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A NOTE ON PARACOMPACT p-SPACES AND THE MONOTONE D-PROPERTY

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Abstract

For any generalized ordered space X with the underlying linearly ordered topological space X_u, let X^* be the minimal closed linearly ordered extension of X and $\tilde{X}$ be the minimal dense linearly ordered extension of X. The following results are obtained.

(1) The projection mapping $\pi : X^* \to X$, $\pi(\langle x, i\rangle) = x$, is closed.
(2) The projection mapping $\phi : \tilde{X} \to X_u$, $\phi(\langle x, i\rangle) = x$, is closed.
(3) X^* is a monotone D-space if and only if X is a monotone D-space.
(4) $\tilde{X}$ is a monotone D-space if and only if X_u is a monotone D-space.
(5) For the Michael line M, $\tilde{M}$ is a paracompact p-space, but not continuously Urysohn.

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1. Preliminaries

A topological space X is continuously Urysohn if there is a continuous function $\varphi : X^2 \setminus \triangle \to C(X)$ such that $\varphi(x, y)(x) \neq \varphi(x, y)(y)$, where $\triangle = \{\langle x, x\rangle : x \in X\}$ and $C(X)$ is the space of bounded continuous real-valued functions with the norm topology. The concept was first explored in [10] and was named in [4]. The family $\{\varphi(x, y) : (x, y) \in X^2 \setminus \triangle\}$ is called a continuous separating family for X. A submetrizable space is continuously Urysohn, and paracompact p-spaces in the sense of Arhangel'skii (the preimages of metric spaces under perfect mappings) are metrizable if and only if they are continuously Urysohn [10].

A topological space $X = (X, \tau)$ is a D-space [11] if for each neighborhood assignment $\varphi : X \to \tau$, $x \in \varphi(x)$ for all $x \in X$, there is a closed discrete subset $F(\varphi)$ of X such that $X = \bigcup\{\varphi(x) : x \in F(\varphi)\}$, where τ is the topology on X. If, moreover, for any two neighborhood assignments φ and ψ for X satisfying $\varphi(x) \subset \psi(x)$ for all $x \in X$, $F(\psi) \subset F(\varphi)$, then X is said to be monotonically D [6]. Monotone D-spaces are D-spaces, but the converse does not hold.

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Let X be a generalized ordered (GO) space with the underlying linearly ordered topological space X_u, let X^* be the minimal closed linearly ordered extension of X and $\tilde{X}$ be the minimal dense linearly ordered extension of X. Clearly, separability, countable chain condition and Lindelöfness of X can be preserved by $\tilde{X}$. It is well known that paracompactness, metrizability and quasi-developability of X are preserved by X^*. However, these properties of X are not hereditary to $\tilde{X}$ (see [5]). Since the Sorgenfrey line S and the Michael line M are submetrizable they have the continuous Urysohn property $\mathscr{P}$. M^* has $\mathscr{P}$ [2], but S^* and $\tilde{S}$ do not have $\mathscr{P}$ [8].

Recall that a GO space is a triple $(X, \tau, <)$ where τ is a topology on the linearly ordered set $(X, <)$ with $\tau \supset \lambda$ and has a base consisting of convex sets, here λ is the open interval topology of the order $<$. If $\tau = \lambda$, $(X, \lambda, <)$ is called a linearly ordered topological space (LOTS) and, for any GO space $(X, \tau, <)$, $(X, \lambda, <)$ is called its underlying LOTS. Let $R_\tau = \{x \in X : [x, \rightarrow) \in \tau - \lambda\}$, $L_\tau = \{x \in X : (\leftarrow, x] \in \tau - \lambda\}$ and $\mathbb{Z}$ be the set of integers. Put

$$\tilde{X} = (X \times \{0\}) \cup (R_\tau \times \{-1\}) \cup (L_\tau \times \{1\}),$$
$$X^* = (X \times \{0\}) \cup (R_\tau \times \{i \in \mathbb{Z} : i < 0\}) \cup (L_\tau \times \{i \in \mathbb{Z} : i > 0\}).$$

Equip $\tilde{X}$ (respectively X^*) with the open interval topology of the lexicographical order on it; then $\tilde{X}$ has a dense subspace $X \times \{0\}$ homeomorphic to X and X^* has a closed subspace $X \times \{0\}$ homeomorphic to X. By [5, Proposition 2.7], $\tilde{X}$ is the minimal dense linearly ordered extension of X and, by [7, Theorem 9], X^* is the minimal closed linearly ordered extension of X.

