ON THE IRREDUCIBLE LATTICES OF ORDERS

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1. Introduction. We shall use the following notation:

- \( R = \) Dedekind domain;
- \( K = \) quotient field of \( R; \)
- \( R_p = \) ring of \( p \)-adic integers in \( K, \) \( p \) being a prime ideal in \( R; \)
- \( A = \) finite-dimensional separable \( K \)-algebra;
- \( G = \) \( R \)-order in \( A \) (for the definition cf. (3)).

All modules that occur are assumed to be finitely generated unitary left modules, unless otherwise specified. By a \( G \)-lattice we mean a \( G \)-module which is torsion-free as \( R \)-module. A \( G \)-lattice is called irreducible if it does not contain a proper \( G \)-submodule of smaller \( R \)-rank. If \( \mathfrak{p} \) is a prime ideal in \( R \) we shall write \( G_{\mathfrak{p}} = R_{\mathfrak{p}} \otimes_R G; \) \( M_{\mathfrak{p}} = R_{\mathfrak{p}} \otimes_R M \) for a \( G \)-lattice \( M, \) and \( K_M = K \otimes_R M. \) Two \( G \)-lattices \( M \) and \( N \) are said to lie in the same genus (notation \( M \sim^G N \)) if \( M_{\mathfrak{p}} \cong N_{\mathfrak{p}} \) for every prime ideal \( \mathfrak{p} \) in \( R. \)

For any \( A \)-module \( L, \) let \( S(L) \) be the collection of \( G \)-lattices \( M, \) for which \( K_M = L. \) Suppose that \( S(L) \) splits into \( r_g(L) \) genera, and into \( r_t(L) \) classes under \( G \)-isomorphism. Maranda (6) has shown: If \( L \) is an absolutely irreducible \( A \)-module, then

\[
(1) \quad r_t(L) = h \cdot r_g(L),
\]

where \( h \) is the class number of \( K. \) Moreover, he listed all \( G \)-lattices which are in the same genus as \( M \in S(L). \)

Our aim in this paper is to extend the results of Maranda (6). We shall describe (for a certain type of \( R \)-orders) all irreducible \( G \)-lattices in terms of irreducible lattices over maximal orders containing \( G. \) In § 2 we show that for considerations of irreducible \( G \)-lattices it suffices to look at orders in simple separable algebras. In § 3 we show that the irreducible \( G \)-lattices are also lattices over maximal orders in \( A, \) if for all irreducible \( G \)-lattices, \( \text{End}_G(M) \) is the same maximal order. In § 4 we apply the results of § 3 to extend Maranda’s results; if \( L \) is an absolutely irreducible \( G \)-lattice, then we describe \( S(L) \) explicitly. However, the applications are not restricted to absolutely irreducible \( A \)-modules.

Convention. Homomorphisms will be written opposite to the scalars.

2. Reduction to orders in simple algebras. If \( H \) is any \( R \)-order in \( A \) containing \( G, \) and if \( M \) is an \( H \)-lattice, we write \( M_H \) and \( M_G \) to indicate whether \( M \) should be considered as an \( H \)-lattice or as a \( G \)-lattice, respectively.

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PROPOSITION 1. If $M$ and $N$ are $H$-lattices, then
\[ \text{Hom}_H(M, N) = \text{Hom}_G(M, N). \]

Proof. We have the inclusion
\[ \text{Hom}_H(M, N) \subset \text{Hom}_G(M, N). \]

To show the reverse inclusion, we pick $0 \neq r \in R$ such that $rH \subset G$. For $f \in \text{Hom}_G(M, N)$ we have:
\[ r((xm)f) = (rxm)f = rx(mf), \quad x \in H, m \in M. \]

Since $N$ is $R$-torsion-free, $f \in \text{Hom}_H(M, N)$.

For the remainder of this section we shall denote by $\text{Irr}(G)$ the set of isomorphism classes of irreducible $G$-lattices.

PROPOSITION 2. We have an injection
\[ F: \text{Irr}(H) \rightarrow \text{Irr}(G), \quad F: (M) \rightarrow (M), \]
where $(M)$ denotes the isomorphism class of $M$.

Proof. This map is well-defined, and $(M) \in \text{Irr}(G)$ if $(M) \in \text{Irr}(H)$, since $M$ is an irreducible $G$-lattice if and only if $KM$ is an irreducible $A$-module. Using Proposition 1, we conclude that $F$ is injective.

LEMMA 3. Let $e_i$, $i = 1, \ldots, n$, be the set of mutually orthogonal central primitive idempotents in $A$. Then
\[ H = \sum_{i=1}^{n} \oplus Ge_i \]
is an $R$-order in $A$ containing $G$, and $F: \text{Irr}(H) \rightarrow \text{Irr}(G)$ is a bijection.

