

Finite basis problem for semigroups of order six

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Dedicated to Professor Norman R. Reilly on the occasion of his 75th birthday

ABSTRACT

Two semigroups are *distinct* if they are neither isomorphic nor anti-isomorphic. Although there exist 15 973 pairwise distinct semigroups of order six, only four are known to be non-finitely based. In the present article, the finite basis property of the other 15 969 distinct semigroups of order six is verified. Since all semigroups of order five or less are finitely based, the four known non-finitely based semigroups of order six are the only examples of minimal order.

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PART I. INTRODUCTION

1. Main result

1.1. Finite basis problem for semigroups of order five or less

An algebra is *finitely based* if the identities it satisfies are finitely axiomatizable. In the 1950s, Lyndon initiated the investigation of identities satisfied by finite algebras; he first proved that all algebras of order two are finitely based [25], and shortly after exhibited a non-finitely based example with only seven elements [26]. Smaller examples of non-finitely based algebras have since been discovered [31, 46], the smallest of which are groupoids of order three, due to Ježek [11] and Murskiĭ [29]. In view of the pioneering result of Lyndon [25], non-finitely based algebras of order three are minimal with respect to being non-finitely based, or *minimal non-finitely based*.

The class of finite groups was the first significant class of finite algebras for which the finite basis property was established; this result, due to Oates and Powell [30], was published in 1964. In the years that followed, the finite basis property was also established for other classical finite algebras such as lattices [28], rings [13, 24], and Lie algebras [2]. The fundamental result of Oates and Powell [30] may have led to the conjecture that all finite semigroups are finitely based. But any hope for such a generalization was quickly extinguished when only a few years later, Perkins demonstrated that the monoid B_2^1 , obtained by adjoining a unit element to the Brandt semigroup

$$B_2 = \langle a, b \mid a^2 = b^2 = 0, aba = a, bab = b \rangle$$

of order five, is non-finitely based [31]. The discovery of a non-finitely based semigroup with only six elements naturally focused attention upon the finite basis problem for semigroups of order less than six. This problem, first raised by Tarski [40] in 1966, attracted considerable interest [4, 8, 12, 41, 42]. A solution to the problem was eventually completed by Trahtman [43] in the early 1980s and published a few years later [45].

THEOREM 1.1. *Every semigroup of order five or less is finitely based.*

Refer to Shevrin and Volkov [39] for a more complete historical account of results leading to the proof of Theorem 1.1. This theorem is also a consequence of a recently established generalization [18].

1.2. Minimal non-finitely based semigroups

It follows from Theorem 1.1 that the semigroup B_2^1 and any other minimal non-finitely based semigroup are of order six. Since Perkins's publication of the semigroup B_2^1 , only three other minimal non-finitely based semigroups have been discovered: the monoid A_2^1 obtained by adjoining a unit element to the 0-simple semigroup

$$A_2 = \langle a, b \mid a^2 = aba = a, bab = b, b^2 = 0 \rangle,$$

the semigroup A_2^g obtained by adjoining a new element g to A_2 where multiplication is given by $g^2 = 0$ and $gx = xg = g$ for all x in A_2 , and the \mathcal{J} -trivial semigroup

$$L = \langle a, b \mid a^2 = a, b^2 = b, aba = 0 \rangle.$$

These four minimal non-finitely based semigroups can also be given by the following multiplication tables.

B_2^1	1	2	3	4	5	6	A_2^1	1	2	3	4	5	6	A_2^g	1	2	3	4	5	6	L	1	2	3	4	5	6
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	6	1	1	1	1	1	1	
2	1	1	1	2	2	3	2	1	1	1	2	2	3	2	1	1	1	2	3	6	2	1	1	1	1	2	
3	1	2	3	1	3	1	3	1	2	3	2	3	3	3	1	2	3	2	3	6	3	1	1	1	1	3	
4	1	1	1	4	4	6	4	1	1	1	4	4	6	4	1	1	1	4	5	6	4	1	1	2	1	4	2
5	1	2	3	4	5	6	5	1	2	3	4	5	6	5	1	4	5	4	5	6	5	1	1	3	1	5	3
6	1	4	6	1	6	1	6	1	4	6	4	6	6	6	6	6	6	6	6	1	6	1	2	2	4	4	6

The non-finite basis property of the semigroup A_2^1 was independently established by Sapir [38] and Trahtman [44] in the 1980s. The semigroup A_2^g , discovered by Volkov (private communication, 3 April 2006), is non-finitely based by either Mashevitskiĭ [27] or Volkov [47]. The semigroup L was recently shown to be non-finitely based by Zhang and Luo [50].

1.3. Finite basis problem for semigroups of order six

Two semigroups are *distinct* if they are neither isomorphic nor anti-isomorphic. There exist 15 973 pairwise distinct semigroups of order six [33], among which 1373 are monoids [6] and 14 600 are non-unital. In view of these large numbers, it is natural to question the existence of minimal non-finitely based semigroups that are distinct from the four known examples. The answer to this fundamental question clearly depends on a complete solution to the following problem.

PROBLEM 1.2. Identify all finitely based semigroups of order six.

This problem was solved by Lee and Li for the special case of monoids. Specifically, they proved that among the 1373 pairwise distinct monoids of order six, only B_2^1 and A_2^1 are non-finitely based semigroups [19]. The objective of the present article is to continue with the investigation by verifying that among the 14 600 pairwise distinct non-unital semigroups of order six, only A_2^g and L are non-finitely based. A complete solution to Problem 1.2 is thus established.

MAIN THEOREM. *Among all distinct semigroups of order six, only B_2^1 , A_2^1 , A_2^g , and L are non-finitely based.*

Consequently, there exist precisely four distinct minimal non-finitely based semigroups. This result was first announced in Lee *et al.* [20].

1.4. Organization

The set of all sections in the present article, under the prerequisite relation, constitutes the directed tree in Figure 1; every section that follows § 2 can be read independently. Notation and background information are given in § 2. An outline of the proof of the main theorem is given in § 3, while the finer details are deferred to §§ 4–27.

