Bull. Aust. Math. Soc. **96** (2017), 487–495
doi:10.1017/S0004972717000351

SOME CONVERGENCE THEOREMS IN FOURIER ALGEBRAS

HEYBETKULU MUSTAFAYEV

(Received 23 February 2017; accepted 6 March 2017; first published online 3 July 2017)

Abstract

Let G be a locally compact amenable group and $A(G)$ and $B(G)$ be the Fourier and the Fourier–Stieltjes algebras of G, respectively. For a power bounded element u of $B(G)$, let $\mathcal{E}_u := \{g \in G : |u(g)| = 1\}$. We prove some convergence theorems for iterates of multipliers in Fourier algebras.

(a) If $\|u\|_{B(G)} \leq 1$, then $\lim_{n\to\infty} \|u^n v\|_{A(G)} = \mathrm{dist}(v, I_{\mathcal{E}_u})$ for $v \in A(G)$, where $I_{\mathcal{E}_u} = \{v \in A(G) : v(\mathcal{E}_u) = \{0\}\}$.
(b) The sequence $\{u^n v\}_{n\in\mathbb{N}}$ converges for every $v \in A(G)$ if and only if $\mathcal{E}_u$ is clopen and $u(\mathcal{E}_u) = \{1\}$.
(c) If the sequence $\{u^n v\}_{n\in\mathbb{N}}$ converges weakly in $A(G)$ for some $v \in A(G)$, then it converges strongly.

2010 *Mathematics subject classification*: primary 43A30; secondary 46J20.

Keywords and phrases: locally compact group, Fourier algebra, Fourier–Stieltjes algebra, convergence.

1. Introduction and preliminaries

The main purpose of this note is to prove some convergence theorems for iterates of multipliers in Fourier algebras.

We begin with some notations and definitions. Let X be a complex Banach space and let $B(X)$ be the algebra of all bounded linear operators on X. Let $X_{(1)}$ denote the closed unit ball of X.

Let G be a locally compact group with a fixed left Haar measure. The Fourier–Stieltjes algebra $B(G)$ and the Fourier algebra $A(G)$ of G, introduced by Eymard in [3], are central objects in harmonic analysis. The Fourier–Stieltjes algebra $B(G)$ is the linear span of the set of all continuous positive-definite functions on G. In fact, for every $u \in B(G)$, there exist a unitary representation π of G and vectors ξ and η in the representation space of π such that $u(g) = \langle \pi(g)\xi, \eta\rangle$ for all $g \in G$. Equipped with pointwise multiplication and the norm

$$\|u\|_{B(G)} = \inf\{\|\xi\| \cdot \|\eta\|\},$$

where the infimum is taken over all pairs (ξ, η) of such representations of u, $B(G)$ is a commutative semisimple Banach algebra [3].

The Fourier algebra $A(G)$ is the linear space of all functions of the form $f := h * \widetilde{k}$, where $h, k \in L^2(G)$ and $\widetilde{k}(g) = \overline{k(g^{-1})}$. With pointwise multiplication and the norm

$$\|f\|_{A(G)} = \inf\{\|h\|_2\|k\|_2 : f = h * \widetilde{k}\},$$

$A(G)$ is a commutative semisimple regular Tauberian Banach algebra. The Gelfand space of $A(G)$ can be identified with G via Dirac measures. Moreover, $A(G)$ is a closed ideal of $B(G)$ [3]. If $h \in L^2(G)$ and $s \in G$, define $L_s h(g) = h(s^{-1}g)$. Let $VN(G)$ denote the closure in the weak operator topology of the linear span of $\{L_g : g \in G\}$ in $B(L^2(G))$. The algebra $A(G)$ is the unique predual of the von Neumann algebra $VN(G)$. Each $f = h * \widetilde{k}$ in $A(G)$ can be regarded as an ultraweakly continuous linear functional on $VN(G)$ defined by

$$\langle S, h * \widetilde{k}\rangle = \langle Sh, \widetilde{k}\rangle, \quad S \in VN(G).$$

It follows that $\langle L_g, f\rangle = f(g)$ for all $f \in A(G)$ and $g \in G$.

