm r}$ from which the result follows. □

3.7. Noether–Lefschetz numbers

Let $(\mathcal{X}, \mathcal{L}_1, \ldots, \mathcal{L}_\ell, \pi)$ be a one-parameter family of L-quasipolarized holomorphic-symplectic varieties of $K3^{[n]}$-type. We have the associated classifying morphism

$$\iota_\pi : C \longrightarrow \mathcal{M}_L.$$

We define the Noether–Lefschetz numbers of the family by

$$\mathrm{NL}^\pi_{s,d,\pm r} = \int_C \iota_\pi^* \mathrm{NL}^\pi_{s,d,\pm r}.$$

Intuitively, the Noether–Lefschetz numbers are the number of fibers of π for which there exists a Hodge class β with prescribed norm $\beta \cdot \beta = s$, degree $\beta \cdot L_i = d_i$ and residue $[\beta] = \pm r$.

In $K3^{[2]}$-type we will also often write $\Phi^\pi(q) = \iota_\pi^* \Phi(q)$.

The families of holomorphic-symplectic varieties we will encounter in geometric constructions often come with mildly singular fibers. The definition of Noether–Lefschetz numbers can be extended to these families as follows. Let $\pi : \mathcal{X} \to C$ be a projective flat morphism to a smooth curve, and let $\mathcal{L}_1, \ldots, \mathcal{L}_\ell \in \mathrm{Pic}(\mathcal{X})$. We assume that over a nonempty open subset of C this defines a one-parameter family of L-quasipolarized holomorphic-symplectic varieties of $K3^{[n]}$ type. We also assume that around every singular point the monodromy is finite. Then there exists a cover

$$f : \widetilde{C} \longrightarrow C$$

such that the pullback family $f^*\mathcal{X} \to \widetilde{C}$ is bimeromorphic to a one-parameter family of L-quasipolarized holomorphic-symplectic varieties of $K3^{[n]}$-type,

$$\widetilde{\pi} : \tilde{\mathcal{X}} \longrightarrow \widetilde{C}.$$

See, for example, [28]. Concretely, around each basepoint of a singular fiber, after a cover that trivializes the monodromy, the rational map $C \dashrightarrow \mathcal{M}_L$ can be extended. (In the examples we will consider, we can construct the cover $\tilde{C} \to C$ and the birational model $\tilde{\mathcal{X}}$ explicitly). We define the Noether–Lefschetz numbers of π by

$$\mathrm{NL}^\pi_{s,d,\pm r} := \frac{1}{k}\mathrm{NL}^{\widetilde{\pi}}_{s,d,\pm},$$

where k is the degree of the cover $\widetilde{C} \to C$. Since the Noether–Lefschetz divisors are pulled back from the separated period domain, the definition is independent of the choice of cover.

3.8. Example: Prime discriminant in $K3^{[2]}$-type

Let

$$V = E_8(-1)^{\oplus 2} \oplus U^{\oplus 3} \oplus \mathbb{Z}\delta, \quad \delta^2 = -2$$

Table 1. *The dimension of the space of modular forms of weight* 11 *for the Weil representation associated to M.*

p	3	7	11	19	23
dim	2	4	6	9	12

be the $K3^{[2]}$-lattice and consider a primitive vector $H \in V$ satisfying

○ $H \cdot H = 2p$ for a prime p with $p \equiv 3 \bmod 4$,
○ $\langle H, V \rangle = 2\mathbb{Z}$.

Equivalently, $H/2$ defines a primitive vector in $V^\vee$ and has norm $p/2$. By Eichler's criterion [19, Lemma 3.5], there exists a unique $O(V)$ orbit of vectors H satisfying these condition. To be concrete, we choose

$$H = 2\left(e' + \frac{p+1}{4}f'\right) + \delta,$$

where e', f' is a basis of one of the summands U. In V one then has

$$M = H^\perp \cong E_8(-1)^{\oplus 2} \oplus U^{\oplus 2} \oplus \begin{pmatrix} -2 & -1 \\ -1 & -\frac{p+1}{2} \end{pmatrix},$$

a lattice of discriminant group $\mathbb{Z}/p\mathbb{Z}$.

We consider H-quasipolarized holomorphic-symplectic varieties X of $K3^{[2]}$-type. Examples are the Fano varieties of lines ($p = 3$) or the Debarre–Voisin fourfolds ($p = 11$); see below. For these varieties, the Borcherds modular forms and the relationship between between Noether–Lefschetz divisors of first and second type can be described very explicitly.

3.8.1. The Borcherds modular forms

Consider the series of Noether–Lefschetz numbers of second type

$$\Phi^\pi(q) = \sum_\gamma \Phi_\gamma^\pi(q)e_\gamma$$

for a one-parameter family π of holomorphic-symplectic varieties of this polarization type. This is a modular form of weight 11 for the Weil representation on $M^\vee/M$. The space of such forms is easily computed through [53], and the first values are given in the following table.

If we write y_1, y_2 for the standard basis of the lattice $\begin{pmatrix} -2 & -1 \\ -1 & -\frac{p+1}{2} \end{pmatrix}$, then the discrimimant of M is generated by

$$y' = \frac{1}{p}(2y_2 - y_1)$$

which has norm $y' \cdot y' = -2/p$. Hence, for any element v of $M^\vee$, written as

$$v = w + ky' \in M^\vee, \quad w \in M, k \in \mathbb{Z},$$

we have $-\frac{1}{2}pv \cdot v = k^2$ modulo p. In particular, this determines $[v] \in \mathbb{Z}/p\mathbb{Z}$ up to multiplication by ± 1. Thus, for any $v \in M^\vee$, we see that

(i) $D := -\frac{p}{2}v \cdot v$ is a square modulo p, and
(ii) $r = [v]$ is determined from D via $r^2 \equiv D \bmod p$, up to multiplication by ± 1.

By the redundancy of Heegner divisors $y_{n,\gamma} = y_{n,-\gamma}$, the coefficient $q^n e_\gamma$ of $\Phi(q)$ is thus determined by n alone. It is, hence, enough to consider

$$\varphi^\pi(q) = \frac{1}{2}\Phi_0^\pi(q) + \frac{1}{2}\sum_{\gamma \in M^\vee/M} \Phi_\gamma^\pi(q). \tag{15}$$

Let χ_p be the Dirichlet character given by the Legendre symbol $\left(\frac{\cdot}{p}\right)$.

Proposition 4. *The series $\Phi_0^\pi(q)$ and $\sum_{\gamma \in M^\vee/M} \Phi_\gamma^\pi(q^p)$ are modular forms of weight 11 and character χ_p for the congruence subgroup $\Gamma_0(p)$.*

Proof. The modularity of the first series is well-known [4]. The second is one direction of the Bruinier–Bundschuh isomorphism [6]. □

The generators of the ring of modular forms for the character χ_p is easily computable (see, e.g., [4, Sec.12]) which yields explicit formulas for φ^π. One example for Fano varieties can be found in [31]. We will consider the case of Debarre–Voisin fourfolds below.

Finally, the Noether–Lefschetz numbers of the family are given by

$$\mathrm{NL}_{s,d}^\pi = \varphi^\pi\left[-\frac{1}{4p}\det\begin{pmatrix} 2p & d \\ d & s \end{pmatrix}\right].$$

3.8.2. Noether–Lefschetz divisors the first type

The relationship between Noether–Lefschetz divisors of the first and second type is not so easy to state in general. However, here the situation simplifies. For any $w \in H^\perp \subset V$, we consider the intersection of $w^\perp$ with the period domain $\mathcal{D}_H$,

$$\mathcal{D}_{w^\perp} = \{x \in \mathcal{D}_H | \langle x, w \rangle = 0\}.$$

The image of this divisor under the quotient map $\mathcal{D}_H \to \mathcal{D}_H/\Gamma_M$ defines an irreducible divisor that by a result of Debarre and Macrì [14] only depends on the discriminant

$$-2e := \mathrm{disc}(w^\perp \subset M).$$

Moreover, e is a square modulo p. We write $\mathcal{C}_{2e}$ for this divisor.

The relationship between Noether–Lefschetz divisors of first and second type is given as follows.

Proposition 5. *Let $D \geq 1$ be a square modulo p, and let $\alpha \in \mathbb{Z}/p\mathbb{Z}$ such that $\alpha^2 \equiv D \bmod p$. The associated Heegner divisor $y_{-D/p,\alpha}$, denoted also by $\mathrm{NL}(D)$, is given by*

$$\mathrm{NL}(D) = \sum_{\substack{a_0 \geq 0, k \in \{0,\ldots,\lfloor\frac{p}{2}\rfloor\} \\ e = pa_0 + k^2 \geq 1}} \left|\left\{c \in \mathbb{Z} \,\middle|\, c^2 = \frac{D}{e}, \; kc \equiv \alpha \bmod p\right\}\right| \mathcal{C}_{2e}.$$

In particular, we have

$$\mathrm{NL}(D) = \begin{cases} \mathcal{C}_{2D} + \ldots & \text{if } D \neq 0 \bmod 11 \\ 2\mathcal{C}_{2D} + \ldots & \text{if } D = 0 \bmod 11, \end{cases}$$

where ... stands for terms $\mathcal{C}_{2e}$ with $e < D$. This shows that the Noether–Lefschetz divisors of the first type are related to the Heegner divisors by an invertible upper triangular matrx. If D is square free, then $\mathrm{NL}(D)$ and $\mathcal{C}_{2D}$ agree up to a constant.