Proof. The $e_i$ are integral over $R$, and $\sum_{i=1}^{n} e_i = 1$; therefore $H$ is an $R$-order in $A$ containing $G$. Because of Proposition 2, it only remains to show that $F$ is surjective. Let $M$ be an irreducible $G$-lattice such that $KM$ corresponds to $e_k$. Then
\[ e_i m' = \delta_{ik} m' \quad \text{for every } m' \in KM, \]
\[ \delta_{ik} \text{ is the Kronecker symbol.} \]
Since $1 \otimes_R M$ is canonically isomorphic to $M$, we may assume that $M \subset KM$, so that
\[ e_i m = \delta_{ik} m \quad \text{for every } m \in M, \]
i.e., $M$ is an $H$-lattice, and $F$ is surjective.

Remark 4. By means of Lemma 3, one knows all irreducible $G$-lattices once the irreducible $H$-lattices are known, where
\[ H = \sum_{i=1}^{n} \oplus Ge_i. \]
However, \[ \text{Irr}(H) = \bigcup_{i=1}^{n} \text{Irr}(Ge_i) \]
is the disjoint union of a finite number of sets. Therefore we may restrict our
attention to orders in simple algebras.

Example 5. Let \( \mathfrak{G} \) be a finite abelian group of order \( g \), and suppose that \( K \)
splits \( \mathfrak{G} \). If \( \chi \approx \mathfrak{G} \) is the character group of \( \mathfrak{G} \), then
\[ \text{Irr}(R\mathfrak{G}) = \{ (I_k e_x) : x \in X, I_k \text{ are representatives of the different ideal classes in } R, \text{ and } e_x \text{ is the primitive idempotent to } \chi \}. \]

Proof.
\[ e_x = \frac{1}{g} \sum_{\bar{g} \in \mathfrak{G}} \chi(\bar{g}^{-1}) e_x, \quad \chi \in X. \]

We use the bijection in Lemma 3:
\[ \text{Irr}(H) \to \text{Irr}(R\mathfrak{G}), \]
where \( H = \sum_{\chi \in X} \oplus R\mathfrak{G} e_x \). However, \( R\mathfrak{G} e_x = Re_x \) is the maximal \( R \)-order in \( Ke_x \). Thus
\[ \text{Irr}(Re_x) = \{ (I_k e_x), k = 1, \ldots \text{ (class number of } R) \}, \]
and by Remark 4 we conclude that
\[ \text{Irr}(R\mathfrak{G}) = \{ (I_k e_x) : x \in X, k = 1, \ldots \text{ (class number of } R) \}. \]

3. Irreducible lattices of orders in simple algebras. Let \( G \) be an \( R \)-order in the simple separable finite-dimensional \( K \)-algebra \( A = (D)_n \), \( D \) a skew-
field of finite dimension over \( K \). We put \( C = G \cap D \), viewing \( D \) as embedded
in \( A \). Then \( C \) is an \( R \)-order in \( D \). Let
\[ \{ B_j \} (j \in J) = \text{different maximal } R \text{-orders in } A \text{ containing } G, \]
\[ M_j = \text{a fixed irreducible } B_j \text{-lattice, for every } j \in J. \]

Then
\[ \text{End}_{B_j}(M_j) \text{ is a maximal } R \text{-order in } D; \]
\[ \{ I_k \}, k \in J(C_j) = \text{representatives of the different classes of left } C_j \text{-ideals in } D. \]

With this notation we can write down a full set of non-isomorphic irreducible
\( B_j \)-lattices for every \( j \in J \):
\[ (2) \quad \text{Irr}(B_j) = \{ (M_j \otimes_{C_j} I_k) : k \in J(C_j) \}; \]
cf. (1; 8).
THEOREM 6. Let \( \text{Irr}(G) \) denote the set of isomorphism classes of irreducible \( G \)-lattices. Then

(i) \( \text{card}(\text{Irr}(G)) \geq \sum_{j \in J} \text{card}(J(C)_j) \);

(ii) We have equality in (i) if \( C = \text{End}_G(M) \) for every irreducible \( G \)-lattice \( M \);

(iii) In the latter case, we can give all irreducible \( G \)-lattices explicitly: Let \( \{I_k\}, k \in J(C), \) be representatives of the different classes of left \( C \)-ideals in \( D \); then

\[
\text{Irr}(G) = \{(M \otimes_C I_k): j \in J, k \in J(C)\}.
\]

Moreover, in this case we have:

\( \text{card}(\text{Irr}(G)) = \text{card}(J)(\text{card}(J(C))) \);

(iv) If we have equality in (i), then there are \( \text{card}(J) \) genera of irreducible \( G \)-lattices, and in each genus there are \( \text{card}(J(C)) \) different isomorphism classes of irreducible \( G \)-lattices. Moreover,

\[
\{M \otimes_C I_k: k \in J(C)\}
\]

are the non-isomorphic irreducible \( G \)-lattices which lie in the same genus as the irreducible \( G \)-lattice \( M \), and representatives of the different genera of irreducible \( G \)-lattices are the \( G \)-lattices

\[
\{M_j: j \in J\}.
\]

The proof of Theorem 6 is done in several steps, as follows.