Let A be a commutative Banach algebra. We will denote by Σ_A the Gelfand space of A equipped with the w^*-topology and by $\widehat{a}$, where $\widehat{a}(\gamma) = \gamma(a)$ $(\gamma \in \Sigma_A)$, the Gelfand transform of $a \in A$. A linear mapping $T : A \to A$ is called a *multiplier* of A if

$$T(ab) = (Ta)b \ (= a(Tb)) \quad \text{for all } a, b \in A.$$

When A is semisimple, the set of all multipliers of A is a commutative, unital, closed and full subalgebra of $B(A)$ [9].

For each $u \in B(G)$, the operator $L_u : A(G) \to A(G)$, defined by $L_u v = uv$ $(v \in A(G))$, is a multiplier of $A(G)$. If G is amenable, then every multiplier of $A(G)$ is of this form and the map $u \mapsto L_u$ is isometric [1].

A commutative Banach algebra A is said to be *regular* if given a closed subset K of Σ_A and $\gamma \in \Sigma_A \setminus K$, there exists an $a \in A$ such that $\widehat{a}(\gamma) \neq 0$ and $\widehat{a}(K) = \{0\}$. A semisimple regular Banach algebra A is said to be *Tauberian* if $\overline{A_{00}} = A$, where

$$A_{00} := \{a \in A : \ \operatorname{supp} \widehat{a} \text{ is compact}\}.$$

The Tauberian condition for A implies that every proper closed ideal of A is contained in a maximal modular ideal.

Let A be a regular semisimple Banach algebra. For a closed subset K of Σ_A, there are two distinguished closed ideals in A with hull equal to K:

$$I_K := \{a \in A : \widehat{a}(K) = \{0\}\}$$

is the largest closed ideal whose hull is K and $J_K := \overline{J_K^0}$ is the smallest closed ideal whose hull is K, where

$$J_K^0 := \{a \in A_{00} : \ \operatorname{supp} \widehat{a} \cap K = \emptyset\}.$$

The set K is said to be a *set of synthesis* for A if $I_K = J_K$ [10, Section 8.3].

An element a of a Banach algebra A (not necessarily commutative) is said to be *power bounded* if $\sup_{n\geq 0} \|a^n\| < \infty$.

The following results are well known (the assertion (d) is contained in [2]).

https://doi.org/10.1017/S0004972717000351 Published online by Cambridge University Press

Proposition 1.1. *Let X be a Banach space and let $T \in B(X)$.*

(a) *For every $x \in X$,*

$$\operatorname{dist}(x, \overline{(I-T)X}) = \sup\{|\langle \varphi, x\rangle| : T^*\varphi = \varphi, \varphi \in X^*_{(1)}\}.$$

(b) *If T is power bounded, then*

$$\overline{(I-T)X} = \Big\{x \in X : \lim_{n\to\infty}\Big\|\frac{1}{n}\sum_{k=0}^{n-1} T^k x\Big\| = 0\Big\}.$$

(c) *If T is a contraction, then, for every $x \in X$,*

$$\lim_{n\to\infty}\Big\|\frac{1}{n}\sum_{k=0}^{n-1} T^k x\Big\| = \operatorname{dist}(x, \overline{(I-T)X}).$$

(d) *If T is a contraction, then, for every $x \in X$,*

$$\lim_{n\to\infty}\|T^n x\| = \sup\Big\{|\langle \varphi, x\rangle| : \varphi \in \bigcap_{n=0}^{\infty} T^{*n}(X^*_{(1)})\Big\}.$$

(e) *If T is power bounded and $x \in X$, then $(1/n)\sum_{k=0}^{n-1} T^k x \to 0$ weakly implies $(1/n)\sum_{k=0}^{n-1} T^k x \to 0$ strongly as $n \to \infty$.*

2. Convergence theorems

In this section, we present some results concerning convergence in Fourier algebras. If $u \in B(G)$, then

$$\mathcal{J}_u := \overline{(1-u)A(G)}$$

is a closed ideal in $A(G)$ associated with u and hull $(\mathcal{J}_u) = \mathcal{F}_u$, where

$$\mathcal{F}_u = \{g \in G : u(g) = 1\}.$$

If $u \in B(G)$ is power bounded, then

$$\mathcal{I}_u := \{v \in A(G) : \lim_{n\to\infty}\|u^n v\|_{A(G)} = 0\}$$

is another closed ideal in $A(G)$ associated with u. Notice also that $|u(g)| \le 1$ for all $g \in G$. We put

$$\mathcal{E}_u := \{g \in G : |u(g)| = 1\}.$$

As proved in [7, Theorem 2.6] and [11, Proposition 2.1], hull$(\mathcal{I}_u) = \mathcal{E}_u$. Since the algebra $A(G)$ is Tauberian, $\mathcal{E}_u = \emptyset$ if and only if $\|u^n v\|_{A(G)} \to 0$ for all $v \in A(G)$. Hence, we may assume that $\mathcal{E}_u \neq \emptyset$.