Proof. For any positive $e = pa_0 + k^2$ with $k \in \{0, 1 \ldots, \lfloor \frac{p}{2} \rfloor\}$ and $a \geq 1$, we choose a lattice $K_e \subset V$ containing H and such that $\mathrm{disc}(K_e^\perp) = -2e$. The lattice is unique up to an automorphism of V that fixes H [14]. Fix $s \in \frac{1}{2}\mathbb{Z}$ with $2s \equiv 3(4)$ and $d \geq 1$ such that $D = -\frac{1}{4} \det \begin{pmatrix} 2p & d \\ d & s \end{pmatrix} = \frac{1}{4}(d^2 - 2ps)$. Then by Proposition 3, Remark 7 and the definition we have

$$\mathsf{NL}(D) = \mathsf{NL}_{s,d} = \sum_e \mu(K_e, s, d) \mathcal{C}_{2e},$$

where the multiplicity is given by

$$\mu(K_e, s, d) = |\{\beta \in K_e \otimes \mathbb{Q} | \beta \in V^\vee, \beta \cdot \beta = s, \beta \cdot H = d\}|. \tag{16}$$

It remains to calculate the multiplicity. We first embed V into the Mukai lattice Λ as the orthogonal of $e + f$ such that $\delta = -e + f$. Here, e, f is a symplectic basis of a not previously used copy of U. One finds that

$$\widehat{L} = (M^\perp \subset \Lambda) \cong \begin{pmatrix} 2 & 1 \\ 1 & \frac{p+1}{2} \end{pmatrix}$$

which has the integral basis

$$x_1 = e + f, \quad x_2 = e' + \frac{p+1}{4} f' + f.$$

Let us next choose

$$K_e = \mathbb{Z}H \oplus \mathbb{Z}(kf' + e'' - a_0 f''),$$

where e'', f'' is a symplectic basis of a third copy of U. The saturation of $K_e \oplus \mathbb{Z}(e + f)$ inside Λ is then given by

$$\widetilde{K}_e \cong \begin{pmatrix} 2 & 1 & 0 \\ 1 & \frac{p+1}{2} & k \\ 0 & k & -2a_0 \end{pmatrix},$$

where the lattice is generated by x_1, x_2 and $x_3 = kf' + e'' - a_0 f''$.

We follow the recipe of the proof of Lemma 5, that is we compare the multiplicity (16) with a simpler multiplicity for $\tilde{K}_e$. If $s \in 2\mathbb{Z}$, then for D to be an integer, we must have d even. Then as in Lemma 5, one gets

$$\mu(K_e, s, d) = \left| \left\{ \beta \in \widetilde{K}_e \middle| \beta \cdot x_1 = 0, \ \beta \cdot x_2 = \frac{d}{2}, \ \beta \cdot \beta = s \right\} \right|.$$

If $s + \frac{1}{2} \in \mathbb{Z}$, then d is odd, and

$$\mu(K_e, s, d) = \left| \left\{ \beta \in \widetilde{K}_e \middle| \beta \cdot x_1 = 1, \ \beta \cdot x_2 = \frac{d+1}{2}, \ \beta \cdot \beta = s + \frac{1}{2} \right\} \right|.$$

The result follows from this by a direct calculation. For exposition, we evaluate the multiplicity in the first case. Using that $\beta \cdot x_1 = 0$, any element $\beta \in \tilde{K}_e$ as on the right-hand side is given by

$$\beta = a(-x_1 + 2x_2) + cx_e.$$

Let $\tilde{d} = d/2$. The condition $\beta \cdot x_2 = \tilde{d}$ yields $ap + kc = \tilde{d}$ which can be solved if and only if $kc \equiv \tilde{d}$ mod p, in which case $a = (\tilde{d} - kc)/p$. Inserting this expression into $\beta \cdot \beta$ yields

$$c^2 e = \tilde{d}^2 - \frac{p}{2}s = -\frac{1}{4}(2ps - d^2) = D.$$

Finally, $D = \tilde{d}^2$ mod p, and hence, if $\alpha^2 = D$ mod p, then $\alpha = \pm\tilde{d}$. If $\alpha \equiv 0$ mod p, then the result follows. In the other case, among $c \in \{\pm\sqrt{D/e}\}$, there is precisely one solution to $kc \equiv \tilde{d}$ mod p if and only if there is precisely one solution to $kc \equiv \alpha$ mod p. □

3.9. Example: Cubic fourfolds

We consider Fano varieties of lines

$$X \subset \mathrm{Gr}(2, 6)$$

of a cubic fourfold. By [2], the Plücker polarization is of square 6 and pairs evenly with any class in $H^2(X, \mathbb{Z})$. Hence, their deformation type is governed by the discussion in Section 3.8 for $p = 3$. The Borcherds modular form for the generic pencil of Fano varieties is computed in [31].

Let $U \subset \mathbb{P}(H^0(\mathbb{P}^5, \mathcal{O}(3)))$ be the open locus corresponding to cubic fourfolds with at worst ADE singularities. There is a period mapping

$$p : U \longrightarrow \mathcal{M}_H$$

to the corresponding moduli space. The pullback of the divisors $\mathcal{C}_{2e}$ under this mapping are the special cubic fourfolds of discriminant $d = 2e$; see [21, 31]. (A cubic fourfold $Y \subset \mathbb{P}^5$ is special if it contains an algebraic surface S such that the saturation of $[S]$ and h^2 is of discriminant d).

For the one-parameter family π of Fano varieties of lines of a generic pencil of cubic fourfolds, the Noether–Lefschetz numbers of the second type $\mathrm{NL}^\pi_{s,d}$ and of first type

$$\mathrm{NL}^\pi(D) = \deg \iota_\pi^* \mathrm{NL}(D) \tag{17}$$

are then related to the classical geometry of special cubic fourfolds. For example,

$$\mathrm{NL}^\pi_{-2,0} = \mathrm{NL}^\pi(D = 3) = 192$$
$$\mathrm{NL}^\pi_{-2,4} = \mathrm{NL}^\pi(D = 7) = 917,568$$

are the degrees of the (closure of the) divisors in $\mathbb{P}(H^0(\mathbb{P}^5, \mathcal{O}(3)))$ parametrizing nodal and Pfaffian cubics, respectively. The locus of determinantal cubic fourfolds $p^{-1}\mathcal{C}_2$ is of codimension ≥ 2; see, e.g., [23, Rmk 3.23], and hence,

$$\mathrm{NL}^\pi_{-1/2,1} = \mathrm{NL}^\pi(D = 1) = 0.$$

Thus, one gets that

$$\mathrm{NL}^\pi_{-5/2,1} = \mathrm{NL}^\pi(D = 4) = 3402$$

which is the degree of the locus $p^{-1}\mathcal{C}_8$ of cubics containing a plane. The equalities of the Noether–Lefschetz numbers of first and second type above follow from Proposition 5: In the first three cases, since D is square free, and in the last case we use that $\mathcal{C}_2$ does not meet the curve defined by π.

3.10. Example II: Debarre–Voisin fourfolds

A Debarre–Voisin fourfold [15] is the holomorphic-symplectic variety

$$X \subset \mathrm{Gr}(6, 10)$$

given as the vanishing locus of a section of $\Lambda^3 \mathcal{U}^\vee$, where $\mathcal{U} \subset \mathbb{C}^{10} \otimes \mathcal{O}$ is the universal subbundle on the Grassmannian. These varieties are of $K3^{[2]}$-type, and the Plücker polarization is of degree $H^2 = 22$ and pairs evenly with any class in $H^2(X, \mathbb{Z})$. Hence, we are in the situation of Section 3.8 for $p = 11$. The Noether–Lefschetz numbers for a generic pencil of these varieties will be computed below.

3.11. Refined Noether–Lefschetz divisors

We will need refined Noether–Lefschetz divisors which also depend on the divisibility $m \geq 1$ of the curve class. Refined Noether–Lefschetz numbers are then defined as usually by intersection with Noether–Lefschetz divisors. As before, we assume that we are in $K3^{[n]}$-type. Let $s \in \mathbb{Q}$, $d = (d_1, \ldots, d_\ell) \in \mathbb{Z}^\ell$ and $r \in \mathbb{Z}_{2n-2}$ be fixed.

If $\Delta(s, d) \neq 0$, we set

$$\mathrm{NL}_{m,s,d,\pm r} = \sum_{L \subset \tilde{L} \subset V} \mu(m, s, d, r | L \subset \tilde{L} \subset V) \cdot \mathrm{NL}_{\tilde{L}},$$

where the refined multiplicity $\mu(\ldots)$ is the number of classes $\beta \in V^\vee$ which are contained in $\tilde{L} \otimes \mathbb{Q}$, satisfy $\beta \cdot \beta = s$, $\beta \cdot L_i = d_i$ and such that the following new conditions hold:

$$\mathrm{div}(\beta) = m, \qquad \left[\frac{\beta}{\mathrm{div}(\beta)} \right] = \pm r \in \mathbb{Z}/(2n-2)\mathbb{Z}.$$

Note that we treat the residue different from the nonrefined case.

If $\Delta(s, d) = 0$, we define

$$\mathrm{NL}_{m,s,d,\pm r} := \mathrm{NL}_{s,d,\pm m \cdot r}$$

if m is the gcd of $d_1, \ldots, d_r$ and the unique class $\beta \in L \otimes \mathbb{Q}$ with $\beta \cdot L_i = d_i/m$ lies in $V^\vee$ and has residue $[\beta] = \pm r$. Otherwise, we set $\mathrm{NL}_{m,s,d,\pm r} = 0$.