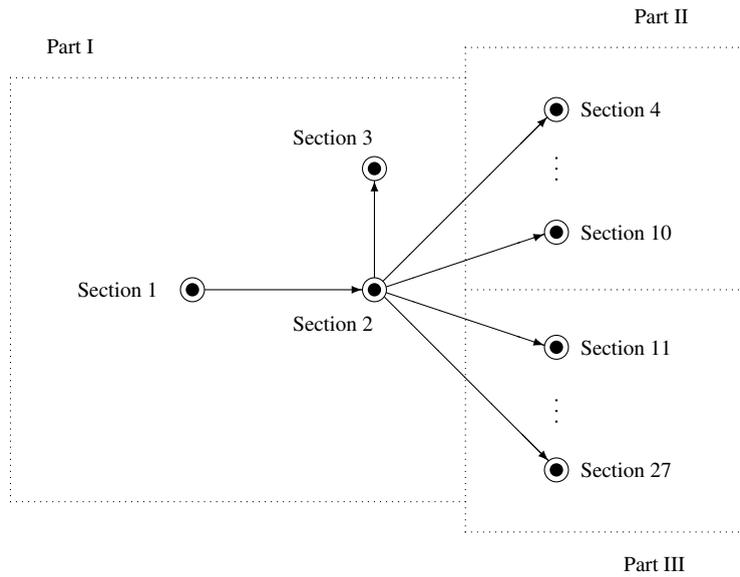


FIGURE 1. Prerequisites of sections.

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2. Preliminaries

2.1. Words, identities, and varieties

Let \mathcal{X} be a countably infinite alphabet throughout. For any subset \mathcal{Y} of \mathcal{X} , let \mathcal{Y}^+ and \mathcal{Y}^* denote the free semigroup and free monoid over \mathcal{Y} , respectively. Elements of \mathcal{X} are called *letters* and elements of \mathcal{X}^* are called *words*. For any word \mathbf{w} :

- the *head* of \mathbf{w} , denoted by $h(\mathbf{w})$, is the first letter occurring in \mathbf{w} ;
- the *tail* of \mathbf{w} , denoted by $t(\mathbf{w})$, is the last letter occurring in \mathbf{w} ;
- the *initial part* of \mathbf{w} , denoted by $ini(\mathbf{w})$, is the word obtained by retaining the first occurrence of each letter in \mathbf{w} ;
- the *content* of \mathbf{w} , denoted by $con(\mathbf{w})$, is the set of letters occurring in \mathbf{w} ;
- the number of occurrences of a letter x in \mathbf{w} is denoted by $occ(x, \mathbf{w})$;
- a letter x is *simple* in \mathbf{w} if $occ(x, \mathbf{w}) = 1$;
- the set of simple letters of \mathbf{w} is denoted by $sim(\mathbf{w})$.

For any letters x and y of \mathbf{w} , let $occ(x, y, \mathbf{w})$ denote the number of occurrences of x before the first y in \mathbf{w} . In other words, if $occ(x, y, \mathbf{w}) = m$, then retaining only the letters x and y in \mathbf{w} results in a word in $x^m y \{x, y\}^*$. Two words \mathbf{w} and \mathbf{w}' are *disjoint* if $con(\mathbf{w}) \cap con(\mathbf{w}') = \emptyset$.

An identity $\mathbf{w} \approx \mathbf{w}'$ is *nontrivial* if $\mathbf{w} \neq \mathbf{w}'$. A semigroup S satisfies an identity $\mathbf{w} \approx \mathbf{w}'$ if, for any substitution $\varphi : \mathcal{X} \rightarrow S$, the elements $\mathbf{w}\varphi$ and $\mathbf{w}'\varphi$ of S are equal. An identity $\mathbf{w} \approx \mathbf{w}'$ containing one or more of the sans-serif letters from $\{H, K, T\}$, with or without indices, represents the system that contains $\mathbf{w} \approx \mathbf{w}'$ and any identity that can be obtained by eliminating all occurrences of some letters from $\{H, K, T\}$. For instance, $xHxKy \approx xHyKx^2$

represents the system

$$xhaxy \approx xhykx^2, \quad xhxy \approx xhyx^2, \quad x^2ky \approx xykx^2, \quad x^2y \approx xyx^2.$$

Let Σ be any set of identities. By Birkhoff's completeness theorem of equational logic [3], an identity $\mathbf{w} \approx \mathbf{w}'$ is *deducible from* or *implied by* Σ if some deduction sequence

$$\mathbf{w} = \mathbf{z}_0 \Rightarrow \mathbf{z}_1 \Rightarrow \cdots \Rightarrow \mathbf{z}_m = \mathbf{w}'$$

exists, where each deduction $\mathbf{z}_i \Rightarrow \mathbf{z}_{i+1}$ involves an identity from Σ , that is, there exist words $\mathbf{e}_i, \mathbf{f}_i \in \mathcal{X}^*$, an identity $\mathbf{u}_i \approx \mathbf{v}_i$ in Σ , and an endomorphism φ_i of \mathcal{X}^+ such that $\mathbf{z}_i = \mathbf{e}_i(\mathbf{u}_i\varphi_i)\mathbf{f}_i$ and $\mathbf{z}_{i+1} = \mathbf{e}_i(\mathbf{v}_i\varphi_i)\mathbf{f}_i$. The deducibility of an identity $\mathbf{w} \approx \mathbf{w}'$ from Σ is indicated by $\mathbf{w} \stackrel{\Sigma}{\approx} \mathbf{w}'$.

For any set Σ of identities, the variety *defined* by Σ is the class of all semigroups that satisfy all identities in Σ ; in this case, Σ is a *basis* for the variety. A variety is *finitely based* if it has a finite basis, and a semigroup is *finitely based* if it generates a finitely based variety. For any variety \mathbf{V} and any set Σ of identities, let $\mathbf{V}\Sigma$ denote the subvariety of \mathbf{V} defined by Σ . For more information on universal algebra and semigroup varieties, refer to Burris and Sankappanavar [5], Shevrin and Volkov [39], and Volkov [49].

2.2. Identities satisfied by some small semigroups

For any nonempty subset U of a semigroup S , the semigroup S/U obtained from S by identifying all elements in U is called a *quotient* of S . A quotient of a subsemigroup of a semigroup S is called a *divisor* of S . Any identity satisfied by a semigroup S is also satisfied by all subsemigroups, quotients, and divisors of S .

The following small semigroups are required throughout the article:

$$\begin{aligned} J &= \langle a, b \mid ab = 0, ba = a, b^2 = b \rangle, \\ L_2 &= \langle a, b \mid a^2 = ab = a, b^2 = ba = b \rangle, \\ R_2 &= \langle a, b \mid a^2 = ba = a, b^2 = ab = b \rangle, \\ N_2 &= \langle a \mid a^2 = 0 \rangle, \\ N_3 &= \langle a \mid a^3 = 0 \rangle, \\ \mathbb{Z}_2 &= \langle a \mid a^2 = 1 \rangle. \end{aligned}$$

For any non-unital semigroup S , let S^1 denote the monoid obtained by adjoining a unit element to S . Let \mathbf{J} , \mathbf{L}_2^1 , \mathbf{N}_2^1 , \mathbf{N}_3 , \mathbf{N}_3^1 and \mathbf{Z}_2 denote the varieties generated by the semigroups J , L_2^1 , N_2^1 , N_3 , N_3^1 , and \mathbb{Z}_2 , respectively.