https://doi.org/10.1017/S0004972717000351 Published online by Cambridge University Press

The *coset ring* of a locally compact group G, denoted by $\mathcal{R}(G)$, is the smallest Boolean algebra of subsets of G containing left cosets of all subgroups of G. As in [5], define the *closed coset ring* $\mathcal{R}_c(G)$ of G by

$$\mathcal{R}_c(G) = \{E \in \mathcal{R}(G_d) : E \text{ is closed in } G\},$$

where G_d is the algebraic group G with the discrete topology. From [7, Theorem 4.1], if $u \in B(G)$ is power bounded, then $\mathcal{E}_u \in \mathcal{R}_c(G)$. On the other hand, if G is amenable, then every subset in $\mathcal{R}_c(G)$ is a set of synthesis for $A(G)$ [5, Lemma 2.2]. Consequently, if $u \in B(G)$ is power bounded, then $\mathcal{E}_u$ is a set of synthesis for $A(G)$ in the case when G is amenable. Furthermore, since $(1 + u)/2$ is power bounded and $\mathcal{F}_u = \mathcal{E}_{(1+u)/2}$, the set $\mathcal{F}_u$ is also a set of synthesis for $A(G)$.

Proposition 2.1. *If G is amenable, then, for arbitrary $u \in B(G)_{(1)}$ and $v \in A(G)$,*

$$\lim_{n\to\infty} \Big\| \frac{1}{n} \sum_{k=0}^{n-1} u^k v \Big\|_{A(G)} = \operatorname{dist}(v, I_{\mathcal{F}_u}),$$

where $I_{\mathcal{F}_u} = \{v \in A(G) : v(\mathcal{F}_u) = \{0\}\}$.

Proof. Applying Proposition 1.1(c) to the operator L_u on the space $A(G)$,

$$\lim_{n\to\infty} \Big\| \frac{1}{n} \sum_{k=0}^{n-1} u^k v \Big\|_{A(G)} = \operatorname{dist}(v, \mathcal{J}_u).$$

On the other hand, since $\mathcal{F}_u$ is a set of synthesis for $A(G)$ and $\operatorname{hull}(\mathcal{J}_u) = \mathcal{F}_u$, we have $\mathcal{J}_u = I_{\mathcal{F}_u}$, where $I_{\mathcal{F}_u} = \{v \in A(G) : v(\mathcal{F}_u) = \{0\}\}$. □

Proposition 2.2. *If $u \in B(G)$ is power bounded, then the sequence*

$$\Big\{ \frac{1}{n} \sum_{k=0}^{n-1} u^k v \Big\}_{n\in\mathbb{N}}$$

converges in $A(G)$ for every $v \in A(G)$ if and only if $\mathcal{F}_u$ is clopen (closed and open).

Proof. Notice that

$$\mathcal{K}_u := \Big\{ v \in A(G) : \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} u^k v \text{ exists} \Big\}$$

is a closed ideal in $A(G)$. Let

$$\mathcal{L}_u := \{v \in A(G) : uv = v\}.$$

By [8, Ch. 2, Theorem 1.3], we can write $\mathcal{K}_u = \mathcal{J}_u \oplus \mathcal{L}_u$, where $\mathcal{J}_u = \overline{(1 - u)A(G)}$. Further, if $v \in \mathcal{L}_u$, then it follows from the identity

$$[(1 - u(g))]v(g) = 0 \quad \text{for all } g \in G$$

https://doi.org/10.1017/S0004972717000351 Published online by Cambridge University Press

that

$$\mathcal{L}_u = \{v \in A(G) : v(g) = 0, \text{ for all } g \in G \setminus \mathcal{F}_u\}.$$