PROPOSITION 7. Let \( M \) be an irreducible \( B_j \)-lattice, \( N \) an irreducible \( B_k \)-lattice, \( j, k \in J, j \neq k \), then \( M \otimes \alpha N \) and \( N \otimes \alpha M \) are not isomorphic as \( G \)-lattices.

Proof. Assume that \( M \otimes \alpha N \) and \( N \otimes \alpha M \) are isomorphic, and let \( \phi: M \otimes \alpha N \to N \otimes \alpha M \) be a \( G \)-isomorphism. Then we make \( M \) into a \( B_k \)-lattice, denoted by \( M_k \), by defining

\[
b_k m_k = (b k(m f)) f^{-1}, \quad b_k \in B_k, m_k \in M_k, m_k = m.
\]

It is easily checked that the action of \( B_j \) on \( M \) and the action of \( B_k \) on \( M_k \) coincide on \( B_j \cap B_k \supset G \). From (1, Theorem 3.9) it follows that

\[
C_j = \text{End}_{B_j}(M), \quad B_j = \text{End}_{C_j}(M),
\]

\[
C_k = \text{End}_{B_k}(M_k), \quad B_k = \text{End}_{C_k}(M_k).
\]

Now we apply Proposition 1 and conclude that

\[
C_j = \text{End}_{B_j}(M) = \text{End}_\alpha(M) = \text{End}_{B_k}(M_k) = C_k;
\]

thus \( B_j = B_k \), and we have deduced a contradiction.

Proof of Theorem 6(i). Because of (2) and Proposition 7, the \( G \)-lattices

\[
\{M_j \otimes_{C_j} I_k, k \in J(C)_j, j \in J\}
\]

are non-isomorphic irreducible \( G \)-lattices, whence the inequality (i) in Theorem 6 follows.
Proof of Theorem 6(ii). If \( C = \text{End}_G(M) \) for every irreducible \( G \)-lattice \( M \), then we have equality in Theorem 6(i). The hypothesis implies that \( C \) is maximal: Let \( M \) be an irreducible \( B_j \)-lattice for some \( j \in J \); then \( \text{End}_{B_j}(M) = \text{End}_G(M) = C \) is a maximal \( R \)-order in \( D \). To prove Theorem 6(ii) we have to show that every irreducible \( G \)-lattice is a \( B_j \)-lattice for some maximal order \( B_j, j \in J \). Let \( M \) be an irreducible \( G \)-lattice. Then \( M \) is a right \( C \)-lattice, since \( C = \text{End}_G(M) \), and \( B = \text{End}_C(M) \) is a maximal \( R \)-order in

\[
K \otimes_R \text{End}_C(M) = \text{End}_D(KM) = A;
\]
cf. (1, Theorem 3.9). Since \( M \) was a \( G \)-lattice to start with, \( G \subset B = \text{End}_C(M) \), and \( M \) is a \( B \)-lattice in the usual fashion.

Proof of Theorem 6(iii). If Theorem 6(ii) holds, then \( C_j = C \) for every \( j \in J \) (\( C_j = \text{End}_{B_j}(M_j) \)), cf. the beginning of § 3), and a full set of non-isomorphic irreducible \( G \)-lattices is given by

\[
\{ M_j \otimes_C I_k; j \in J, k \in J(C) \}.
\]

Proof of Theorem 6(iv). We shall prove the following lemma, which is of interest in itself.

Lemma 8. Let \( M \) be an irreducible \( G \)-lattice such that \( M \) is also a \( B_j \)-lattice for some \( j \in J \); let \( C_j = \text{End}_{B_j}(M) \). Then

\[
\{ M \otimes_{C_j} I_k; k \in J(C_j) \}
\]
are all the non-isomorphic \( G \)-lattices in the same genus as \( M \).

For the notation, compare the beginning of § 3.

Proof. Since \( C_j \) is a maximal \( R \)-order in \( D \), all the \( G \)-lattices \( M \otimes_{C_j} I_k \) are non-isomorphic, and they lie in the same genus as \( M \). Now let \( N \) be a \( G \)-lattice in the same genus as \( M \). Then \( N_p \) is a \( (B_j)_p \)-lattice for every prime ideal \( p \) in \( R \). However, this can only be if \( N \) is a \( B_j \)-lattice itself. Therefore, \( N \cong M \otimes_{C_j} I_k \) for some \( k \in J(C_j) \).