LEMMA 2.1. *Let $\mathbf{w} \approx \mathbf{w}'$ be any identity. Then:*

- (i) L_2 satisfies $\mathbf{w} \approx \mathbf{w}'$ if and only if $\mathbf{h}(\mathbf{w}) = \mathbf{h}(\mathbf{w}')$;
- (ii) R_2 satisfies $\mathbf{w} \approx \mathbf{w}'$ if and only if $\mathbf{t}(\mathbf{w}) = \mathbf{t}(\mathbf{w}')$;
- (iii) L_2^1 satisfies $\mathbf{w} \approx \mathbf{w}'$ if and only if $\mathbf{ini}(\mathbf{w}) = \mathbf{ini}(\mathbf{w}')$;
- (iv) N_2 satisfies $\mathbf{w} \approx \mathbf{w}'$ if and only if either $|\mathbf{w}|, |\mathbf{w}'| \geq 2$ or $\mathbf{w} = \mathbf{w}'$;
- (v) N_2^1 satisfies $\mathbf{w} \approx \mathbf{w}'$ if and only if $\mathbf{con}(\mathbf{w}) = \mathbf{con}(\mathbf{w}')$ and $\mathbf{sim}(\mathbf{w}) = \mathbf{sim}(\mathbf{w}')$;
- (vi) N_3 satisfies $\mathbf{w} \approx \mathbf{w}'$ if and only if either $|\mathbf{w}|, |\mathbf{w}'| \geq 3$ or $\mathbf{occ}(x, \mathbf{w}) = \mathbf{occ}(x, \mathbf{w}')$ for all $x \in \mathcal{X}$;
- (vii) N_3^1 satisfies $\mathbf{w} \approx \mathbf{w}'$ if and only if for each $x \in \mathcal{X}$, either $\mathbf{occ}(x, \mathbf{w}) = \mathbf{occ}(x, \mathbf{w}') \leq 2$ or $\mathbf{occ}(x, \mathbf{w}), \mathbf{occ}(x, \mathbf{w}') \geq 3$;
- (viii) \mathbb{Z}_2 satisfies $\mathbf{w} \approx \mathbf{w}'$ if and only if $\mathbf{occ}(x, \mathbf{w}) \equiv \mathbf{occ}(x, \mathbf{w}') \pmod{2}$ for all $x \in \mathcal{X}$.

Proof. These are easily verified. For instance, see Petrich and Reilly [32, Theorem V.1.9] for parts (i)–(iii) and Almeida [1, Lemma 6.1.4] for parts (v) and (vii). \square

LEMMA 2.2 [10, Lemma 7]. *The semigroup J satisfies an identity $\mathbf{w} \approx \mathbf{w}'$ if and only if $\text{con}(\mathbf{w}) = \text{con}(\mathbf{w}')$ and either one of the following conditions holds:*

- (i) $\text{t}(\mathbf{w}) \in \text{sim}(\mathbf{w})$, $\text{t}(\mathbf{w}') \in \text{sim}(\mathbf{w}')$, and $\text{t}(\mathbf{w}) = \text{t}(\mathbf{w}')$;
- (ii) $\text{t}(\mathbf{w}) \notin \text{sim}(\mathbf{w})$ and $\text{t}(\mathbf{w}') \notin \text{sim}(\mathbf{w}')$.

For any letters x and y of a word \mathbf{w} , write $x \prec_{\mathbf{w}} y$ to indicate that within \mathbf{w} , each occurrence of x precedes the first y . In other words, if $x \prec_{\mathbf{w}} y$, then retaining only the letters x and y in \mathbf{w} results in a word in $\{x\}^+\{y\}^+$. The *simple part* of a word \mathbf{w} , denoted by \mathbf{w}_{sim} , is the word obtained by retaining the simple letters of \mathbf{w} . For example, if $\mathbf{w} = x^3abycxzy^2ext$, then $\mathbf{w}_{\text{sim}} = abczet$.

LEMMA 2.3. *Suppose that $\mathbf{w} \approx \mathbf{w}'$ is any identity satisfied by J^1 . Then:*

- (i) $\text{con}(\mathbf{w}) = \text{con}(\mathbf{w}')$ and $\text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$;
- (ii) for any letters $x \in \text{con}(\mathbf{w}) = \text{con}(\mathbf{w}')$ and $y \in \text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$, the conditions $x \prec_{\mathbf{w}} y$ and $x \prec_{\mathbf{w}'} y$ are equivalent;
- (iii) $\mathbf{w}_{\text{sim}} = \mathbf{w}'_{\text{sim}}$.

Proof. (i) This follows from Lemma 2.1(v) because the subsemigroup $\{0, a, 1\}$ of J^1 is isomorphic to N_2^1 .

(ii) Seeking a contradiction, suppose $x \prec_{\mathbf{w}} y$ and $x \not\prec_{\mathbf{w}'} y$ where $x \in \text{con}(\mathbf{w}) = \text{con}(\mathbf{w}')$ and $y \in \text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$. Then retaining the letters x and y in the identity $\mathbf{w} \approx \mathbf{w}'$ results in an identity of the form $x^p y \approx x^q y x^r$ with $p, r \geq 1$ and $q \geq 0$. Since J^1 has a unit element, the semigroup J satisfies $x^p y \approx x^q y x^r$, but this is impossible because $b^p a \neq b^q a b^r$ in J .

(iii) This is a consequence of part (ii). □

2.3. Connected words

A word that is a single letter is called a *singleton*. A non-singleton word is *connected* if it cannot be written as a product of two nonempty disjoint words.

Let \mathbf{A}_0 and \mathbf{B}_0 denote the varieties generated by the following semigroups of order four:

$$A_0 = \langle a, b \mid a^2 = a, b^2 = b, ba = 0 \rangle,$$

$$B_0 = \langle a, b, c \mid a^2 = a, b^2 = b, ab = ba = 0, ac = cb = c \rangle.$$

LEMMA 2.4. *Let \mathbf{w} and \mathbf{w}' be any words where \mathbf{w} is connected. Then A_0 satisfies $\mathbf{w} \approx \mathbf{w}'$ if and only if \mathbf{w}' is connected and $\text{con}(\mathbf{w}) = \text{con}(\mathbf{w}')$.*

Proof. This follows from Edmunds [8, proof of part 4 of the first proposition]. □

A semigroup S is *left idempotent-separable* if, for any distinct $x, y \in S$, there exists an idempotent $e \in S$ such that $ex \neq ey$. A *right idempotent-separable* semigroup is dually defined. A semigroup is *idempotent-separable* if it is both left idempotent-separable and right idempotent-separable.