Since $A(G)$ is regular, $\mathrm{hull}(\mathcal{L}_u) = \overline{G \setminus \mathcal{F}_u}$. Now assume that the sequence

$$\left\{\frac{1}{n}\sum_{k=0}^{n-1} u^k v\right\}_{n\in\mathbb{N}}$$

converges for every $v \in A(G)$. As $\mathcal{K}_u = A(G)$, we have $A(G) = \mathcal{J}_u \oplus \mathcal{L}_u$, so that $\mathrm{hull}(\mathcal{J}_u) \cap \mathrm{hull}(\mathcal{L}_u) = \emptyset$. Since $\mathrm{hull}(\mathcal{J}_u) = \mathcal{F}_u$, we can write

$$\overline{G \setminus \mathcal{F}_u} = \mathrm{hull}(\mathcal{L}_u) \subseteq G \setminus \mathrm{hull}(\mathcal{J}_u) = G \setminus \mathcal{F}_u.$$

It follows that $\mathcal{F}_u$ is a clopen set.

Assume that $\mathcal{F}_u$ is clopen. Then $\mathrm{hull}(\mathcal{L}_u) = G \setminus \mathcal{F}_u$ and, therefore,

$$\mathrm{hull}(\mathcal{K}_u) = \mathrm{hull}(\mathcal{J}_u) \cap \mathrm{hull}(\mathcal{L}_u) = \emptyset.$$

Since the algebra $A(G)$ is Tauberian, we have $\mathcal{K}_u = A(G)$. □

The main result of this note is the following theorem.

Theorem 2.3. *If G is amenable, then, for arbitrary $u \in B(G)_{(1)}$ and $v \in A(G)$,*

$$\lim_{n\to\infty} \|u^n v\|_{A(G)} = \mathrm{dist}(v, I_{\mathcal{E}_u}),$$

where $I_{\mathcal{E}_u} = \{v \in A(G) : v(\mathcal{E}_u) = \{0\}\}$.

Proof. For arbitrary $u \in B(G)$ and $S \in VN(G)$, define $u \circ S \in VN(G)$ by

$$\langle u \circ S, v\rangle = \langle S, uv\rangle, \quad v \in A(G).$$

Clearly, $u \circ S = L_u^*(S)$. Now let $u \in B(G)_{(1)}$ be given. Applying Proposition 1.1(d) to the operator L_u on the space $A(G)$,

$$\lim_{n\to\infty} \|u^n v\| = \sup\left\{|\langle S, v\rangle| : S \in \bigcap_{n=0}^{\infty} u^n \circ VN(G)_{(1)}\right\} \quad \text{for all } v \in A(G).$$

Let us show that

$$\bigcap_{n=0}^{\infty} u^n \circ VN(G)_{(1)} = \{S \in VN(G)_{(1)} : |u|^2 \circ S = S\}.$$

Let $S \in VN(G)_{(1)}$ be such that $|u|^2 \circ S = S$. Since

$$S = |u|^{2n} \circ S = u^n \circ (\overline{u}^n \circ S) \quad (n = 0, 1, 2, \ldots)$$

and $\overline{u}^n \circ S \in VN(G)_{(1)}$, we see that $S \in \bigcap_{n=0}^{\infty} u^n \circ VN(G)_{(1)}$.

https://doi.org/10.1017/S0004972717000351 Published online by Cambridge University Press

For the reverse inclusion, let

$$S \in \bigcap_{n=0}^{\infty} u^n \circ VN(G)_{(1)}.$$

For arbitrary $v \in A(G)$, the function $w := (1 - |u|^2)v$ vanishes on $\mathcal{E}_u$ and therefore $w \in I_{\mathcal{E}_u}$, where

$$I_{\mathcal{E}_u} = \{v \in A(G) : v(\mathcal{E}_u) = \{0\}\}.$$

As we have noted above, $\mathrm{hull}(\mathcal{I}_u) = \mathcal{E}_u$ and $\mathcal{E}_u$ is a set of synthesis for $A(G)$. Consequently, $I_{\mathcal{E}_u} = \mathcal{I}_u$ and, therefore, $w \in \mathcal{I}_u$. So, $\|u^n w\|_{A(G)} \to 0$. Further, for every $n \in \mathbb{N}$, there exists $S_n \in VN(G)_{(1)}$ such that $S = u^n \circ S_n$. Thus,

$$|\langle S, w\rangle| = |\langle u^n \circ S_n, w\rangle| = |\langle S_n, u^n w\rangle| \le \|u^n w\|_{A(G)} \to 0.$$