Let $X = (X, \tau, <)$ be a GO space with the underlying LOTS $X_u = (X, \lambda, <)$. The main results of the note are as follows.

(1) The projection mapping $\pi : X^* \to X$, $\pi(\langle x, i\rangle) = x$, is closed.
(2) The projection mapping $\phi : \tilde{X} \to X_u$, $\phi(\langle x, i\rangle) = x$, is closed.
(3) X^* is a monotone D-space if and only if X is a monotone D-space.
(4) $\tilde{X}$ is a monotone D-space if and only if X_u is a monotone D-space.
(5) For the Michael line M and the Cantor line C, $\tilde{M}$ (respectively $\tilde{C}$) is a paracompact p-space, but not continuously Urysohn. Hence any dense linearly ordered extension of M and C is not continuously Urysohn.

Throughout this paper, spaces are topological spaces and are *Hausdorff*, mappings are continuous and surjective and $[0, 1]$ is the usual unit closed interval. By $\mathbb{R}$, $\mathbb{P}$ and $\mathbb{Q}$ we mean the sets of reals, irrationals and rationals, respectively. The Sorgenfrey line S is $\mathbb{R}$ with half-open intervals of the form $[a, b)$ as a basis for the topology and the Michael line M is $\mathbb{R}$ with points in $\mathbb{P}$ isolated and points in $\mathbb{Q}$ having their usual neighborhoods. By the spaces $[0, \omega_1)$ and $[0, \omega_1]$ we mean the usual ordinal spaces, where ω_1 is the first uncountable ordinal.

2. Results

Let $X = (X, \tau, <)$ be a GO space with the underlying LOTS $X_u = (X, \lambda, <)$, and R_τ and L_τ be defined as in Section 1. For the sake of simplicity, we use the following notation: $\mathtt{L} = L_\tau - R_\tau$, $\mathtt{R} = R_\tau - L_\tau$, $\mathtt{I} = R_\tau \cap L_\tau$, $\mathtt{E} = X - (R_\tau \cup L_\tau)$ and, for $A \subset X$, $A^* = \{\langle x, k\rangle \in X^* : x \in A\}$.

THEOREM 2.1. *Let X be a GO space with the underlying LOTS X_u. Then:*

(1) *the projection mapping $\pi : X^* \to X$, $\pi(\langle x, i\rangle) = x$, is closed;*
(2) *the projection mapping $\phi : \tilde{X} \to X_u$, $\phi(\langle x, i\rangle) = x$, is closed.*

PROOF. (1) Let $X = (X, \tau, <)$ and $X_u = (X, \lambda, <)$. To show the continuity of π, let V be a nonempty open convex subset of X. Note that $\pi^{-1}(V) = V^*$, and we will show that V^* is open in X^*. If $V = \{c\}$, then $\langle c, 0\rangle$ has both an immediate predecessor and an immediate successor and thus $\{\langle c, 0\rangle\}$ is open in X^*. Clearly if $k \neq 0$ and $\langle c, k\rangle \in X^*$, then $\{\langle c, k\rangle\}$ is open. Let $|V| \geq 2$ and $x^* = \langle x, k\rangle \in V^*$. If $k \neq 0$, x^* is an interior point of V^*. Let $k = 0$. If x is a minimal point of V, $\langle x, 0\rangle$ has an immediate predecessor a^* in X^*. Take $y \in V$ such that $x < y$. Since V is convex, $x^* \in (a^*, \langle y, 0\rangle) \subset V^*$ and so x^* is an interior point of V^*. Similarly, if x is a maximal point of V, x^* is an interior point of V^*. If $x \in V$ is neither minimal nor maximal, there are $x_1, x_2 \in V$ such that $x_1 < x < x_2$ and $x^* \in (\langle x_1, 0\rangle), \langle x_2, 0\rangle) \subset V^*$ and thus x^* is also an interior point of V^*.