LEMMA 2.5 [21, §3]. *The following semigroups are idempotent-separable: A_0 , B_0 , any monoid, and the direct product of any two idempotent-separable semigroups.*

LEMMA 2.6. *Let \mathbf{V} be the variety generated by any idempotent-separable semigroup S .*

- (i) *If $A_0 \in \mathbf{V}$, then S has a basis that consists of identities formed by connected words.*
- (ii) *If $B_0 \in \mathbf{V}$ and $A_0 \notin \mathbf{V}$, then S has a basis that consists of identities formed by words that are products of pairwise disjoint connected words.*

Proof. This follows from Lee and Volkov [21, Proposition 3.2]. \square

2.4. Exclusion identities

Let S be any semigroup in a variety \mathbf{V} . An identity $\mathbf{w} \approx \mathbf{w}'$ is called an *exclusion identity* for S in \mathbf{V} if the following are all satisfied:

(E1) S does not satisfy $\mathbf{w} \approx \mathbf{w}'$;

(E2) any subvariety of \mathbf{V} not containing S satisfies $\mathbf{w} \approx \mathbf{w}'$.

For any fixed integer $n \geq 2$, let \mathbf{V}_n denote the variety defined by the identity

$$\nu_n : x^{2n} \approx x^n$$

and let $\psi_n : \mathcal{X} \rightarrow \mathcal{X}^+$ denote the substitution $x \mapsto h^n x h^n$ for all $x \in \mathcal{X}$.

LEMMA 2.7 [15, Theorem 2]. *Let S be any non-unital semigroup such that $S, S^1 \in \mathbf{V}_n$. Suppose that $\mathbf{w} \approx \mathbf{w}'$ is an exclusion identity for S in \mathbf{V}_n with $h \notin \text{con}(\mathbf{w}\mathbf{w}')$. Then the identity $\mathbf{w}\psi_n \approx \mathbf{w}'\psi_n$ is an exclusion identity for S^1 in \mathbf{V}_n .*

LEMMA 2.8. *For each $n \geq 2$, the following is an exclusion identity for A_0 in \mathbf{V}_n :*

$$(x^n y^n)^{n+1} \approx x^n y^n. \quad (2.1)$$

Proof. By Lemma 2.4, the identity (2.1) is not satisfied by A_0 . Suppose that a subvariety \mathbf{V} of \mathbf{V}_n does not satisfy the identity (2.1). Then there exists some semigroup $S \in \mathbf{V}$ with elements $a, b \in S$ such that

(a) $(a^n b^n)^{n+1} \neq a^n b^n$.

Let $T = \langle a^n, b^n \rangle$ be the subsemigroup of S generated by the elements a^n and b^n . Since $a^{2n} \stackrel{\nu_n}{=} a^n$ and $b^{2n} \stackrel{\nu_n}{=} b^n$, it is easily shown by (a) that

(b) $a^n, b^n, a^n b^n$, and $b^n a^n$ are distinct elements of T .

Suppose that there exists some $x \in \{a^n, b^n, a^n b^n\}$ such that $x \in T b^n a^n T$. Then

$$a^n x b^n \in a^n \{a^n, b^n, a^n b^n\} b^n \stackrel{\nu_n}{=} \{a^n b^n\},$$

so that $a^n x b^n = a^n b^n$. Therefore

$$a^n b^n = a^n x b^n \in a^n (T b^n a^n T) b^n \stackrel{\nu_n}{\subseteq} \{(a^n b^n)^2, (a^n b^n)^3, \dots\},$$

whence $a^n b^n = (a^n b^n)^{r+1}$ for some $r \geq 1$. But

$$a^n b^n = (a^n b^n)^{r+1} = (a^n b^n)^{2r+1} = \dots = (a^n b^n)^{nr+1} \stackrel{\nu_n}{=} (a^n b^n)^{n+1},$$

which is impossible by (a). Therefore x does not exist, so that $\{a^n, b^n, a^n b^n\}$ and $T b^n a^n T$ are disjoint sets. It follows that

(c) $I = T b^n a^n T$ is an ideal of T with $a^n, b^n, a^n b^n \notin I$.

By (b) and (c), the divisor $T/(T b^n a^n T)$ of S is isomorphic to A_0 . Hence $A_0 \in \mathbf{V}$. \square

LEMMA 2.9. *For each $n \geq 2$, the following is an exclusion identity for N_2^1 in \mathbf{V}_n :*

$$h^n (x h^n)^{n+1} \approx h^n x h^n. \quad (2.2)$$

Proof. Suppose that \mathbf{V} is any subvariety of \mathbf{V}_n such that $N_2 \notin \mathbf{V}$. Then by Lemma 2.1(iv), the variety \mathbf{V} satisfies an identity $\alpha : x \approx x^{r+1}$ for some $r \geq 1$. The variety \mathbf{V} also satisfies $x^{n+1} \approx x$ because

$$x \stackrel{\alpha}{\approx} x^{r+1} \stackrel{\alpha}{\approx} x^{2r+1} \stackrel{\alpha}{\approx} \dots \stackrel{\alpha}{\approx} x^{nr+1} \stackrel{\nu_n}{\approx} x^{n+1}.$$

Hence the identity $x^{n+1} \approx x$ is an exclusion identity for N_2 in \mathbf{V}_n . By Lemma 2.7, the identity $(x^{n+1})\psi_n \approx x\psi_n$ is an exclusion identity for N_2^1 in \mathbf{V}_n ; this identity can be converted into the identity (2.2) by ν_n . □

2.5. *Factors of length two formed by simple letters*

For any word \mathbf{w} , let $F_{SS}(\mathbf{w})$ denote the set of factors of \mathbf{w} of length two that are formed by simple letters:

$$F_{SS}(\mathbf{w}) = \{xy \in \mathcal{X}^2 \mid \mathbf{w} \in \mathcal{X}^*xy\mathcal{X}^*, x, y \in \text{sim}(\mathbf{w})\}.$$

LEMMA 2.10. *Let S be any semigroup that satisfies ν_n for some $n \geq 2$. Suppose that S satisfies an identity $\mathbf{w} \approx \mathbf{w}'$ with $\text{con}(\mathbf{w}) = \text{con}(\mathbf{w}')$, $\text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$, and $F_{SS}(\mathbf{w}) \neq F_{SS}(\mathbf{w}')$. Then S satisfies the identity*

$$x^n y x^n z x^n \approx x^n y z x^n. \tag{2.3}$$

Proof. Generality is not lost by assuming $yz \in F_{SS}(\mathbf{w}) \setminus F_{SS}(\mathbf{w}')$. Let $\varphi : \mathcal{X} \rightarrow \mathcal{X}^+$ denote the substitution

$$t \mapsto \begin{cases} x^n y & \text{if } t = y, \\ z x^n & \text{if } t = z, \\ x^n & \text{otherwise.} \end{cases}$$