Now, since

$$\langle S, (1 - |u|^2)v\rangle = 0 \quad \text{for all } v \in A(G),$$

we obtain $|u|^2 \circ S = S$. Consequently,

$$\lim_{n\to\infty} \|u^n v\| = \sup\{|\langle S, v\rangle| : |u|^2 \circ S = S, S \in VN(G)_{(1)}\}.$$

On the other hand, by Proposition 1.1(a),

$$\sup\{|\langle S, v\rangle| : |u|^2 \circ S = S, S \in VN(G)_{(1)}\} = \mathrm{dist}(v, \mathcal{J}_{|u|^2}),$$

where

$$\mathcal{J}_{|u|^2} = \overline{(1 - |u|^2)A(G)}.$$

Since $\mathrm{hull}(\mathcal{J}_{|u|^2}) = \mathcal{E}_u$ and $\mathcal{E}_u$ is a set of synthesis for $A(G)$, we have $\mathcal{J}_{|u|^2} = I_{\mathcal{E}_u}$. Thus,

$$\lim_{n\to\infty} \|u^n v\|_{A(G)} = \mathrm{dist}(v, \mathcal{J}_{|u|^2}) = \mathrm{dist}(v, I_{\mathcal{E}_u}). \qquad \square$$

Let $u \in B(G)$ be power bounded and $C_u := \sup_{n\ge 0} \|u^n\|_{B(G)}$. Define a new norm $\|v\|_1$ on $A(G)$ by $\|v\|_1 = \sup_{n\ge 0} \|u^n v\|_{A(G)}$. Then

$$\|v\|_{A(G)} \le \|v\|_1 \le C_u \|v\|_{A(G)},$$

so that the norms $\|v\|_{A(G)}$ and $\|v\|_1$ on $A(G)$ are equivalent.

The following result is an immediate consequence of Theorem 2.3.

Corollary 2.4. *Suppose that G is amenable and $u \in B(G)$ is power bounded. Define $C_u := \sup_{n\ge 0} \|u^n\|_{B(G)}$. For arbitrary $v \in A(G)$,*

$$\frac{1}{C_u}\mathrm{dist}(v, I_{\mathcal{E}_u}) \le \varliminf_{n\to\infty} \|u^n v\|_{A(G)} \le \varlimsup_{n\to\infty} \|u^n v\|_{A(G)} \le C_u \mathrm{dist}(v, I_{\mathcal{E}_u}),$$

where $I_{\mathcal{E}_u} = \{v \in A(G) : v(\mathcal{E}_u) = \{0\}\}$.

https://doi.org/10.1017/S0004972717000351 Published online by Cambridge University Press

From Proposition 1.1(e), if $u \in B(G)$ is power bounded, then $(1/n)\sum_{k=0}^{n-1} u^k v \to 0$ weakly in $A(G)$ implies $(1/n)\sum_{k=0}^{n-1} u^k v \to 0$ strongly as $n \to \infty$.

Corollary 2.5. *Let G be amenable, $u \in B(G)$ be power bounded and $v \in A(G)$. If the sequence $\{u^n v\}_{n\in\mathbb{N}}$ converges weakly in $A(G)$, then it converges strongly.*

Proof. It suffices to show that $u^n v \to 0$ weakly implies $\|u^n v\|_{A(G)} \to 0$. Since

$$|u(g)|^n |v(g)| = |\langle L_g, u^n v\rangle| \to 0 \quad \text{for all } g \in G,$$

it follows that v vanishes on $\mathcal{E}_u$, that is, $v \in I_{\mathcal{E}_u}$. By Corollary 2.4,

$$\lim_{n\to\infty} \|u^n v\|_{A(G)} \to 0. \qquad \square$$

Proposition 2.6. *Let G be amenable and $u \in B(G)$ be power bounded. The sequence $\{u^n v\}_{n\in\mathbb{N}}$ converges for every $v \in A(G)$ if and only if $\mathcal{E}_u$ is clopen and $u(\mathcal{E}_u) = \{1\}$.*

Proof. Assume that $\mathcal{E}_u$ is clopen and $u(\mathcal{E}_u) = \{1\}$. Note that the condition $u(\mathcal{E}_u) = \{1\}$ means that the sets $\mathcal{F}_u$ and $\mathcal{E}_u$ coincide. As $\mathcal{F}_u$ is clopen, by Proposition 2.2, for arbitrary $v \in A(G)$, there exists $w \in A(G)$ such that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} u^k v = w.$$