To show that π is closed, let F be a closed subset of X^* and $x \in X - \pi(F)$. Then $\pi^{-1}(x) \cap F = \emptyset$. We will find an open neighborhood of x in X without meeting $\pi(F)$. If $x \in \mathtt{I}$, $\{x\}$ is open and $\{x\} \cap \pi(F) = \emptyset$. If $x \in \mathtt{R}$, $\pi^{-1}(x) = \{\langle x, i\rangle : i \leq 0\}$ and $(\langle x, -1\rangle, \langle b, 0\rangle) \cap F = \emptyset$ for some $b > x$. Then $[x, b)$ is open and $[x, b) \cap \pi(F) = \emptyset$. If $x \in \mathtt{L}$, $\pi^{-1}(x) = \{\langle x, i\rangle : i \geq 0\}$ and $(\langle a, 0\rangle, \langle x, 1\rangle) \cap F = \emptyset$ for some $a < x$. Then $(a, x]$ is open and $(a, x] \cap \pi(F) = \emptyset$. If $x \in \mathtt{E}$, $\pi^{-1}(x) = \{\langle x, 0\rangle\}$ and there are c, d with $c < x < d$ such that $(\langle c, 0\rangle, \langle d, 0\rangle) \cap F = \emptyset$ and so $(c, d) \cap \pi(F) = \emptyset$.

(2) To show that ϕ is continuous, let (a, b) be an open interval in X_u. Then $U = \phi^{-1}((a, b))$ must be one of the four open intervals of $\tilde{X} : U = (\langle a, 0\rangle, \langle b, 0\rangle)$ if $a \notin L_\tau$, $b \notin R_\tau$; $U = (\langle a, 1\rangle, \langle b, 0\rangle)$ if $a \in L_\tau$, $b \notin R_\tau$; $U = (\langle a, 0\rangle, \langle b, -1\rangle)$ if $a \notin L_\tau, b \in R_\tau$; $U = (\langle a, 1\rangle, \langle b, -1\rangle)$ if $a \in L_\tau, b \in R_\tau$.

We will now show that ϕ is closed. Let F be a closed subset of $\tilde{X}$ and $x \in X - \phi(F)$. Then $\phi^{-1}(x) \cap F = \emptyset$. We will find an open interval in X_u containing x without meeting F. If $x \in \mathtt{E}$, $\phi^{-1}(x) = \{\langle x, 0\rangle\}$ and thus there are $a_x, b_x \in X$ with $a_x < x < b_x$ such that $(\langle a_x, 0\rangle, \langle b_x, 0\rangle) \cap F = \emptyset$. Hence $(a_x, b_x) \cap \phi(F) = \emptyset$. If $x \in \mathtt{R}$, then $\phi^{-1}(x) = \{\langle x, -1\rangle, \langle x, 0\rangle\}$ and there are $c_x, d_x \in X$ with $c_x < x < d_x$ such that $(\langle c_x, 0\rangle, \langle x, 0\rangle) \cap F = \emptyset$ and $(\langle x, -1\rangle, \langle d_x, 0\rangle) \cap F = \emptyset$ and thus $(c_x, d_x) \cap \phi(F) = \emptyset$. If $x \in \mathtt{L}$, then $\phi^{-1}(x) = \{\langle x, 0\rangle, \langle x, 1\rangle\}$ and there are $e_x, f_x \in X$ with $e_x < x < f_x$ such that $(\langle e_x, 0\rangle, \langle x, 1\rangle) \cap F = \emptyset$ and $(\langle x, 0\rangle, \langle f_x, 0\rangle) \cap F = \emptyset$. So $(e_x, f_x) \cap \phi(F) = \emptyset$. If $x \in \mathtt{I}$, $\phi^{-1}(x) = \{\langle x, -1\rangle, \langle x, 0\rangle, \langle x, 1\rangle\}$. Take $g_x, h_x \in X$ with $g_x < x < h_x$ such that $(\langle g_x, 0\rangle, \langle x, 0\rangle) \cap F = \emptyset$ and $(\langle x, 0\rangle, \langle h_x, 0\rangle) \cap F = \emptyset$. Therefore $(g_x, h_x) \cap \phi(F) = \emptyset$. □

Let K be the Cantor set and λ the usual open interval topology on $\mathbb{R}$. Put $T = \bigcup\{K_q : q \in \mathbb{Q}\}$, where $K_q = \{x + q : x \in K\}$. Define a topology ν on $\mathbb{R}$ with the base $\{[x, x + \epsilon) : \epsilon > 0, x \notin T\} \cup \{\{x\} : x \in T\}$ and call the space $(\mathbb{R}, \nu, <)$ the

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Cantor line C. Having a σ-discrete dense subset T, by [1, Proposition 3.1] C is perfect (open sets are F_σ-sets). By [9], any orderable dense extension of C is not perfect.

In the following corollary, $\mathbb{R}$ is the Euclidean line, α is an inverse invariant under countable-to-one closed mappings and β is an inverse invariant under finite-to-one closed mappings.