Then the deduction $x^n(\mathbf{w}\varphi)x^n \stackrel{\nu_n}{\approx} x^n y z x^n$ holds because yz is a factor of \mathbf{w} and $y, z \in \text{sim}(\mathbf{w})$. But since yz is not a factor of \mathbf{w}' and $y, z \in \text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$, either

$$x^n(\mathbf{w}'\varphi)x^n \stackrel{\nu_n}{\approx} x^n y x^n z x^n \quad \text{or} \quad x^n(\mathbf{w}'\varphi)x^n \stackrel{\nu_n}{\approx} x^n z x^n y x^n$$

holds. Hence S satisfies either the identity (2.3) or the identity $\alpha : x^n y z x^n \approx x^n z x^n y x^n$. Since

$$x^n y z x^n \stackrel{\alpha}{\approx} x^n z (x^n y) x^n \stackrel{\alpha}{\approx} x^n x^n y x^n z x^n \stackrel{\nu_n}{\approx} x^n y x^n z x^n,$$

the semigroup S satisfies the identity (2.3) in either case. □

3. *Proof of the main theorem*

The following twenty sufficient conditions for the finite basis property are required.

CONDITION 1 [37]. *Any finite semigroup that satisfies the identities*

$$x^{13} \approx x, \quad x^{12}y^{12}x^{12}y^{12} \approx x^{12}y^{12}$$

is finitely based.

CONDITION 2 [9, Theorem 6.1]. *Any semigroup that satisfies the identities*

$$xy^3 \approx xy, \quad xyx^2 \approx x^3y, \quad xyxy \approx xy^2x$$

is finitely based.

CONDITION 3. *Any semigroup that satisfies the identities*

$$x^3y \approx x^2y, \quad xy^3 \approx xy^2, \quad yxy \approx yxy, \quad yxy \approx xyxy$$

is finitely based.

CONDITION 4 [36, Proposition C]. Any semigroup that satisfies the identity

$$xyzy \approx xy^2z$$

is finitely based.

CONDITION 5 [16, Theorem 1.3]. Any semigroup that satisfies the identities

$$x^3 \approx x^2, \quad xyxyx \approx xyx, \quad yxzx \approx xzyx$$

is finitely based.

CONDITION 6 [17, Theorem 3.3]. Any semigroup that satisfies the identities

$$x^3 \approx x^2, \quad x^2yx \approx xyx, \quad yx^2 \approx xyx, \quad yxzx \approx xzyx$$

is finitely based.

CONDITION 7. Any semigroup that satisfies the identities

$$x^4 \approx x^2, \quad x^3yx \approx xyx, \quad xy^2x \approx yx^2y, \quad yxzx \approx xzyx$$

is finitely based.

CONDITION 8. Any semigroup that satisfies the identities

$$x^3yz \approx xyz, \quad x^3y \approx xy^3, \quad yx^2 \approx x^3y, \quad xyxy \approx xy^2x$$

is finitely based.

CONDITION 9. Any semigroup that satisfies the identities

$$x^4 \approx x^2, \quad x^3yx \approx xyx, \quad xyxy \approx x^2y^2, \quad xy^2z \approx xzy^2$$

is finitely based.

CONDITION 10. Any semigroup that satisfies the identities

$$x^3 \approx x^2, \quad x^2yx \approx xyx, \quad yx^2 \approx xyx, \quad x^2y^2 \approx y^2x^2, \quad x^2yz^2 \approx xyz^2x$$

is finitely based.

CONDITION 11 [18, Theorem 5.1]. Any semigroup that satisfies the identities

$$x^4 \approx x^2, \quad x^3yx \approx xyx, \quad x^2yx \approx xyx^2, \quad x^2yz^2 \approx xyz^2x, \quad yxzx \approx xzyx$$

is finitely based.

CONDITION 12 [22, Proposition 3.2]. Any semigroup that satisfies the identities

$$x^4 \approx x^2, \quad x(yx)^3 \approx xyx, \quad x^2y^3x^2 \approx x^2yx^2, \quad (x^2y^2)^3 \approx (x^2y^2)^2, \quad yxzx \approx xzyx$$

is finitely based.

CONDITION 13 [22, Proposition 3.2]. Any semigroup that satisfies the identities

$$x^5 \approx x^2, \quad x(yx)^4 \approx xyx, \quad x^2y^4x^2 \approx x^2yx^2, \quad (x^3y^3)^3 \approx (x^3y^3)^2, \quad yxzx \approx xzyx$$

is finitely based.

CONDITION 14. Any semigroup that satisfies the identities

$$x^4 \approx x^2, \quad xy^3x \approx xyx, \quad xyx^3 \approx xyx, \quad x^2yx \approx xyx^2, \quad yxzx \approx x^2yzx, \quad xyxy \approx xy^2x$$

is finitely based.

CONDITION 15 [23, Corollary 4.6]. Any semigroup that satisfies the identities

$$x^8y \approx x^2y, \quad xy^8 \approx xy^2, \quad x^7yx \approx xyx, \quad x^2yx \approx xyx^2, \quad yxzx \approx x^2yzx, \quad xyxy \approx x^2y^2$$

is finitely based.

CONDITION 16 [31, Theorem 22]. Any finite semigroup that satisfies the identity

$$hxyk \approx hyxk$$

is finitely based.

CONDITION 17 [35, Theorem A]. Any semigroup that satisfies the identity

$$hxyk \approx hxy^2k$$

is finitely based.

CONDITION 18 [35, Theorem A]. Any semigroup that satisfies the identity

$$hxxkxyt \approx hx^2ky^2t$$

is finitely based.

CONDITION 19 [19, Condition 9]. Any semigroup that satisfies the identities

$$x^4 \approx x^2, \quad x^3yx \approx xyx, \quad yx^2 \approx x^3y, \quad xyxy \approx xy^2x$$

but violates both of the identities

$$xyxy \approx x^2y^2, \quad xy^3x \approx xyx$$

is finitely based.

CONDITION 20. Any semigroup that satisfies the identities

$$x^3 \approx x^2, \quad xhx^2 \approx xhx, \quad x^2hax \approx x^2hak, \\ xhykxy \approx xhykyx, \quad xhyxy \approx xhy^2x, \quad xykxy \approx xykyx, \quad xyxy \approx xy^2x$$

but violates the identity

$$h^2xyxy \approx h^2x^2y^2$$

is finitely based.