We then have

$$\mathrm{NL}_{s,d,\pm r} = \sum_{m \geq 1} \sum_{\substack{\pm r' \\ \pm m \cdot r' = \pm r}} \mathrm{NL}_{m,s,d,\pm r'} \tag{18}$$

and

$$\mathrm{NL}_{m,s,d,\pm r} = \mathrm{NL}_{1, s/m^2, d/m, \pm r}.$$

By a simple induction argument as in [26, Lemma 1], these two equations show that the data of the unrefined Noether–Lefschetz numbers are equivalent to the the data of the refined Noether–Lefschetz numbers/divisors.

Remark 8. If the residue of a class is determined by d and s, the inverse relation between refined and unrefined is easy to state. We simply have

$$\mathsf{NL}_{1,s,d} = \sum_{k|\gcd(d_1,\ldots,d_\ell)} \mu(k) \cdot \mathsf{NL}_{s/k^2,d/k},$$

parallel to the multiple cover rule we study in this paper.

4. Gromov–Witten theory and Noether–Lefschetz theory

Let V be the $K3^{[n]}$-lattice, and let $L \subset V$ be a fixed primitive sublattice with integral basis L_i. We consider a one-parameter family

$$\pi : \mathcal{X} \longrightarrow C, \quad \mathcal{L}_1, \ldots, \mathcal{L}_\ell \in \mathrm{Pic}(\mathcal{X})$$

of L-quasipolarized holomorphic-symplectic varieties of $K3^{[n]}$-type.

The goal of this section is to relate Gromov–Witten invariants of $\mathcal{X}$ in fiber classes to the Noether–Lefschetz numbers of the family and the reduced Gromov–Witten invariants in $K3^{[n]}$-type.

4.1. Gromov–Witten invariants of the family

Let $\gamma_i \in H^*(\mathcal{X})$ be cohomology classes which can be written in terms of polynomials p_i in the Chern classes of $\mathcal{L}_i$,

$$\gamma_i = p_i(c_1(\mathcal{L}_1), \ldots, c_1(\mathcal{L}_\ell)).$$

Let $\overline{M}_{g,N}(\mathcal{X}, d)$ for $d \in \mathbb{Z}^\ell$ be the moduli space of N-marked genus g stable maps $f : C \to \mathcal{X}$ such that

○ f maps into the fibers of $\mathcal{X}$, that is $\pi_* f_*[C] = 0$, and
○ f is of degree d_i against L_i,

$$\int_{[C]} f^*(c_1(\mathcal{L}_i)) = d_i.$$

We consider the invariants

$$\langle \alpha; \gamma_1, \ldots, \gamma_N \rangle_{g,d}^{\mathcal{X}} = \int_{[\overline{M}_{g,n}(\mathcal{X},d)]} \tau^*(\alpha) \, \mathrm{ev}_1^*(\gamma_1) \cdots \mathrm{ev}_N^*(\gamma_N),$$

where $\alpha \in H^*(\overline{M}_{g,n})$ is tautological and τ is the forgetful map.

4.2. Gromov–Witten invariants of the fiber

Let X be any holomorphic-symplectic variety of $K3^{[n]}$-type, and let $\beta \in H_2(X, \mathbb{Z})$ be an effective curve class. Assume there exists an embedding

$$L \otimes \mathbb{R} \hookrightarrow H^2(X, \mathbb{R})$$

which is an isometry onto its image such that $\beta \cdot L_i = d_i$ for all i. As usual, we let $L_i \in H^2(X, \mathbb{R})$ denote the image of $L_i \in L$ under this map. Let also

$$\gamma_i = p_i(L_1, \ldots, L_\ell).$$

By deformation invariance and the invariance property of Section 2.2, the reduced Gromov–Witten invariant $\langle \alpha; \gamma_1, \ldots, \gamma_N \rangle_{g,\beta}^X$ only depends on the degree $d = (d_1, \ldots, d_\ell)$, the polynomials p_i, $s = \beta \cdot \beta$ and the curve invariants $m = \mathsf{div}(\beta)$ and the residue set $\pm r = \pm [\beta/\mathsf{div}(\beta)]$. We write

$$\langle \alpha; \gamma_1, \ldots, \gamma_N \rangle_{g,\beta}^X = \langle \alpha; p_1, \ldots, p_N \rangle_{g,m,s,d,\pm r}^X.$$

4.3. The relation

Consider the refined Noether–Lefschetz numbers of π,

$$\mathsf{NL}_{m,s,d,\pm r}^\pi := \int_C \iota_\pi^* \mathsf{NL}_{m,s,d,\pm r'},$$

where $\iota_\pi : C \to \mathcal{M}_L$ is the morphism defined by the family.

Proposition 6. *Let* $\gamma_i = p_i(\mathcal{L}_1, \ldots, \mathcal{L}_\ell) \in H^*(\mathcal{X})$. *Then we have*

$$\langle \alpha; \gamma_1, \ldots, \gamma_N \rangle_{g,d}^{\mathcal{X}} = \sum_{m,s,\pm r} \mathsf{NL}_{m,s,d,\pm r}^\pi \cdot \langle \alpha \cdot (-1)^g \lambda_g; p_1, \ldots, p_N \rangle_{g,m,s,d,\pm r}^X.$$

Here, λ_i are the i-th Chern classes of the Hodge bundle on the moduli space of stable curves. The proposition can be extended to more general classes γ_i. It is enough to assume that γ_i is the product of some polynomial in the $\mathcal{L}_i$ and a class which restricts to a monodromy invariant class on each fiber, for example, a Chern class.

Proof. The proof follows by the identical argument as for the K3 surfaces, as discussed in [37, Section 3.2]. The above equality in the K3 case is [37, Eqn. (17)]. As in [37], for each $\xi \in C$, we want to group together all curve classes in $H_2(\mathcal{X}_\xi, \mathbb{Z})$ of degree d which have the same Gromov–Witten invariants. By Corollary 2, we, hence, may group together classes of the same square, the same divisibility and the same residue. Thus, we replace the set $B_\xi(m, h, d)$ of [37, Sec.3.2] by

$$B_\xi(m, s, d, \pm r) = \left\{ \beta \in H_2(\mathcal{X}_\xi, \mathbb{Z}) \, \middle| \, \begin{array}{c} (\beta, \beta) = s, \ \mathsf{div}(\beta) = m, \\ [\beta/\mathsf{div}(\beta)] = \pm r, \ \beta \cdot L_i = d_i \\ \beta \perp H^{2,0}(\mathcal{X}_\xi, \mathbb{C}) \end{array} \right\}.$$

The rest of the argument of [37, Sec.3] goes through without change. □

4.4. Reformulation

We can rewrite Proposition 6 in terms of invariants where we have formally subtracted multiple cover contrubtions. For simplicity, assume that for $\beta \in H_2(X, \mathbb{Z})$ the residue $r([\beta])$ is determined by the degrees $d_i = \beta \cdot L_i$. Write $r(d)$ for the residue. Proposition 6 then says that

$$\langle \alpha; \gamma_1, \ldots, \gamma_N \rangle_{g,d}^{\mathcal{X}} = \sum_{m,s} \mathsf{NL}_{m,s,d}^\pi \cdot \langle \alpha(-1)^g \lambda_g; p_1, \ldots, p_N \rangle_{g,m,s,d}^X.$$

Let us subtract formally the multiple cover contributions from the invariants of $\mathcal{X}$,

$$\langle \alpha; \gamma_1, \ldots, \gamma_N \rangle_{g,d}^{\mathcal{X},\mathrm{mc}} := \sum_{k|d} (-1)^{r(d)+r(d/k)} \mu(k) k^{2g-3+N-\deg(\alpha)} \langle \alpha; \gamma_1, \ldots, \gamma_N \rangle_{g,d/k}^{\mathcal{X}}$$

as well as from the the reduced Gromov–Witten invariants,

$$\left\langle \alpha(-1)^g \lambda_g; p_1, \ldots, p_N \right\rangle_{g,m,s,d}^{X,\mathrm{mc}}$$
$$:= \sum_{k \mid m} (-1)^{r(d)+r(d/k)} k^{2g-3+N-\deg(\alpha)} \mu(k) \left\langle \alpha(-1)^g \lambda_g; p_1, \ldots, p_N \right\rangle_{g,m/k,s/k^2,d/k}^{X}.$$

The following is the result of a short calculation.

Lemma 6. *We have*

$$\left\langle \alpha; \gamma_1, \ldots, \gamma_N \right\rangle_{g,d}^{\mathcal{X},mc} = \sum_{m,s} \mathsf{NL}_{m,s,d} \cdot \left\langle \alpha(-1)^g \lambda_g; p_1, \ldots, p_N \right\rangle_{g,m,s,d}^{X,mc}.$$

In particular, if the multiple cover conjecture (Conjecture B) holds, after subtracting the multiple cover contributions, the invariants of X do not depend on the divisibility m, and so with equation (18), we obtain

$$\left\langle \alpha; \gamma_1, \ldots, \gamma_N \right\rangle_{g,d}^{\mathcal{X},\mathrm{mc}} = \sum_s \left\langle \alpha(-1)^g \lambda_g; p_1, \ldots, p_N \right\rangle_{g,1,s,d}^{X,\mathrm{mc}} \sum_{m,s} \mathsf{NL}_{m,s,d}$$
$$= \sum_s \mathsf{NL}_{s,d} \cdot \left\langle \alpha(-1)^g \lambda_g; p_1, \ldots, p_N \right\rangle_{g,1,s,d}^{X,\mathrm{mc}}. \tag{19}$$

5. Mirror symmetry

5.1. Overview

In this section, we review how to use mirror symmetry formulas to compute the genus 0 Gromov–Witten invariants for the total space $\mathcal{X}$ of generic pencils of Fano varieties of lines of cubic fourfolds and of Debarre–Voisin varieties.