COROLLARY 2.2. *Let X be a GO space with the underlying LOTS X_u. Then:*

(1) *X^* has α whenever X has α and $\tilde{X}$ has β whenever X_u has β;*
(2) *whenever X_u is $\mathbb{R}$, $\tilde{X}$ is a locally compact paracompact p-space;*
(3) *$\tilde{M}$ and $\tilde{C}$ are not continuously Urysohn, where M is the Michael line and C is the Cantor line.*

PROOF. (1) follows directly from Theorem 2.1 since π is countable-to-one and ϕ is finite-to-one. For (2), note that paracompact p-spaces are precisely preimages of metric spaces under perfect mappings and $\tilde{X} = \phi^{-1}(\mathbb{R})$. (3) Since $\tilde{M}$ has a dense subspace $M \times \{0\}$ (homeomorphic to M) which is not perfect, $\tilde{M}$ is not perfect and thus it is not metrizable. By [5] $\tilde{C}$ is not perfect and thus not metrizable. Since M and C have the underlying LOTS $\mathbb{R}$, by Theorem 2.1, the projection mappings $\phi_1 : \tilde{M} \to \mathbb{R}$ and $\phi_2 : \tilde{C} \to \mathbb{R}$ are (finite-to-one) closed mappings. Hence $\tilde{M} = \phi_1^{-1}(\mathbb{R})$ and $\tilde{C} = \phi_2^{-1}(\mathbb{R})$ are preimages of metric spaces and thus are paracompact p-spaces. Hence they are not continuously Urysohn. □

COROLLARY 2.3. *Let X be a GO space with the underlying LOTS X_u. If $\mathscr{P}$ is local compactness, countable compactness, compactness, paracompactness or Lindelöfness, then $\tilde{X}$ has $\mathscr{P}$ if and only if X_u has $\mathscr{P}$.*

REMARK 2.4. (1) Let X be a GO space with the underlying LOTS X_u. The projection mapping $\phi' : \tilde{X} \to X$ may not be continuous. The projection mapping $\pi_u : X^* \to X_u$ is continuous, but it may be not open or closed. In Theorem 2.1 the projection mappings $\pi : X^* \to X$ and ϕ: $\tilde{X} \to X_u$ need not be open. In fact, the continuity of π_u is clear. Let S be the Sorgenfrey line, M be the Michael line and $\mathbb{R}$ be the Euclidean space. Let $X_u = \mathbb{R}$. If $X = S$, $\{\langle x, -1\rangle\}$ is open in S^*, but $\{x\}$ is not open in S or $\mathbb{R}$ and thus π and π_u are not open. If $X = M$, for $p \in \mathbb{P}$, $\{\langle p, 0\rangle\}$ is open in $\tilde{M}$, but $\{p\}$ is not open in $\mathbb{R}$ and thus ϕ is not open. $\mathbb{Q} \times \{0\}$ is closed in M^*, but $\mathbb{Q}$ is not closed in $\mathbb{R}$. Hence π_u is not closed.

(2) By Theorem 2.1 and Corollary 2.2, the continuous Urysohn property is not an inverse invariant under finite-to-one closed mappings.

LEMMA 2.5. *Let X be a GO space and $F \subset X$. If F is a closed discrete subspace of X, then F^* is a closed discrete subspace of X^*.*

PROOF. Let $X = (X, \tau, <)$. By Theorem 2.1 the projection mapping $\pi : X^* \to X$, $\pi(\langle x, i\rangle) = x$, is continuous, so $F^* = \pi^{-1}(F)$ is closed in X^*. To show that F^* is discrete, let $x^* = \langle x, k\rangle \in F^*$. If $k \neq 0$, $\{x^*\}$ is open. Let $k = 0$. Then there is an open convex $V \subset X$ with $x \in V$ and $V \cap F = \{x\}$. If $V = \{x\}$, $\{x^*\}$ is clearly open. Let $|V| \geq 2$. If x is the minimal point, x^* has an immediate predecessor p^*

in X^*. If $x \in L_\tau$, $(p^*, \langle x, 1\rangle) = \{x^*\}$ is open. If $x \notin L_\tau$, $(p^*, \langle q, 0\rangle) \cap F^* = \{x^*\}$ for some $q \in V$ with $x < q$. The proof for x being maximal is analogous. Now let x be neither minimal nor maximal. We can take an open U with $x^* \in U$ and $U \cap F^* = \{x^*\}$ as follows: $U = (\langle x, -1\rangle, \langle x, 1\rangle)$ if $x \in \mathtt{I}$; $U = (\langle x, -1\rangle, \langle q, 0\rangle)$ if $x \in \mathtt{R}$, where $q \in V$ and $x < q$; $U = (\langle p, 0\rangle, \langle x, 1\rangle)$ if $x \in \mathtt{L}$, where $p \in V$ and $p < x$; $U = (\langle p, 0\rangle, \langle q, 0\rangle)$ if $x \in \mathtt{E}$, where $p, q \in V$ and $p < x < q$. □