Conditions 3, 7, 8, 9, 10, 14, and 20 are established in §§ 4–10.

It is routinely verified, with the aid of a computer, that 14534 of the 14600 pairwise distinct non-unital semigroups of order six are finitely based by Conditions 1–20 or their dual conditions. Multiplication tables of the 66 sporadic cases to which the twenty sufficient conditions do not apply are listed below in lexicographical order; the underlying set of each

semigroup is $\{1, 2, 3, 4, 5, 6\}$, and each multiplication table is represented by a 6×6 array where the (i, j) -entry denotes the product of the elements $i, j \in \{1, 2, 3, 4, 5, 6\}$.

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
111113	111123	113156	113156	113333	113356	123156	123333	123456	123456	
111314	444444	121411	444444	114444	123456	444444	123456	124365	444444	(\mathcal{A})
555555	111123	555555	113656	123456	555555	123656	125555	555555	444444	
123456	123456	556513	666666	124365	556613	666666	125634	666666	456123	
§11	§12	§13	§14	§13	§13	§14	§13	§13	§13	
111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
111111	111111	111112	111112	111112	111112	111112	111112	111112	111112	
123456	333333	111112	111113	111113	111113	111113	111113	111113	111113	
444444	333456	111214	111214	111214	111214	111214	111214	111224	111224	(\mathcal{B})
445444	555555	555555	112111	112111	112112	113151	113153	112224	112224	
446445	553456	123456	121456	122456	122456	121416	121416	121456	122456	
§11	§14	§11	§15	§15	§15	§12	§12	§15	§15	
111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
111112	111112	111112	111112	111112	111112	111112	111112	111112	111112	
111113	111113	111113	111113	111113	111113	111113	111113	111113	111113	
112111	112111	112112	112112	112121	112122	112141	112141	112142	113454	(\mathcal{C})
555555	555555	555555	555555	112123	112123	113151	113151	113153	133454	
121456	122456	121456	122456	121446	122446	121416	122446	122446	113456	
§16	§16	§16	§16	§17	§17	§18	§19	L	§17	
111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
111112	111112	111112	111112	111112	111112	111112	111112	111112	111112	
111113	111121	111121	111122	111122	111141	111141	111142	111142	112123	
113455	123444	444444	123444	444444	444444	444444	444444	444444	444444	(\mathcal{D})
133455	123445	444445	123445	444445	555555	555555	555555	555555	112123	
133456	123446	123446	123446	123446	123416	123446	123416	123446	123456	
§17	§20	§16	§20	§16	§21	§21	§21	§21	§11	
111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
111112	111112	111112	111113	111113	111113	111113	111113	111113	111113	
112125	333333	333333	111133	111133	111133	333333	333333	333333	111133	
112224	333313	333313	111344	123444	444444	123451	123451	123453	444444	(\mathcal{E})
112225	555555	555555	123455	123455	123455	123453	123541	123543	113156	
123556	123416	123436	123466	123466	123466	666666	666666	666666	133156	
§15	§21	§21	§20	§22	§22	§22	§23	§23	§23	
111111	111111	111111	111111	111111	111111	111111	111111	111111	111111	
111122	111122	111122	111122	111122	111122	111122	111122	111123	111123	
111133	112133	123111	123122	333333	333333	333333	333333	333333	333333	
444444	444444	444444	444444	333344	444444	444444	444444	123411	123423	(\mathcal{F})
113456	123456	111456	111456	121156	111456	121456	333456	111156	111156	
133456	123456	444165	444165	121256	114456	124456	443456	666666	666666	
§24	§24	§25	§25	§23	§26	§26	§26	§27	§27	
111111	111111	111111	111111	111111	111116					
111123	111223	121256	122256	122446	111236					
333333	333333	333333	122256	123456	123236					
123433	123456	121256	123456	644221	111456					(\mathcal{G})
333356	123456	126256	555555	644221	145456					
666666	666666	666666	566612	666666	666661					
§27	§27	§14	§13	§13	A_2^g					

For each $X \in \{\mathcal{A}, \mathcal{B}, \dots, \mathcal{G}\}$, let X_n denote the n th semigroup in row (X) above. The semigroups \mathcal{C}_9 and \mathcal{G}_6 are isomorphic to the non-finitely based semigroups L and A_2^g , respectively. The remaining 64 semigroups are shown to be finitely based in §§ 11–27. Specifically, it is indicated below each multiplication table the section in which the semigroup is shown to be finitely based. Consequently, among the 14 600 pairwise distinct non-unital semigroups of order six, only A_2^g and L are non-finitely based.

As observed in § 1, the monoids B_2^1 and A_2^1 are the only non-finitely based semigroups among the 1373 pairwise distinct monoids of order six [19]. The proof of the main theorem is thus complete.

REMARK 3.1. There is no redundancy with Conditions 1–20 since they are distinguished by semigroups with the following multiplication tables.

111111	111111	111111	111111	111111	111111	111111	111111	111111	111111
121111	111111	111111	111111	111111	111111	111112	111111	111122	111122
113156	111133	111113	111113	111134	111133	111113	112111	111133	111211
444444	444444	111211	111114	113434	113434	113453	111456	111144	111144
113656	111456	113153	111335	111156	111155	113543	555555	122456	111155
666666	444165	111116	113456	115656	113456	121116	555614	144265	123156
S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}
111111	111116	111111	111111	111111	111111	111111	111111	111111	111111
111111	111236	111112	111112	111111	111111	111111	122456	111112	111111
111114	123116	123452	123152	111133	111113	111112	122456	113451	111134
111144	111456	124532	444444	444444	111114	123453	122456	444444	444444
113455	145116	125342	125132	123456	111135	555555	124456	445134	113456
113466	666661	111116	111416	123465	123156	111112	666666	121116	666666
S_{11}	S_{12}	S_{13}	S_{14}	S_{15}	S_{16}	S_{17}	S_{18}	S_{19}	S_{20}

Specifically, the semigroup S_m is finitely based by Condition n if and only if $m = n$.

PART II. SUFFICIENT CONDITIONS FOR THE FINITE BASIS PROPERTY

4. On Condition 3

Let $\mathbf{X}_{(4.1)}$ denote the variety defined by the identities

$$x^3y \approx x^2y, \quad xy^3 \approx xy^2, \tag{4.1a}$$

$$xyx \approx yxy, \tag{4.1b}$$

$$xyx \approx xyxy. \tag{4.1c}$$

Then Condition 3 is equivalent to the following result.