Since $uw = w$, this implies that

$$\lim_{n\to\infty} \Big\| \frac{1}{n}\sum_{k=0}^{n-1} u^k (v-w)\Big\|_{A(G)} = 0.$$

On the other hand, applying Proposition 1.1(b) to the operator L_u on the space $A(G)$,

$$\mathcal{J}_u = \Big\{ v \in A(G) : \lim_{n\to\infty} \Big\| \frac{1}{n}\sum_{k=0}^{n-1} u^k v\Big\|_{A(G)} = 0\Big\},$$

so that $v - w \in \mathcal{J}_u$. Since $\mathcal{E}_u = \mathcal{F}_u$ and the set $\mathcal{E}_u$ (or $\mathcal{F}_u$) is a set of synthesis for $A(G)$, the identities $\mathrm{hull}(\mathcal{I}_u) = \mathcal{E}_u = \mathcal{F}_u = \mathrm{hull}(\mathcal{J}_u)$ yield $\mathcal{I}_u = \mathcal{J}_u$. Consequently, $v - w \in \mathcal{I}_u$ and, therefore,

$$u^n v - w = u^n (v - w) \to 0.$$

Assume that the sequence $\{u^n v\}_{n\in\mathbb{N}}$ converges for every $v \in A(G)$. It follows from Proposition 2.2 that $\mathcal{F}_u$ is clopen. Further, since

$$\lim_{n\to\infty} \|u^{n+1} v - u^n v\|_{A(G)} = 0,$$

we have

$$|u(g)^{n+1} v(g) - u(g)^n v(g)| \to 0$$

for all $v \in A(G)$ and $g \in G$. On the other hand, for every $g \in G$, there exists $v \in A(G)$ such that $v(g) \neq 0$. It follows that

$$|u(g)|^n |u(g) - 1| \to 0 \quad \text{for all } g \in G.$$

For $g \in \mathcal{E}_u$, since $|u(g)| = 1$, we have $u(g) = 1$. Hence, $\mathcal{E}_u \subseteq \mathcal{F}_u$ and so $\mathcal{E}_u = \mathcal{F}_u$. $\square$

https://doi.org/10.1017/S0004972717000351 Published online by Cambridge University Press

As usual, $M(G)$ and $L^1(G)$ denote the measure algebra and the group algebra of G, respectively. When G is abelian, $L^1(G) \simeq A(\widehat{G})$, $M(G) \simeq B(\widehat{G})$ and $L^\infty(G) \simeq VN(\widehat{G})$, where $\widehat{G}$ is the dual group of G. Here, $\simeq$ stands for 'isometrically isomorphic'. Consequently, every result about $A(G)$ or $B(G)$ entails a corresponding statement for the L^1 or the measure algebra, respectively.

Let $\widehat{f}$ and $\widehat{\mu}$ denote the Fourier and the Fourier–Stieltjes transforms of $f \in L^1(G)$ and $\mu \in M(G)$, respectively. For arbitrary $\mu \in M(G)$, set

$$\mathcal{E}_\mu := \{\chi \in \widehat{G} : |\widehat{\mu}(\chi)| = 1\}.$$

For $n \in \mathbb{N}$, let μ^n denote the nth convolution power of $\mu \in M(G)$, where $\mu^0 := \delta_0$ is the Dirac measure concentrated at $\{0\}$. The classical Foguel theorem [4] states that a power bounded measure $\mu \in M(G)$ is mixing by convolution in the sense that $\|\mu^n * f\|_1 \to 0$ for all $f \in L^1(G)$ with $\widehat{f}(0) = 0$ if and only if $\mathcal{E}_\mu = \{0\}$. In [6, Theorem 2], Granirer proved that if $\mu \in M(G)$ is power bounded and $f \in L^1(G)$, then $\|\mu^n * f\|_1 \to 0$ if and only if $\widehat{f}$ vanishes on $\mathcal{E}_\mu$.

We have the following quantitative generalisations of these results.