THEOREM 2.6. *Let X be any GO space with the underlying LOTS X_u. Then:*

(1) *X^* is a monotone D-space if and only if X is a monotone D-space;*
(2) *$\tilde{X}$ is a monotone D-space if and only if X_u is a monotone D-space.*

PROOF. Let $X = (X, \tau, <)$ and $X_u = (X, \lambda, <)$. If X^* is monotonically D, then since the projection mapping $\pi : X^* \to X$ is closed, by [6, Theorem 1.7(c)] X is monotonically D. Let X be monotonically D. To show that X^* is monotonically D, let φ' be a neighborhood assignment for X^*. Define a neighborhood assignment φ^* for X^* such that $\varphi^*(z) \subset \varphi'(z)$ for $z \in X^*$ as follows. Let $x^* = \langle x, k\rangle \in X^*$. If $k \neq 0$, define $\varphi^*(x^*) = \{x^*\}$. Let $k = 0$. If $x \in \mathtt{I}$, define $\varphi^*(x^*) = \{x^*\}$. If $x \notin \mathtt{I}$, let $C_x = \bigcup\{C : x^* \in C \subset \varphi'(x^*),\ C \text{ is open and convex}\}$. Define $\varphi^*(x^*) = \{\langle y, i\rangle : t < y < x$ for some t for which there is j with $\langle t, j\rangle \in C_x\} \cup \{x^*\}$ if $x \in \mathtt{L}$; $\varphi^*(x^*) = \{x^*\} \cup \{\langle y, i\rangle : x < y < z$ for some z for which there is k with $\langle z, k\rangle \in C_x\}$ if $x \in \mathtt{R}$; $\varphi^*(x^*) = \{\langle y, i\rangle : t < y < z$ for some t for which there is j with $\langle t, j\rangle, \langle z, k\rangle \in C_x\}$ if $x \in \mathtt{E}$. Define $\varphi(\langle x, 0\rangle) = \varphi^*(\langle x, 0\rangle) \cap (X \times \{0\})$ for $x \in X$. Then φ is a neighborhood assignment for the subspace $X \times \{0\}$ (homeomorphic to the monotone D-space X) of X^*. So there is a closed discrete subset F_φ of X such that $X \times \{0\} = \varphi(F_\varphi \times \{0\})$. If ψ is also a neighborhood assignment for $X \times \{0\}$ such that $\varphi(w) \subset \psi(w)$, then $F_\psi \times \{0\} \subset F_\varphi \times \{0\}$. By Lemma 2.5, $(F_\varphi)^*$ is closed and discrete in X^*. Put $F_{\varphi^*} = (F_\varphi)^*$. We will show that $\{\varphi^*(y^*) : y^* \in F_{\varphi^*}\}$ covers X^*. Let $y^* = \langle y, k\rangle \in X^* - F_{\varphi^*}$. Since $y \notin F_\varphi$ and $\varphi(F_\varphi \times \{0\}) = X \times \{0\}$, there is $\langle x, 0\rangle \in F_\varphi \times \{0\} \subset F_{\varphi^*}$ such that $\langle y, 0\rangle \in \varphi(\langle x, 0\rangle)$ with $x \neq y$. Without loss of generality we assume that $y < x$. By the definition of φ, $y^* = \langle y, k\rangle \in \varphi^*(\langle x, 0\rangle)$. So $\varphi^*(F_{\varphi^*}) = X^*$. Put $F_{\varphi'} = F_{\varphi^*}$. Then $\{\varphi'(y^*) : y^* \in F_{\varphi'}\}$ covers X^* since $\varphi^*(z) \subset \varphi'(z)$ for $z \in X^*$. If ψ' is also a neighborhood assignment for X^* such that $\varphi'(z) \subset \psi'(z)$, then we have $\varphi^*(z) \subset \psi^*(z)$ for $z \in X^*$. So $F_\psi \subset F_\varphi$ and thus $F_{\psi'} \subset F_{\varphi'}$. Hence X^* is monotonically D.