PROPOSITION 4.1. *All subvarieties of $\mathbf{X}_{(4.1)}$ are finitely based.*

It is shown in § 4.1 that $\mathbf{X}_{(4.1)} = \mathbf{A}_0 \vee \mathbf{N}_3$. The proof of Proposition 4.1 is then given in § 4.2.

4.1. Identities satisfied by $\mathbf{X}_{(4.1)}$

LEMMA 4.2. *For any $k \geq 2$, the identity (4.1c) implies the identity*

$$(x_1x_2 \dots x_k)^3 \approx (x_1x_2 \dots x_k)^2. \tag{4.2}$$

Proof. The identity (4.1c) clearly implies the identity $(x_1x_2)^3 \approx (x_1x_2)^2$, which in turn implies the identity (4.2) for any $k \geq 2$. □

LEMMA 4.3 [21, Theorem 4.1]. *The variety \mathbf{A}_0 is defined by the identities (4.1) and*

$$x^3 \approx x^2. \tag{4.3}$$

PROPOSITION 4.4. *The equality $\mathbf{X}_{(4.1)} = \mathbf{A}_0 \vee \mathbf{N}_3$ holds.*

Proof. It is routinely verified, either directly or by Lemmas 2.1(vi) and 4.3, that A_0 and N_3 satisfy the identities (4.1). Hence the inclusion $\mathbf{A}_0 \vee \mathbf{N}_3 \subseteq \mathbf{X}_{(4.1)}$ holds. To complete the proof, it remains to show that any nontrivial identity $\mathbf{w} \approx \mathbf{w}'$ satisfied by $\mathbf{A}_0 \vee \mathbf{N}_3$ is implied by the identities (4.1). Since \mathbf{A}_0 satisfies the identity $\mathbf{w} \approx \mathbf{w}'$, the identities {(4.1), (4.3)} imply the identity $\mathbf{w} \approx \mathbf{w}'$. Hence there exists a deduction sequence

$$\mathbf{w} = \mathbf{z}_0 \Rightarrow \mathbf{z}_1 \Rightarrow \cdots \Rightarrow \mathbf{z}_r = \mathbf{w}', \quad (4.4)$$

where each deduction $\mathbf{z}_j \Rightarrow \mathbf{z}_{j+1}$ involves an identity from {(4.1), (4.3)}. For each j , since A_0 satisfies the identity $\mathbf{w} \approx \mathbf{z}_j$, it is easily shown that $\text{con}(\mathbf{w}) = \text{con}(\mathbf{z}_j)$.

Let $\text{con}(\mathbf{w}) = \{x_1, \dots, x_k\}$. If $k = 1$, then the assumption $\mathbf{w} \neq \mathbf{w}'$ and Lemma 2.1(vi) imply $|\mathbf{w}|, |\mathbf{w}'| \geq 3$, whence the identity $\mathbf{w} \approx \mathbf{w}'$ is implied by the identities (4.1a). Thus assume $k \geq 2$. Suppose that some deduction $\mathbf{z}_j \Rightarrow \mathbf{z}_{j+1}$ in (4.4) involves the identity (4.3). Then $\mathbf{z}_j = \mathbf{e}\mathbf{u}^p\mathbf{f}$ and $\mathbf{z}_{j+1} = \mathbf{e}\mathbf{u}^q\mathbf{f}$ for some $\mathbf{e}, \mathbf{f} \in \mathcal{X}^*$, $\mathbf{u} \in \mathcal{X}^+$, and $\{p, q\} = \{2, 3\}$. If either $\mathbf{e} \neq \emptyset$ or $\mathbf{f} \neq \emptyset$, then

$$\mathbf{z}_j = \mathbf{e}\mathbf{u}^p\mathbf{f} \stackrel{(4.1a)}{\approx} \mathbf{e}\mathbf{u}^q\mathbf{f} = \mathbf{z}_{j+1}.$$

If $\mathbf{e} = \mathbf{f} = \emptyset$, then since $|\mathbf{u}| \geq |\text{con}(\mathbf{w})| \geq 2$,

$$\mathbf{z}_j = \mathbf{u}^p \stackrel{(4.2)}{\approx} \mathbf{u}^q = \mathbf{z}_{j+1}.$$

Therefore any deduction $\mathbf{z}_j \Rightarrow \mathbf{z}_{j+1}$ in (4.4) that involves the identity (4.3) can be replaced by a deduction that involves an identity from (4.1a) or (4.2). Hence the identity $\mathbf{w} \approx \mathbf{w}'$ is implied by the identities {(4.1), (4.2)}. It follows from Lemma 4.2 that the identity $\mathbf{w} \approx \mathbf{w}'$ is implied by the identities (4.1). \square

COROLLARY 4.5. *The variety $\mathbf{X}_{(4.1)}$ satisfies the identities*

$$x^2yx \approx xyx^2 \approx xyx, \quad (4.5a)$$

$$xyxzx \approx x^2yzx. \quad (4.5b)$$

Proof. It is routinely shown, either directly or by Lemmas 2.1(vi) and 2.4, that A_0 and N_3 satisfy the identities (4.5). The result now follows from Proposition 4.4. \square

4.2. Proof of Proposition 4.1

Let \mathbf{V} be any proper subvariety of $\mathbf{X}_{(4.1)}$. Then by Proposition 4.4, either $A_0 \notin \mathbf{V}$ or $N_3 \notin \mathbf{V}$.

Case 1: $A_0 \notin \mathbf{V}$. Then by Lemma 2.8, the variety \mathbf{V} satisfies the identity $\alpha : (x^2y^2)^3 \approx x^2y^2$. Since

$$x^2y^2 \stackrel{\alpha}{\approx} x^2y^2x^2y^2x^2y^2 \stackrel{(4.5b)}{\approx} x^4y^4x^2y^2 \stackrel{(4.5a)}{\approx} xyxy,$$

the variety \mathbf{V} satisfies the identity $xyxy \approx x^2y^2$. Since \mathbf{V} satisfies the identities {(4.1), (4.5)}, it satisfies the identities in Condition 15 and so is finitely based.

Case 2: $N_3 \notin \mathbf{V}$. Then it follows from Lemma 2.1(vi) that the variety \mathbf{V} satisfies the identity $x^3 \approx x^2$ and so, by Lemma 4.3, is a subvariety of \mathbf{A}_0 . Since every subvariety of \mathbf{A}_0 is finitely based [14, Corollary 4.3], the variety \mathbf{V} is finitely based.