Corollary 2.7. *Let G be a locally compact abelian group, $\mu \in M(G)$ be power bounded and $C_\mu := \sup_{n\geq 0} \|\mu^n\|_1$. For arbitrary $f \in L^1(G)$,*

$$\frac{1}{C_\mu}\mathrm{dist}(f, I_{\mathcal{E}_\mu}) \leq \varliminf_{n\to\infty} \|\mu^n * f\|_1 \leq \varlimsup_{n\to\infty} \|\mu^n * f\|_1 \leq C_\mu \mathrm{dist}(f, I_{\mathcal{E}_\mu}).$$

In particular, if $\mu \in M(G)_{(1)}$, then

$$\lim_{n\to\infty} \|\mu^n * f\|_1 = \mathrm{dist}(f, I_{\mathcal{E}_\mu}) \quad \text{for all } f \in L^1(G),$$

where $I_{\mathcal{E}_\mu} = \{f \in L^1(G) : \widehat{f}(\mathcal{E}_\mu) = \{0\}\}$.

If G is a compact abelian group, then $L^p(G)$ $(1 \leq p < \infty)$ with the convolution as multiplication and the usual norm is a commutative, semisimple and regular Banach algebra. The Gelfand space of $L^p(G)$ is $\widehat{G}$ and the Gelfand transform of $f \in L^p(G)$ is just the Fourier transform of f. As $\widehat{G}$ is discrete, every subset of $\widehat{G}$ is a set of synthesis for $L^p(G)$.

The proof of the following proposition is similar to the proof of Theorem 2.3.

Proposition 2.8. *Let G be a compact abelian group, $\mu \in M(G)$ be power bounded and $C_\mu := \sup_{n\geq 0} \|\mu^n\|_1$. For arbitrary $f \in L^p(G)$ $(1 < p < \infty)$,*

$$\frac{1}{C_\mu}\mathrm{dist}(f, I_{\mathcal{E}_\mu}) \leq \varliminf_{n\to\infty} \|\mu^n * f\|_p \leq \varlimsup_{n\to\infty} \|\mu^n * f\|_p \leq C_\mu \mathrm{dist}(f, I_{\mathcal{E}_\mu}).$$

In particular, if $\mu \in M(G)_{(1)}$, then

$$\lim_{n\to\infty} \|\mu^n * f\|_p = \mathrm{dist}(f, I_{\mathcal{E}_\mu}) \quad \text{for all } f \in L^p(G),$$

where $I_{\mathcal{E}_\mu} = \{f \in L^p(G) : \widehat{f}(\mathcal{E}_\mu) = \{0\}\}$.

Acknowledgement

I am grateful to the referee for his helpful remarks and suggestions.

References

[1] A. Derighetti, 'Some results on the Fourier–Stieltjes algebra of a locally compact group', *Comment. Math. Helv.* **45** (1970), 219–228.
[2] Y. Derriennic and M. Lin, 'Sur le comportement asymptotique des puissances de convolution d'une probabilité', *Ann. Inst. H. Poincaré* **20** (1984), 127–132.
[3] P. Eymard, 'L'algébre de Fourier d'un groupe localement compact', *Bull. Soc. Math. France* **92** (1964), 181–236.
[4] S. R. Foguel, 'On iterates of convolutions', *Proc. Amer. Math. Soc.* **47** (1975), 368–370.
[5] B. Forrest, E. Kaniuth, A. T. Lau and N. Spronk, 'Ideals with bounded approximate identities in Fourier algebras', *J. Funct. Anal.* **203** (2003), 286–304.
[6] E. E. Granirer, 'On some properties of the Banach algebras $A_p(G)$ for locally compact groups', *Proc. Amer. Math. Soc.* **95** (1985), 375–381.
[7] E. Kaniuth, A. Lau and A. Ülger, 'Multipliers of commutative Banach algebras, power boundedness and Fourier–Stieltjes algebras', *J. Lond. Math. Soc.* **81** (2010), 255–275.
[8] U. Krengel, *Ergodic Theorems* (De Gruyter, Berlin, 1985).
[9] R. Larsen, *An Introduction to the Theory of Multipliers* (Springer, New York, 1971).
[10] R. Larsen, *Banach Algebras* (Marcel Dekker, New York, 1973).
[11] H. Mustafayev, 'Distance formulas in group algebras', *C. R. Acad. Sci. Paris* **354** (2016), 577–582.

HEYBETKULU MUSTAFAYEV, Department of Mathematics,
Faculty of Science, Yuzuncu Yil University, Van, Turkey
e-mail: hsmustafayev@yahoo.com

https://doi.org/10.1017/S0004972717000351 Published online by Cambridge University Press