If $\tilde{X}$ is monotonically D, then since the projection mapping $\phi : \tilde{X} \to X_u$ is closed, by [6, Theorem 1.7(c)] X_u is monotonically D. If X_u is monotonically D, since the projection mapping $\phi : \tilde{X} \to X_u$ is closed and finite-to-one and the monotone D-property is an inverse invariant under finite-to-one closed mappings [3, Theorem 4], $\tilde{X}$ is monotonically D. □

EXAMPLE 2.7. The lexicographic rectangle $X = \mathbb{R} \times [0, 1]$ (respectively, the lexicographic square $Y = [0, 1] \times [0, 1]$) is a paracompact p-space, but not continuously Urysohn.

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PROOF. Since the projection mappings $f_1 : X \to \mathbb{R}$ and $f_2 : X \to [0, 1])$, $f_1(\langle x, y\rangle) = x$ and $f_2(\langle x, y\rangle) = x$, are perfect, X and Y are paracompact p-spaces. Because the subspace Y of X is not perfect, Y is not continuously Urysohn, so neither is X. □

Recall that the long line L_{ω_1} is the set $[0, \omega_1) \times [0, 1)$ with the linearly ordered topology of the lexicographical order. The extended long line $L^*_{\omega_1}$ is the set $L_{\omega_1} \cup \{\omega_1\}$ (for any $x \in L_{\omega_1}$, $x < \omega_1$) equipped with the open interval topology, or equivalently the one-point compactification of L_{ω_1}. The set $\mathbb{R} \times [0, 1]$ with the lexicographical order topology is called the lexicographic rectangle and its subspace $[0, 1] \times [0, 1]$ is called the lexicographic square. The set $[0, \omega_1) \times \mathbb{Z} \cup \{\langle\omega_1, 0\rangle\}$ with the lexicographical-order topology is denoted by Z_{ω_1}.

EXAMPLE 2.8. Let X be any GO space with the underlying LOTS X_u. Then:

(1) whenever X_u is $[0, \omega_1)$ or L_{ω_1}, $\tilde{X}$ is countably compact;
(2) whenever X_u is $[0, \omega_1]$, $L^*_{\omega_1}$ or the lexicographic square, $\tilde{X}$ is compact;
(3) whenever X_u is Z_{ω_1} or the lexicographic rectangle, $\tilde{X}$ is Lindelöf.

PROOF. Note that the ordinal space $[0, \omega_1]$, the extended long line $L^*_{\omega_1}$ and the lexicographic square $[0, 1] \times [0, 1]$ are compact; the ordinal space $[0, \omega_1)$ and the long line L_{ω_1} are countably compact; Z_{ω_1} and the lexicographic rectangle $\mathbb{R} \times [0, 1]$ are Lindelöf. □

By [10], the one-point compactification $C(\omega_1)$ of the discrete uncountable space $[0, \omega_1)$ is not continuously Urysohn.

EXAMPLE 2.9. The one-point Lindelöfication $L(\omega_1)$ of the discrete uncountable space $[0, \omega_1)$ is continuously Urysohn.

PROOF. Note that $L(\omega_1)$ is homeomorphic to Z_{ω_1} which is continuously Urysohn by [2, Example 3.1]. □

By [8, Theorem 4], any linearly ordered extension the Sorgenfrey line S is not continuously Urysohn, so not submetrizable. Now we have the following example.

EXAMPLE 2.10. Let S be the Sorgenfrey line, M be the Michael line and C be the Cantor line. Then:

(1) $\tilde{S}$, $\tilde{M}$ and $\tilde{C}$ are locally compact paracompact p-spaces, and are Lindelöf;
(2) any dense linearly ordered extension of the Michael line M or the Cantor line C is not continuously Urysohn, so not submetrizable.

PROOF. (1) Noticing that S, M and C have the Euclidean space $\mathbb{R}$ as their underlying LOTS, the conclusion is true from Theorem 2.1(2) and Corollary 2.2. (2) Suppose that a dense linearly ordered extension $L(M)$ of M is continuously Urysohn. By [5], $L(M)$ has a subspace homeomorphic to $\tilde{M}$ and thus $\tilde{M}$ is a continuously Urysohn space. This contradicts Corollary 2.2. The proof for C is similar. □

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