Mirror symmetry here means an application of the following results: Givental's description of the I-function for complete intersections in toric varieties [18], the proof of the abelian/nonabelian correspondence by Webb that relates the I-function of a GIT quotient with that of its abelian quotient [52] and the genus 0 wallcrossing formula between quasi-maps and Gromov–Witten invariants for GIT quotients by Ciocan-Fontanine and Kim [11].

We first determine the small I-function for the cases we are interested in, then we shortly recall how to relate the I and J functions. We assume basic familiarity with the language of [11, 52] throughout.

5.2. I-functions

We work in the following setup: Let V be a vector space over $\mathbb{C}$, and let G be a connected reductive group acting faithfully on V on the left. We also fix a character of G for which we assume that the semistable and stable locus, denoted by $V^s(G)$, agrees. For simplicity, we also assume that the G-action on the stable locus is free. We consider the GIT quotient

$$Y = V /\!/ G = V^s(G)/G.$$

Let $T \subset G$ be a maximal torus, and consider also the abelian quotient $V^s(T)/T$. We have then the following diagram relating the abelian and nonabelian quotients:

$$\begin{array}{ccc} V^s(G)/T & \xrightarrow{\ j\ } & V^s(T)/T \\ \downarrow{\scriptstyle \xi} & & \\ V^s(G)/G. & & \end{array}$$

The Weyl group W of G acts naturally on the cohomology of $V^s(G)/T$, and one has the isomorphism

$$\xi^* : H^*(V^s(G)/G, \mathbb{Q}) \xrightarrow{\cong} H^*(V^s(G)/T, \mathbb{Q})^W.$$

Let E be a G-representation, and consider a smooth zero locus of the associated homogeneous bundle E on Y,

$$X \subset Y.$$

The small I-function of X in Y is a formal series

$$I^X = I^{Y,E} = 1 + \sum_{\beta \neq 0} q^\beta I_\beta(z),$$

where $\beta \in H_2(Y, \mathbb{Z})$ runs over all curve classes, q^β is a formal variable and $I_\beta(z)$ is a formal series in $z^{\pm 1}$ with coefficients in $H^*(Y, \mathbb{Q})$. It can then be determined in the following steps.

Abelian/Nonabelian correspondence ([52]).

$$\xi^* I_\beta^{Y,E} = j^* \sum_{\tilde{\beta} \mapsto \beta} \left(\prod_\alpha \frac{\prod_{k=-\infty}^{\tilde{\beta} \cdot c_1(L_\alpha)} (c_1(L_\alpha) + kz)}{\prod_{k=-\infty}^0 (c_1(L_\alpha) + kz)} \right) I_{\tilde{\beta}}^{V//T, E},$$

where α runs over the roots of G and L_α is the associated line bundle on the abelian quotient, $\tilde{\beta} \in H_2(V//T, \mathbb{Z}) = \operatorname{Hom}(\chi(T), \mathbb{Z})$ runs over the characters of T that restrict to the given character $\beta \in H_2(Y, \mathbb{Z}) = \operatorname{Hom}(\chi(G), \mathbb{Z})$ under the map induced by $\chi(G) \to \chi(T)$. When it is clear from context, we will often omit the pullbacks ξ^* and j^* from the notation.

Twisting ([18]). When restricting the G-representation E to T, it decomposes into a direct sum of one-dimensional representations M_i. We write M_i also for the associated line bundles on $V//T$. Then

$$I_\beta^{V//T, E} = \left(\prod_{i=1}^{\operatorname{rk}(E)} \prod_{k=1}^{c_1(M_i) \cdot \beta} (c_1(M_i) + kz) \right) \cdot I_\beta^X.$$

Toric varieties ([18]). Let $D_i, i = 1, \ldots, n$ be the torus invariant divisors on the toric variety $V//T$.

$$I_\beta^{V//T} = \prod_{i=1}^n \frac{\prod_{k=-\infty}^0 (D_i + kz)}{\prod_{k=-\infty}^{D_i \cdot \beta} (D_i + kz)}.$$

Example 2. (Projective space $\mathbb{P}^{n-1}$) We have $I_d^{\mathbb{P}^{n-1}} = (\prod_{k=1}^d (H + kz)^n)^{-1}$.

Example 3. (Grassmannian) Let $M_{k \times n}$ be the space of $k \times n$-matrices acted on by $\operatorname{GL}(k)$ on the left. Taking the determinant character, the stable locus is the locus of matrices of full rank and the associated GIT quotient is the Grassmannian

$$\operatorname{Gr}(k, n) = M_{k \times n} //_{\det} \operatorname{GL}(k).$$

The stable locus for the maximal torus $T \subset \operatorname{GL}(k)$ of diagonal matrices is given by matrices where each row is nonzero. The abelian quotient is

$$M_{k \times n} //T = \underbrace{\mathbb{P}^{n-1} \times \ldots \times \mathbb{P}^{n-1}}_{k \text{ times}}.$$

The roots of $\operatorname{GL}(k)$ are $e_i^* - e_j^*$ and correspond to $\mathcal{O}(H_i - H_j)$, where H_i is the hyperplane class pulled back from the i-th factor.

The universal subbundle $\mathcal{U} \to \mathbb{C}^n \otimes \mathcal{O}_{\mathrm{Gr}}$ on the Grassmannian corresponds to the inclusion of G-representations

$$M_{k \times n} \times \mathbb{C}^k \longrightarrow M_{k \times n} \times \mathbb{C}^n,$$

where a column vector $w \in \mathbb{C}^k$ is acting on by $g \cdot w := (g^t)^{-1} w$, and $\mathbb{C}^n$ carries the trivial representation. The Plücker polarization on $\mathrm{Gr}(k, n)$ thus corresponds to the line bundle $\mathcal{O}(H_1 + \ldots + H_k)$ on $(\mathbb{P}^{n-1})^k$. Hence, if we consider degree d curves on the Grassmannian, in the abelian/nonabelian correspondence we have to sum over $(d_1, \ldots, d_k)$ adding up to d.

Calculating the I-function is then easy. For example, for $k = 2$ (and dropping the pullbacks ξ^*, j^* from notation), one obtains

$$I_d^{\mathrm{Gr}(2,n)} = \sum_{d = d_1 + d_2} (-1)^d \frac{H_1 - H_2 + (d_1 - d_2)z}{H_1 - H_2} \frac{1}{\prod_{k=1}^{d_1} (H_1 + kz)^n \prod_{k=1}^{d_2} (H_2 + kz)^n},$$

where the division by $H_1 - H_2$ is to take place formally.

Example 4. (Fano variety of a cubic fourfold) The Fano variety of a cubic fourfold $X \subset \mathrm{Gr}(2, 6)$ is a zero locus of a section of $\mathrm{Sym}^3(U^\vee)$. On the abelian quotient $\mathbb{P}^5 \times \mathbb{P}^5$, this vector bundle corresponds to

$$\mathcal{O}(3H_1) \oplus \mathcal{O}(2H_1 + H_2) \oplus \mathcal{O}(H_1 + 2H_2) \oplus \mathcal{O}(3H_2).$$

We find the I-function

$$I^{X \subset \mathrm{Gr}(2,6)} = I_d^{\mathrm{Gr}(2,6)} \cdot \prod_{\substack{3 = i_1 + i_2 \\ i_1, i_2 \geq 0}} \prod_{k=1}^{i_1 d_1 + i_2 d_2} (i_1 H_1 + i_2 H_2 + kz).$$

Example 5. (A pencil of cubic fourfolds) We consider a generic pencil of cubic fourfolds $\mathcal{X} \subset \mathrm{Gr}(2, 6) \times \mathbb{P}^1$. Since $\mathcal{X}$ is the zero locus of a generic section of the globally generated bundle $\mathrm{Sym}^3(U^\vee) \otimes \mathcal{O}_{\mathbb{P}^1}(1)$, it is smooth by a Bertini type argument. The abelian quotient is $\mathbb{P}^5 \times \mathbb{P}^5 \times \mathbb{P}^1$. Let h be the hyperplane class on $\mathbb{P}^1$. Then the I-function for the fiber part reads:

$$I_{(d,0)}^{\mathcal{X}} = (-1)^d \sum_{d = d_1 + d_2} \frac{H_1 - H_2 + (d_1 - d_2)z}{H_1 - H_2} \cdot$$

$$\frac{1}{\prod_{k=1}^{d_1} (H_1 + kz)^6 \prod_{k=1}^{d_2} (H_2 + kz)^6} \prod_{\substack{3 = i_1 + i_2 \\ i_1, i_2 \geq 0}} \prod_{k=1}^{i_1 d_1 + i_2 d_2} (i_1 H_1 + i_2 H_2 + h + kz).$$