Corollary 9. If \( M \) and \( N \) are irreducible \( G \)-lattices such that \( M \) is a \( B_j \)-lattice for some \( j \in J \) and \( N \) is a \( B_k \)-lattice for some \( k \in J \), then \( N_0 \) is in the same genus as \( N \) if and only if \( B_j = B_k \).  

Corollary 10. If \( L \) is an irreducible \( A \)-module, then

\[
r_0(L) \geq \text{card}(J).
\]

For the definition of \( r_0(L) \), compare § 1.

The proof of Theorem 6(iv) follows now easily if one observes that we have equality in Theorem 6(i), i.e. every irreducible \( G \)-lattice is isomorphic to some \( B_j \)-lattice.

This completes the proof of Theorem 6.
4. Applications of Theorem 6 to some special orders. Let \( A \) be a separable finite-dimensional \( K \)-algebra.

**Lemma 11.** If \( R \) is a Dedekind domain such that the class number of \( R \) is finite and such that \((R:p)\) is finite for every prime ideal \( p \) in \( R \), then there are only finitely many different maximal \( R \)-orders in \( A \) containing a fixed \( R \)-order \( G \) in \( A \).

*Proof.* There is only a finite number of non-isomorphic irreducible \( A \)-modules, say \( L_1, \ldots, L_t \). Under the hypotheses on \( R \), the Jordan-Zassenhaus theorem is valid (cf. §10), i.e. for the \( R \)-order \( G \), \( S(L_i) \) (cf. §1) contains only a finite number of non-isomorphic irreducible \( G \)-lattices. Now the result follows from Proposition 7 if one observes that every maximal \( R \)-order in \( A \) decomposes into a direct sum of maximal orders in the simple components of \( A \). The main applications of Theorem 6 can be gained by using the following result.

**Lemma 12.** Let \( G \) be an \( R \)-order in the simple separable \( K \)-algebra \( A = (K')_n \), \( K' \) an extension field of finite dimension over \( K \). If \( G \subseteq K = C \) is the maximal \( R \)-order in \( K' \), then every irreducible \( G \)-lattice is an irreducible lattice for some maximal \( R \)-order in \( A \) containing \( G \), i.e. Theorem 6(iii), (iv) can be applied.

*Proof.* It only remains to show that \( \text{End}_G(M) = C \) for every irreducible \( G \)-lattice \( M \); then the lemma follows from Theorem 6(ii). Since \( C \) is the only maximal \( R \)-order in \( D \), \( \text{End}_G(M) \subseteq C \) for every irreducible \( G \)-lattice \( M \). But since \( C \) is commutative and is contained in the centre of \( G \), \( \text{End}_G(M) = C \).

For the remainder of the paper we adopt the following notation:
\( A \) is a separable finite-dimensional \( K \)-algebra;
\( L \) = irreducible \( A \)-module;
\( D_L = \text{End}_A(L) \);
\( e_L \) = central primitive idempotent corresponding to \( L \);
\( A e_L = \text{End}_G(L) \) = simple component of \( A \) corresponding to \( L \).

For an \( R \)-order \( G \) in \( A \) we let:
\( C_L = Ge_L \cap D_L \);
\( B^L_j, j \in J_L \) = different maximal \( R \)-orders in \( A e_L \) containing \( Ge_L \);
\( M^L_j \) = irreducible \( B^L_j \)-lattice, \( j \in J_L \);
\( I^L_k, k \in J(C_L) \) = representatives of the classes of left \( C_L \)-ideals in \( D \);
\( S(L) = \{ M : M = G \text{-lattice}, KM \cong L \} \).

**Theorem 13.** If \( D_L \) is commutative and if \( C_L \) is the maximal \( R \)-order in \( D \), then

(i) all irreducible non-isomorphic \( G \)-lattices in \( S(L) \) are given by
\[ \{ M^L_j \otimes e_L I^L_k, j \in J_L, k \in J(C_L) \}, \]

(ii) \( S(L) \) splits into \( \text{card}(J_L) \) genera:
\[ \{ M^L_j \otimes e_L I^L_k, k \in J(C_L) \}, \quad j \in J_L, \]

(iii) \( r_t(L) = (\text{card}(J(C))) r_t(L), r_t(L) = \text{card}(J_L), \quad (\text{this is an extension of Maranda's results (6))}. \)

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Remark 14. In the special case where \( L \) is an absolutely irreducible \( A \)-module, we obtain the well-known formula (1).

References


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