Example 6. (A pencil of Debarre–Voisin fourfolds) We consider a pencil $\mathcal{X} \subset \mathrm{Gr}(6, 10) \times \mathbb{P}^1$ of DV fourfolds which is cut out by $\wedge^3 \mathcal{U}^\vee \otimes \mathcal{O}(1)$. The abelian quotient is $(\mathbb{P}^9)^6 \times \mathbb{P}^1$. We let h be the hyperplane class of $\mathbb{P}^1$ and H_i be the hyperplane class pulled back from the i-th copy of $\mathbb{P}^9$. Then the I-function in

the fiber class is

$$
\begin{aligned}
I^{\mathcal{X}}_{(d,0)} = (-1)^d \sum_{d=d_1+\ldots+d_6} &\left(\prod_{1 \le i < j \le 6} \frac{H_i - H_j + (d_i - d_j)z}{H_i - H_j} \right) \\
&\times \prod_{i=1}^{6} \prod_{k=1}^{d_i} \frac{1}{(H_i + kz)^{10}} \\
&\times \prod_{1 \le i_1 < i_2 < i_3 \le 6} \prod_{k=1}^{d_{i_1} + d_{i_2} + d_{i_3}} (H_{i_1} + H_{i_2} + H_{i_3} + h + kz).
\end{aligned}
$$

5.3. *I and J functions*

Given a GIT quotient X as before, let $t \in H^2(X, \mathbb{C})$ be any element (or a formal variable). The big *J*-function (at $\epsilon = \infty$) with insertion t is

$$
J^{\infty}(q, t, z) = e^{\frac{t}{z}} \left[1 + \sum_{\beta \ne 0} e^{\beta \cdot t} q^{\beta} \, \mathrm{ev}_* \left(\frac{[\overline{M}_{0,1}(X, \beta)]^{\mathrm{vir}}}{z(z - \psi)} \right) \right].
$$

Expand the small *I*-function according to degree:

$$
I = I_0(q) + \frac{I_1(q)}{z} + \frac{I_2(q)}{z^2} + \ldots .
$$

Then the mirror theorem [11] states that

$$
J^{\infty}\left(q, \frac{I_1(q)}{I_0(q)}, z \right) = \frac{I(q, z)}{I_0(q)}.
$$

By inverting this relation, we can compute the descendent one-point invariants of X. We sketch the details for the case of the Lefschetz pencil $\mathcal{X} \subset \mathrm{Gr}(2, 6) \times \mathbb{P}^1$ of Fano's. In this case (with H the Plücker polarization on $\mathrm{Gr}(2, 6)$), let us write

$$
I_0(q) = f_0(q), \quad I_1(q) = f_1(q)H + f_2(q)h.
$$

Then with $t = I_1(q)/I_0(q)$ and since β is fiber we get

$$
e^{\beta \cdot t} q^d = \left(q e^{\frac{f_1(q)}{f_0(q)}} \right)^d = Q^d,
$$

where we have identified $q^{\beta} = q^d$ and used the variable

$$
Q = q \exp\left(\frac{f_1(q)}{f_0(q)} \right).
$$

Then we obtain the relation

$$
\exp\left(-\frac{I_1(q)}{I_0(q)} \frac{1}{z} \right) \frac{I^{\mathrm{fib}}(q, z)}{I_0(q)} = 1 + \sum_{\beta \ne 0} Q^{\beta} \, \mathrm{ev}_* \left(\frac{[\overline{M}_{0,1}(X, \beta)]^{\mathrm{vir}}}{z(z - \psi)} \right),
$$

where $I^{\mathrm{fib}}(q, z)$ stands for the *I*-functions involving only fiber classes β.

6. Results

6.1. Cubic fourfolds

We consider a generic pencil of Fano varieties of cubic fourfolds

$$\mathcal{X} \subset \mathrm{Gr}(2,6) \times \mathbb{P}^1.$$

This defines a one-parameter family $\pi : \mathcal{X} \to \mathbb{P}^1$ polarized by the Plücker embeddings. The family has precisely 192 singular fibers $\mathcal{X}_t$, which are irreducible varieties with ordinary double point singularities along a smooth K3 surface (and smooth elsewhere) [12]. The blowup $\mathrm{Bl}_S \mathcal{X}_t$ along the singular locus is isomorphic to the Hilbert scheme $\mathrm{Hilb}_2(S)$, and the blowdown map contracts a $\mathbb{P}^1$-bundle over S along its fibers which are (-2)-curves [21, Sec.6.3]. The map $\mathcal{X} \to \mathbb{P}^1$ is a ordinary double point degeneration to $\mathcal{X}_t$.

To obtain a family of smooth holomorphic-symplectic manifolds, we follow the arguments of Maulik and Pandharipande [37, Sec.5.1]. We choose a double cover

$$\epsilon : C \longrightarrow \mathbb{P}^1$$

which is ramified along the 192 base points of nodal fibers. The family

$$\epsilon^* \mathcal{X} \longrightarrow C$$

then has double point singularities along the surfaces S which can be resolved by a small resolution

$$\tilde{\pi} : \tilde{\mathcal{X}} \longrightarrow C.$$

The family $\tilde{\pi}$ is a one-parameter family of quasi-polarized $K3^{[2]}$-type varieties, polarized by the pullback of the Plücker polarization.

The Noether–Lefschetz numbers of the family π in terms of $\tilde{\pi}$ are then

$$\mathrm{NL}^{\pi}_{s,d} = \frac{1}{2} \mathrm{NL}^{\tilde{\pi}}_{s,d}.$$

We also have the following comparison of Gromov–Witten invariants of the total spaces of $\mathcal{X}$ and $\tilde{\mathcal{X}}$ in fiber classes. We consider one-pointed invariants to simplify the notation.

Lemma 7. *For any i, α,*

$$\langle \alpha; H^i \rangle^{\mathcal{X},mc}_{g,d} = \frac{1}{2} \langle \alpha; H^i \rangle^{\tilde{\mathcal{X}},mc}_{g,d}.$$

Proof. We need to prove that

$$\langle \alpha; H^i \rangle^{\mathcal{X}}_{g,d} = \frac{1}{2} \langle \alpha; H^i \rangle^{\tilde{\mathcal{X}}}_{g,d}.$$

This follows from the same argument as in [37, Lem. 4]: The conifold transition is taken relative to the K3 surface S. The extra components which appear in the degeneration argument is a bundle (with fiber $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1})$ or a quadric in $\mathbb{P}^4$) over the K3 surface S. Because of the existence of the symplectic form, it follows that the curve classes which may contribute nontrivially have to be fiber classes. The argument of [30] then goes through without change. $\square$

By the lemma the Gromov–Witten/Noether–Lefschetz relation of Propsition 6 extends to the family π. Specializing to genus 0 we obtain that

$$\left\langle \alpha; H^i \right\rangle_{0,d}^{\mathcal{X},\mathrm{mc}} = \sum_{m,s} \left\langle \alpha; H^i \right\rangle_{0,m,s,d}^{X,\mathrm{mc}} \mathrm{NL}_{m,s,d}^{\pi}. \tag{20}$$

The left-hand side can be computed using the mirror symmetry formulas of Section 5. The primitive invariants appearing on the right-hand side are given by Remark 5. The Noether–Lefschetz numbers $\mathrm{NL}_{s,d}$, and hence, their refinements $\mathrm{NL}_{m,s,d}$ are determined by [31] and the formulas in Section 3.11. By using a computer (see the author's website for the code), one finds that for degree 6, 8, 9, 15 this equation uniquely determines the invariants f_β, g_β for $\beta = m\alpha$ in cases $(m,\alpha) \in \{(2,0),(2,3/2),(3,3/2),(5,3/2)\}$. Moreover, one checks then that for these degrees we have

$$\left\langle \alpha; H^i \right\rangle_{0,d}^{\mathcal{X},\mathrm{mc}} = \sum_{s} \mathrm{NL}_{s,d} \cdot \left\langle \alpha; H^i \right\rangle_{0,1,s,d}^{X,\mathrm{mc}}. \tag{21}$$

which implies that Conjecture A holds in these cases. Together with Proposition 2 this proves Proposition 1. (As mentioned in the introduction, we have checked equation (21) up to degree 38, which provides plenty of evidence for Conjecture A.)

6.2. Debarre–Voisin fourfolds

We consider a generic pencil of Debarre–Voisin fourfolds

$$\mathcal{X} \subset \mathrm{Gr}(6,10) \times \mathbb{P}^1, \quad \pi : \mathcal{X} \longrightarrow \mathbb{P}^1.$$

The case is very similar to the case of cubic fourfolds. As shown in C (by J. Song), we have the same description of the singular fibers as in the Fano case. In particular, we may use the same double cover construction and conclude the Gromov–Witten/Noether–Lefschetz relation (20) for π.

We want to determine the generating series of Noether–Lefschetz numbers

$$\varphi(q) = \sum_{D \geq 0} q^{D/11} \mathrm{NL}^{\pi}(D),$$

where D runs over squares modulo 11, and we used the notation of Section 3.8. Recall that we have

$$\mathrm{NL}_{s,d}^{\pi} = \mathrm{NL}^{\pi}(D), \quad \text{where} \quad D = -\frac{1}{4} \det \begin{pmatrix} 22 & d \\ d & s \end{pmatrix}.$$

We first prove the following basic invariants.

Lemma 8. $\mathrm{NL}^{\pi}(0) = -10$ *and* $\mathrm{NL}^{\pi}(11) = 640$.

Proof. We have

$$\mathrm{NL}^{\pi}(0) = \int_{\mathbb{P}^1} \iota_{\pi}^* c_1(\mathcal{K}^*),$$

where $\mathcal{K} \to \mathcal{D}_L/\Gamma_M$ is the descent of the tautological bundle $\mathcal{O}(-1)$. It is well-known that $\iota_{\pi}^* \mathcal{K}$ corresponds to the Hodge bundle $\pi_* \Omega_{\pi}^2$. Hence, $\iota_{\pi}^* \mathcal{K}^*$ is isomorphic to

$$\mathcal{L} = R^2 \pi_* \mathcal{O}_{\mathcal{X}}$$

which has fiber $H^2(\mathcal{X}_t, \mathcal{O}_{\mathcal{X}_t})$ over $t \in \mathbb{P}^1$. In K-theory we have

$$R\pi_*\mathcal{O} = \mathcal{O}_{\mathbb{P}^1} + \mathcal{L} + \mathcal{L}^{\otimes 2}.$$

By a Riemann–Roch calculation (using the software package [29]), we find that

$$3c_1(\mathcal{L}) = \int \mathrm{ch}_1(R\pi_*\mathcal{O}) = \int \pi_*(\mathrm{td}_{\mathcal{X}}/\mathrm{td}_{\mathbb{P}^1}) = -30.$$

The number $\mathrm{NL}^\pi(11) = \mathrm{NL}_{-2,0}$ is the number of singular fibers. To compute these, we recall that the singular locus of every singular fiber is a smooth K3 surface and the blowup along the singular locus has exceptional divisor a $\mathbb{P}^1$-bundle over the K3 surface. Hence, the topological Euler characteristic of a singular fiber is 300. By a standard computation (using [29]), the topological Euler number of the total family is $e(\mathcal{X}) = -147, 12$. Hence, if δ is the number of singular fibers we get

$$-14,712 = e(\mathcal{X}) = 324(2 - \delta) + \delta \cdot 300, \quad \text{hence } \delta = 640.$$

The last part also follows from [15, Proof of Prop.3.1].   □

To further constrain the Noether–Lefschetz numbers, we argue as follows. By a computer check (see again the author's webpage), the Gromov–Witten/Noether–Lefschetz relation

$$\left\langle H^3 \right\rangle_{0,d}^{\mathcal{X},\mathrm{mc}} = \sum_{m,s} \left\langle H^3 \right\rangle_{0,m,s,d}^{X,\mathrm{mc}} \mathrm{NL}_{m,s,d}^\pi \tag{22}$$

involves for $d \le 13$ only terms for which the multiple cover conjecture is known by Proposition 2. Hence, for $d \le 13$ we may rewrite it

$$\left\langle H^3 \right\rangle_{0,d}^{\mathcal{X},\mathrm{mc}} = \sum_s \mathrm{NL}_{s,d} \cdot \left\langle H^3 \right\rangle_{0,1,s,d}^{X,\mathrm{mc}}.$$

The left-hand side can be computed using the mirror symmetry formalism. The primitive invariants on the right are given in Remark 5. For $1 \le d \le 5$ one obtains

$$\begin{aligned}
0 &= 264\mathrm{NL}(3) \\
130,680 &= 3,960\mathrm{NL}(1) + 132\mathrm{NL}(12) \\
0 &= 792\mathrm{NL}(5) \\
3,020,160 &= 264\mathrm{NL}(15) + 7,920\mathrm{NL}(4) \\
0 &= 1,320\mathrm{NL}(9)
\end{aligned} \tag{23}$$

Using equations (23) and $\mathrm{NL}(0) = -10$ and employing Williams' program [53], we find that:

$$\mathrm{NL}(1) = \mathrm{NL}(3) = \mathrm{NL}(4) = \mathrm{NL}(5) = \mathrm{NL}(9) = 0. \tag{24}$$

This in turn determines the modular form Φ^π uniquely. (One independently checks that indeed $\mathrm{NL}(11) = 640$ matches the second result of Lemma 8.)

Theorem 2 follows from this, from Proposition 4 and straightforward linear algebra (by [4, Sec.12], we have that E_1, Δ_{11}, E_3 generate the ring of modular forms for character χ_{11}:

$$\bigoplus_{k \ge 0} \mathrm{Mod}_k(\Gamma_0(11), \chi_{11}^k) = \mathbb{C}[E_1, \Delta_{11}, E_3]/(\mathrm{relations})).$$

Proof of Corollary 1. Define the Noether–Lefschetz numbers of first type:

$$\mathcal{C}_{2e}^{\pi} = \frac{1}{2} \int_C \iota_{\tilde{\pi}}^* \mathcal{C}_{2e}.$$

These are related to the $\mathrm{NL}^{\pi}(D)$ by Proposition 5. The proof, hence, follows from equation (24) and Lemma 9 below. □

Lemma 9. *$\mathcal{C}_{2e}$ is HLS if and only if $\mathcal{C}_{2e}^{\pi} = 0$.*

Proof. Let $\mathcal{M}'_{DV} \subset \mathcal{M}_{DV}$ be the open locus consisting of Debarre–Voisin varieties which are either smooth or singular with ordinary double point singularity along a smooth K3 surface (which holds on an open subset of the irreducible discriminant divisor [3]). By [15], the complement of $\mathcal{M}'_{DV}$ has codimension 2. The period map extends to a morphism $p : \mathcal{M}'_{DV} \to \mathcal{M}_H \subset \overline{\mathcal{M}}_H$. Let also $\tilde{p} : \widetilde{\mathcal{M}}_{DV} \to \overline{\mathcal{M}}_H$ be the resolution of the rational period map $\mathcal{M}_{DV} \dashrightarrow \overline{\mathcal{M}}_H$. We view $\mathcal{M}'_{DV}$ as a open subvariety of $\widetilde{\mathcal{M}}_{DV}$,

$$\mathcal{M}'_{DV} \xhookrightarrow{j} \widetilde{\mathcal{M}}_{DV} \xrightarrow{\tilde{p}} \overline{\mathcal{M}}_H.$$

We need the following basic fact: Assume $D = \tilde{p}_* E$ for some irreducible divisor $E \subset \widetilde{\mathcal{M}}_{DV}$. Since $\tilde{p}$ is birational [41], by Zariski's main theorem [20, Cor.11.4], E is the unique irreducible divisor in $\widetilde{\mathcal{M}}_{DV}$ which maps to D (otherwise, the generic point of the image would have more than two preimages; hence, the fiber would not be connected), and by the same argument, the map $E \to D$ is birational. Hence, $\tilde{p}^* \tilde{p}_* E = E$. This yields

$$p^* D = j^* \tilde{p}^* D = j^* \tilde{p}^* \tilde{p}_* E = j^* E.$$

We find that D is HLS (term on the right vanishes for some necessarily unique E) if and only if $p^* D = 0$ in $A^1(\mathcal{M}'_{DV})$.

There is a $\mathrm{SL}(V_{10})$-bundle $\pi : U \to \mathcal{M}'_{DV}$ for an open $U \subset \mathbb{P}(\wedge^3 V_{10}^*)$ with complement of codimension ≥ 2. Hence, we also have $p^* D = 0$ if and only if $\pi^* p^* D = 0$ if and only if $\pi^* p^* D \cdot L = 0$ for a generic line L in U. □

Appendix A. A multiple cover rule for abelian surfaces

In this appendix, we state a conjectural rule that expressed reduced Gromov–Witten invariants of an abelian surfaces for any curve class β in terms of invariants for which β is primitive. The conjectural formula extends a proposal of [7] for the abelian surface analogue of the Katz–Klemm–Vafa formula. As in the hyperkähler case, the conjecture can be reinterpreted as saying that, after subtracting multiple covers, the Gromov–Witten invariants are independent of the divisibility.

A.1. Monodromy

Recall that the cohomology of an abelian surface is described by

$$H^i(A, \mathbb{Q}) = \bigwedge^i H^1(A, \mathbb{Q}).$$

The class of a point $\mathsf{p} \in H^4(A, \mathbb{Z})$ thus defines a canonical element

$$\mathsf{p} \in \bigwedge^4 H^1(A, \mathbb{Q}).$$

An isomorphism of abelian groups $\varphi : H^1(A,\mathbb{Z}) \to H^1(A',\mathbb{Z})$ extends naturally to a morphism of the full cohomology $H^*(A,\mathbb{Z})$ by setting $\varphi|_{H^i(A,\mathbb{Z})} = \wedge^i \varphi$. One has that φ is a parallel transport operator (i.e., the parallel transport along a deformation from A to A' through complex tori) if and only if φ preserves the canonical element [8, Sec.1.10].

The Zariski closure of the space of parallel transport operators is the set of $\mathbb{C}$-vector spaces homomorphisms:

$$M_{A,A'} = \{\varphi : H^1(A,\mathbb{C}) \longrightarrow H^1(A',\mathbb{C}) | \varphi(\mathsf{p}) = \mathsf{p}'\}.$$

It follows that the induced map $\wedge^2 \varphi : H^2(A,\mathbb{C}) \to H^2(A',\mathbb{C})$ preserves the canonical inner product. If $A' = A$, the above just says that the monodromy group is $\mathrm{SL}(H^1(A,\mathbb{Z}))$ and its Zariski closure $\mathrm{SL}(H^1(A,\mathbb{C}))$.

A.2. Multiple cover rule

Let $\beta \in H_2(A,\mathbb{Z})$ be an effective curve class. For any divisor $k|\beta$, choose an abelian variety A_k and a morphism $\varphi_k : H^1(A,\mathbb{R}) \to H^1(A_k,\mathbb{R})$ preserving the canonical element such that the induced morphism

$$\varphi_k = \bigoplus_i \wedge^i \varphi_k : H^*(A,\mathbb{R}) \longrightarrow H^*(A_k,\mathbb{R})$$

takes β/k to a *primitive* effective curve class.

Let also $\alpha \in H^*(\overline{M}_{g,n})$ be a tautological class and $\gamma_i \in H^*(A,\mathbb{R})$ be arbitrary insertions.

Conjecture C. *For any effective curve class $\beta \in H_2(A,\mathbb{Z})$,*

$$\left\langle \alpha; \gamma_1, \ldots, \gamma_n \right\rangle_{g,\beta}^A = \sum_{k|\beta} k^{3g-3+n-\deg(\alpha)} \left\langle \alpha; \varphi_k(\gamma_1), \ldots, \varphi_k(\gamma_n) \right\rangle_{g,\varphi_k(\beta/k)}^{A_k}.$$

A.3. Example

We apply the conjectural multiple cover formula to the analogue of the Katz–Klemm–Vafa formula for abelian surfaces which is the integral

$$N_{g,\beta}^{\mathrm{FLS}} = \int_{[\overline{M}_{g,n}(A,\beta)^{\mathrm{FLS}}]^{\mathrm{red}}} (-1)^{g-2} \lambda_{g-2},$$

where $\overline{M}_{g,n}(A,\beta)^{\mathrm{FLS}}$ is the substack of $\overline{M}_{g,n}(A,\beta)$ that maps with image in a fixed linear system (FLS); see [7].

To apply the multiple cover rule, we specialize to $A = E \times E'$. Consider symplectic bases

$$\alpha_1, \beta_1 \in H^1(E,\mathbb{Z}), \quad \alpha_2, \beta_2 \in H^1(E',\mathbb{Z})$$

which give a basis of $H^1(A,\mathbb{Z})$ (we omit the pullback), and let

$$\omega_1 = \alpha_1\beta_1, \quad \omega_2 = \alpha_2\beta_2 \in H^2(A,\mathbb{Z}).$$

We take $\beta = (d_1,d_2) := d_1\omega_1 + d_2\omega_2$. For every $k|\gcd(d_1,d_2)$, define $\varphi_k \in \mathrm{SL}(H^1(A,\mathbb{Q}))$ by

$$\alpha_1 \longmapsto \alpha_1, \quad \beta_1 \longmapsto \frac{k}{d_1}\beta_1, \quad \alpha_2 \longmapsto \alpha_2, \quad \beta_2 \longmapsto \frac{d_1}{k}\beta_2.$$

The extension to the full cohomology satisfies

$$\varphi_k(\beta/k) = \omega_1 + \frac{d_1 d_2}{k^2}\omega_2 = (1, d_1 d_2/k^2).$$

Recall the result of Bryan ([7, Sec.3.2]) that

$$N_{g,\beta}^{\mathrm{FLS}} = \left\langle (-1)^{g-2}\lambda_{g-2}; \prod_{i=1}^{4}\xi_i \right\rangle_{g,\beta}^{A},$$

where we can take

$$(\xi_1, \xi_2, \xi_3, \xi_3) = (\omega_1\alpha_2, \omega_1\beta_2, \alpha_1\omega_2, \beta_1\omega_2).$$

Conjecture C then implies

$$N_{g,(d,d')}^{\mathrm{FLS}} = \sum_{k|\beta} k^{2g-3+6}\left\langle (-1)^{g-2}\lambda_{g-2}; \prod_{i=1}^{4}\varphi_k(\xi_i) \right\rangle_{g,(1,dd'/k^2)}$$

$$= \sum_{k|\beta} k^{2g+3} N_{g,(1,dd'/k^2)}^{\mathrm{FLS}}$$

which matches precisely Conjecture A in [7].

Appenidx B. Comparision with Gopakumar–Vafa invariants

For a K3 surface S and effective curve class $\beta \in H_2(S, \mathbb{Z})$, consider the Gromov–Witten invariant

$$R_{g,\beta} = \int_{[\overline{M}_{g,n}(S,\beta)]^{\mathrm{red}}} (-1)^g \lambda_g.$$

Fix any primitive effective class $\alpha \in H_2(S, \mathbb{Z})$. Define the generating series

$$F_\alpha = \sum_{g\geq 0}\sum_{m>0} R_{g,m\alpha} u^{2g-2} v^m,$$

where u, v are formal variables. Following [49], the Gopakumar–Vafa invariants $r_{g,m\alpha} \in \mathbb{Q}$ of the K3 surface S are defined by the equality

$$F_\alpha = \sum_{g\geq 0}\sum_{m>0} r_{g,m\alpha} \sum_{k>0} \frac{1}{k}\left(\frac{\sin(ku/2)}{2}\right)^{2g-2} v^{km}. \tag{25}$$

Recall from the introduction (Section 0.1) the numbers

$$\widetilde{r}_{g,\beta} = \sum_{k|\beta} k^{2g-3}\mu(k)R_{g,\beta/k}.$$

We have the following connection between the invariants $r_{g,\beta}$ and $\widetilde{r}_{g,\beta}$: For any g, consider the expansion

$$\left(\frac{1}{2}\sin(u/2)\right)^{2g-2} = \sum_{\tilde{g}} a_{g,\tilde{g}} u^{2\tilde{g}-2}. \tag{26}$$

Lemma 10. *For any $g \geq 0$, we have the upper-triangular relation*

$$\tilde{r}_{\tilde{g},\beta} = \sum_{\tilde{g}} a_{g,\tilde{g}} r_{g,\beta}. \tag{27}$$

In [49], it was shown that the $r_{g,\beta}$ do not depend on the divisibility of the curve class β. By equation (27), we find that also $\tilde{r}_{g,\beta}$ does not depend on the divisibility, as claimed in Theorem 1. Since (27) is upper-triangular, we see that also the converse holds, i.e., $r_{g,\beta}$ does not depend on the divisibility if and only if the same holds for $\tilde{r}_{g,\beta}$.

Proof of Lemma 10. Let us define

$$\hat{r}_{\tilde{g},\beta} := \sum_{\tilde{g}} a_{g,\tilde{g}} r_{g,\beta}.$$

Inserting equation (26) into equation (25), we get

$$\begin{aligned}
F_\alpha &= \sum_{g\geq 0} \sum_{m>0} r_{g,m\alpha} \sum_{k>0} \frac{1}{k} \sum_{\tilde{g}} a_{g,\tilde{g}} k^{2\tilde{g}-2} u^{2\tilde{g}-2} v^{km} \\
&= \sum_{\tilde{g}\geq 0} \sum_{m>0} \sum_{k>0} k^{2\tilde{g}-3} \hat{r}_{g,m\alpha} u^{2\tilde{g}-2} v^{km}.
\end{aligned}$$

Taking the $v^n u^{2g-2}$ coefficient, this shows that

$$R_{g,n\alpha} = \sum_{k|n} k^{2g-3} \hat{r}_{g,n/k\alpha}.$$

By Möbius inversion (i.e., using the identity $\sum_{d|n,d>0} \mu(d) = \delta_{n1}$), we get $\hat{r}_{g,n\alpha} = \sum_{k|n} k^{2g-3} \mu(k) R_{g,n\alpha/k}$, so $\tilde{r}_{g,n\alpha} = \hat{r}_{g,n\alpha}$. □

By [49], all the $r_{g,\beta}$ are integers. It would be interesting to find integer-invariants which underlie the Gromov–Witten invariants of hyperkähler varieties in dimension > 2. For hyperkähler fourfolds, a partial proposal is discussed in [10].

Appendix C. Geometry of a general singular Debarre–Voisin fourfold *by Jieao Song*

We give a description for the singularities of a general singular Debarre–Voisin variety. In the notation of [3], the class of the trivector σ defining a general such Debarre–Voisin variety X_6^σ lies in the divisor $\mathcal{D}^{3,3,10}$: There exists a unique three-dimensional subspace $V_3 \subset V_{10}$ such that σ satisfies the degeneracy condition $\sigma(V_3, V_3, V_{10}) = 0$. Under the period map, this divisor corresponds to the Heegner divisor[12] $\mathcal{D}_{22}$ in the period domain. We obtained the following description for the set-theoretical singular locus of X_6^σ in [3, Proposition 2.4].

Proposition 7. *Let $[\sigma] \in \mathcal{D}^{3,3,10}$ be general so that there exists a unique three-dimensional subspace $V_3 \subset V_{10}$ with $\sigma(V_3, V_3, V_{10}) = 0$. Set-theoretically, the singular locus of X_6^σ is*

$$S := \{ [V_6] \in X_6^\sigma \mid V_6 \supset V_3 \},$$

which is a K3 surface of degree 22.

We prove the following stronger result, following the idea in [21, Lemma 6.3.1], where a similar result is proved for the variety of lines of a nodal cubic hypersurface. We shall see that the two cases share some surprising similarities.

---

[12]The divisor $\mathcal{D}_{22}$ is denoted by $\mathcal{C}_{22}$ in the main body of the text.

Proposition 8. *Let σ be as in the previous proposition. For the associated Debarre–Voisin variety X_6^σ, the singularities along the degree-22 K3 surface S are codimension-2 ordinary double points. More precisely, by blowing up the singular locus S, we get a smooth hyperkähler fourfold of $K3^{[2]}$-type, and the exceptional divisor is a conic fibration over S.*

Proof. We briefly recall the argument for the nodal cubic: For a cubic $X \subset \mathbf{P}^5 = \mathbf{P}(V_6)$ containing a node $p := [V_1]$, the projectivized normal cone $\mathbf{P}C_p X$ is a quadric hypersurface Q in $\mathbf{P}T_p\mathbf{P}^5 = \mathbf{P}(V_6/V_1)$, and the varieties of lines $F \subset \mathrm{Gr}(2, V_6)$ are singular along a K3 surface S parametrizing lines in X passing through p. Instead of blowing up S in F, Hassett considered studying the ambient Grassmannian $\mathrm{Gr}(2, V_6)$ and blowing up the Schubert variety $\Sigma := \mathbf{P}(V_6/V_1) \subset \mathrm{Gr}(2, V_6)$, which parametrizes all lines in $\mathbf{P}(V_6)$ passing through p. This gives the following Cartesian diagram

$$
\begin{array}{ccc}
\tilde{F} := \mathrm{Bl}_S F & \longhookrightarrow & \mathrm{Bl}_\Sigma \mathrm{Gr}(2, V_6) \\
\downarrow & & \downarrow \\
F & \longhookrightarrow & \mathrm{Gr}(2, V_6).
\end{array}
$$

For a given point $x := [V_2] \in S$, we get one distinguished point $y := [V_2/V_1]$ in $\mathbf{P}^4 = \mathbf{P}(V_6/V_1)$ that lies on the quadric Q. The projectivized normal space $\mathbf{P}\mathcal{N}_{\Sigma/\mathrm{Gr}(2,V_6),x}$ can be identified with $\mathbf{P}(V_6/V_2)$, which parametrizes lines in $\mathbf{P}^4 = \mathbf{P}(V_6/V_1)$ passing through the point y, and the projectivized normal cone $\mathbf{P}C_{S,x}F$ is given by the subscheme parametrizing such lines that are also entirely contained in the quadric threefold Q, in other words, lines in Q passing through a given point. This condition gives a smooth conic curve, so the singularities of F along S are indeed codimension-2 ordinary double points.

We use a similar argument to study the singular Debarre–Voisin variety X_6^σ. By assumption, the hyperplane section X_3^σ admits an ordinary double point at $[V_3]$, so its tangent cone at $[V_3]$ is a smooth quadric hypersurface Q in the projectivization of the tangent space

$$
\mathbf{P}T_{[V_3]} \mathrm{Gr}(3, V_{10}) \simeq \mathbf{P}\mathrm{Hom}(V_3, V_{10}/V_3) =: \mathbf{P}(T_{21}) = \mathbf{P}^{20}.
$$

For a given $x := [V_6] \in S$, the projective space $\mathbf{P}\mathrm{Hom}(V_3, V_6/V_3) =: \mathbf{P}(T_9) = \mathbf{P}^8$ gives a distinguished linear subspace contained in Q.

Following the proof of Hassett, instead of blowing up S in X_6^σ, we consider the ambient Grassmannian $\mathrm{Gr}(6, V_{10})$ and blow up the entire Schubert variety

$$
\Sigma := \{[V_6] \in \mathrm{Gr}(6, V_{10}) \mid V_6 \supset V_3\} \simeq \mathrm{Gr}(3, V_{10}/V_3),
$$

which is smooth of codimension 12. We have the following description for its normal bundle in $\mathrm{Gr}(6, V_{10})$:

$$
\mathcal{N}_{\Sigma/\mathrm{Gr}(6,V_{10})} = \mathrm{Hom}(\mathcal{U}_6, \mathcal{Q}_{10/6})/\mathrm{Hom}(\mathcal{U}_6/V_3, \mathcal{Q}_{10/6}) \simeq \mathrm{Hom}(V_3, \mathcal{Q}_{10/6}),
$$

where we denote by $\mathcal{U}_6$ and $\mathcal{Q}_{10/6}$ the restrictions to Σ of the two tautological bundles on $\mathrm{Gr}(6, V_{10})$. For the given point $x \in S$, the projectivization of the normal space is therefore an 11-dimensional projective space

$$
\mathbf{P}\mathcal{N}_{\Sigma/\mathrm{Gr}(6,V_{10}),x} \simeq \mathbf{P}\mathrm{Hom}(V_3, V_{10}/V_6) \simeq \mathbf{P}(T_{21}/T_9),
$$

where we recall that T_{21} is the tangent space of $\mathrm{Gr}(3, V_{10})$ at $[V_3]$, and T_9 is the tangent space of $\mathrm{Gr}(3, V_6)$ at $[V_3]$, viewed as a subspace of T_{21}.

Consider the proper transform of X_6^σ denoted by $\tilde{X}_6^\sigma$. We have the following Cartesian diagram

$$
\begin{array}{ccc}
\tilde{X}_6^\sigma & \lhook\joinrel\longrightarrow & \mathrm{Bl}_\Sigma \, \mathrm{Gr}(6, V_{10}) \\
\downarrow & & \downarrow \\
X_6^\sigma & \lhook\joinrel\longrightarrow & \mathrm{Gr}(6, V_{10}).
\end{array}
$$

Consequently, we get a natural closed embedding of the projectivized normal cone

$$
\mathbf{PC}_{S,x} X_6^\sigma \lhook\joinrel\longrightarrow \mathbf{P}\mathcal{N}_{\Sigma/\mathrm{Gr}(6,V_{10}),x} \simeq \mathbf{P}(T_{21}/T_9).
$$

The total projective space $\mathbf{P}(T_{21}/T_9)$ parametrizes nine-dimensional linear subspaces of $\mathbf{P}(T_{21})$ that contain the distinguished $\mathbf{P}^8 = \mathbf{P}(T_9)$, and the projectivized normal cone $\mathbf{PC}_{S,x} X_6^\sigma$ can then be identified with the subscheme that parametrizes such $\mathbf{P}^9$ that are also contained in the quadric Q. In other words, it parametrizes nine-dimensional linear subspaces in a 19-dimensional quadric containing a fixed $\mathbf{P}^8$. This is again a smooth conic curve, just like in the nodal cubic case. Thus, the singularities of X_6^σ along S are indeed codimension-2 ordinary double points, and $\tilde{X}_6^\sigma$ is smooth.

Finally, we show that the resolution $\tilde{X}_6^\sigma$ that we obtained has trivial canonical class. Since X_6^σ is birational to the Hilbert square $S^{[2]}$, this will then force $\tilde{X}_6^\sigma$ to be a smooth hyperkähler fourfold of $K3^{[2]}$-type.

We denote by E the exceptional divisor for the blowup $\mathrm{Bl}_\Sigma \, \mathrm{Gr}(6, V_{10}) \to \mathrm{Gr}(6, V_{10})$, and by D the exceptional divisor for the blowup $\tilde{X}_6^\sigma \to X_6^\sigma$. The divisor D can be identified with the projectivized normal cone $\mathbf{PC}_S X_6^\sigma$, so the morphism $D \to S$ is a conic fibration by the above analysis. By construction, the Zariski open subset $\tilde{X}_6^\sigma \setminus D$ is isomorphic to the smooth locus $X_6^\sigma \setminus S$. The latter has trivial canonical class since it is the regular zero-locus of σ viewed as a section of the vector bundle $\bigwedge^3 \mathcal{U}_6^\vee$. Therefore, the canonical divisor $K_{\tilde{X}_6^\sigma}$ is linearly equivalent to some multiple of D. We write $K_{\tilde{X}_6^\sigma} = mD$, and it remains to show that $m = 0$.

Since $D \to S$ is a smooth conic fibration in the projectivized normal bundle $E \to \Sigma$, the relative $\mathcal{O}(-1)$ of $E \to \Sigma$ restricts to the relative canonical bundle of $D \to S$. Note that by the Leray–Hirsch theorem, this bundle is necessarily nontrivial. Since E is the exceptional divisor, the relative $\mathcal{O}(-1)$ on E is given by $\mathcal{O}_E(E)$, so we have

$$
\omega_{D/S} \simeq \mathcal{O}_E(E)|_D.
$$

Using the fact that S is a K3 surface and that $\mathcal{O}_E(E)|_{\tilde{X}_6^\sigma} \simeq \mathcal{O}_{\tilde{X}_6^\sigma}(D)$, this gives

$$
\omega_D \simeq \omega_{D/S} \simeq \mathcal{O}_{\tilde{X}_6^\sigma}(D)|_D, \quad \text{hence} \quad K_D = D|_D,
$$

which in particular must be nontrivial.

On the other hand, by the adjunction formula, we have

$$
K_D \simeq (K_{\tilde{X}_6^\sigma} + D)|_D = (m+1)D|_D.
$$

Thus, we may conclude that $m = 0$, and $K_{\tilde{X}_6^\sigma}$ is indeed trivial. $\square$

Remark 9. Contrary to the nodal cubic case, the resolution $\tilde{X}_6^\sigma$ obtained is not isomorphic to the Hilbert scheme $S^{[2]}$, even for a generic member of the family. This can be seen by studying the chamber decomposition for a generic $S^{[2]}$ with Picard rank 2: One may find exactly two chambers in the movable cone, corresponding to $S^{[2]}$ and a second birational model; the Plücker polarization pulled back to $S^{[2]}$ via the birational map is equal to $10H - 33\delta$ and not nef (see, for example, [13, Table 1]), so we may

conclude that $\tilde{X}_6^\sigma$ is the second birational model. The two models are related by a Mukai flop, and it would be interesting to see this geometrically.

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