5. On Condition 7

Let $\mathbf{X}_{(5.1)}$ denote the variety defined by the identities

$$x^3Hx \approx xHx, \tag{5.1a}$$

$$xy^2x \approx yx^2y, \tag{5.1b}$$

$$xyzx \approx xzyx. \tag{5.1c}$$

Then Condition 7 is equivalent to the following result.

PROPOSITION 5.1. All subvarieties of $\mathbf{X}_{(5.1)}$ are finitely based.

It is shown in § 5.1 that $\mathbf{X}_{(5.1)} = \mathbf{A}_0 \vee \mathbf{N}_2^1 \vee \mathbf{Z}_2$. The proof of Proposition 5.1 is then given in § 5.2.

5.1. Identities satisfied by $\mathbf{X}_{(5.1)}$

LEMMA 5.2. The identities (5.1) imply the identities

$$xHxKx \approx x^2HKx, \tag{5.2a}$$

$$xHyzKx \approx xHzyKx, \tag{5.2b}$$

$$xHyKxy \approx xHyKyx, \quad xyHxKy \approx yxHxKy. \tag{5.2c}$$

Proof. It is easily shown that the identity (5.1c) implies the identities (5.2a). Since

$$\begin{aligned} xHyzKx &\stackrel{(5.1a)}{\approx} x^3HyzKx \stackrel{(5.2a)}{\approx} xHxyzKx \stackrel{(5.1c)}{\approx} xHxzyxKx \\ &\stackrel{(5.2a)}{\approx} x^3HzyKx \stackrel{(5.1a)}{\approx} xHzyKx, \end{aligned}$$

the identities (5.1) imply the identities (5.2b). Since

$$\begin{aligned} xHyKxy &\stackrel{(5.1a)}{\approx} x^3Hy^3Kxy \stackrel{(5.2a)}{\approx} x^2Hy^2Kyx^2y \stackrel{(5.1b)}{\approx} x^2Hy^2Kxy^2x \\ &\stackrel{(5.2a)}{\approx} x^3Hy^3Kyx \stackrel{(5.1a)}{\approx} xHyKyx, \end{aligned}$$

the identities (5.1) imply $xHyKxy \approx xHyKyx$. By a symmetrical argument, the identities (5.1) also imply $xyHxKy \approx yxHxKy$. Hence the identities (5.1) imply the identities (5.2c). \square

In this section, a non-simple word \mathbf{w} is said to be in *canonical form* if

$$\mathbf{w} = x_0^{e_0} x_1^{e_1} \dots x_r^{e_r} y_1 \dots y_s x_0,$$

where all of the following are satisfied:

- (I) x_0, x_1, \dots, x_r are distinct non-simple letters of \mathbf{w} in alphabetical order;
- (II) $e_0 \in \{1, 2\}$ and $e_1, \dots, e_r \in \{2, 3\}$;
- (III) y_1, \dots, y_s are distinct simple letters of \mathbf{w} in alphabetical order.

Note that if the above word \mathbf{w} in canonical form does not contain any simple letter, then $s = 0$ and $y_1 \dots y_s = \emptyset$.

LEMMA 5.3. Let \mathbf{w} be any non-simple word with $h(\mathbf{w}) = t(\mathbf{w})$. Then there exists some word $\overline{\mathbf{w}}$ in canonical form such that the identities (5.1) imply the identity $\mathbf{w} \approx \overline{\mathbf{w}}$.

Proof. By Lemma 5.2, it suffices to convert \mathbf{w} , using the identities $\{(5.1), (5.2)\}$, into a word in canonical form. By assumption, $\mathbf{w} = x\mathbf{u}x$ for some $\mathbf{u} \in \mathcal{X}^*$. The identities (5.2a) can be used to group every x in \mathbf{u} with the first x in \mathbf{w} , resulting in a word $x^e\mathbf{u}'x$, where $e = \text{occ}(x, \mathbf{u}) + 1$ and \mathbf{u}' is obtained from \mathbf{u} by removing all occurrences of x . The identities (5.2b) can then be used to arrange the letters of \mathbf{u}' in any order. Specifically, the non-simple letters of \mathbf{u}' can be grouped to the left in alphabetical order to form a prefix of \mathbf{u}' , while the simple letters of \mathbf{u}' can be grouped to the right in alphabetical order to form a suffix of \mathbf{u}' . The resulting word is of the form $x^e x_1^{e_1} \dots x_r^{e_r} y_1 \dots y_s x$ for some $e_1, \dots, e_r \geq 2$. The identities (5.1a) can be used to reduce the exponents, so that $e \in \{1, 2\}$ and $e_1, \dots, e_r \in \{2, 3\}$. The resulting word is in canonical form if x alphabetically precedes x_1 . If x does not alphabetically precede x_1 , then

$$\begin{aligned} x^e x_1^{e_1} x_2^{e_2} \dots x_r^{e_r} y_1 \dots y_s x &\stackrel{(5.2b)}{\approx} x^e x_1^{e_1-1} x_2^{e_2} \dots x_r^{e_r} y_1 \dots y_s x_1 x \\ &\stackrel{(5.2c)}{\approx} x_1^{e_1-1} x^e x_2^{e_2} \dots x_r^{e_r} y_1 \dots y_s x x_1 \\ &\stackrel{(5.2b)}{\approx} x_1^{e_1-1} x^{e+1} x_2^{e_2} \dots x_r^{e_r} y_1 \dots y_s x_1, \end{aligned}$$

and the factors $x^{e+1}, x_2^{e_2}, \dots, x_r^{e_r}$ in the latter word can be alphabetically ordered by the identities (5.2b), resulting in a word in canonical form. \square

LEMMA 5.4. *Let \mathbf{w} be any connected word. Then there exists some word $\bar{\mathbf{w}}$ in canonical form such that the identities (5.1) imply the identity $\mathbf{w} \approx \bar{\mathbf{w}}$.*

Proof. By Lemma 5.2, it suffices to convert \mathbf{w} , using the identities $\{(5.1), (5.2)\}$, into a word in canonical form. Since \mathbf{w} is connected, the letter $x = h(\mathbf{w})$ occurs at least twice in \mathbf{w} . Hence $\mathbf{w} = x\mathbf{u}x\mathbf{v}$ for some $\mathbf{u}, \mathbf{v} \in \mathcal{X}^*$ such that $x \notin \text{con}(\mathbf{v})$. If the factor \mathbf{v} is empty, then the result follows from Lemma 5.3. Therefore assume that \mathbf{v} is nonempty, whence $\mathbf{w} = x\mathbf{u}x\mathbf{y}\mathbf{v}'$ for some $y \in \mathcal{X}$ and $\mathbf{v}' \in \mathcal{X}^*$. There are two cases to consider.

Case 1: $y \notin \text{con}(\mathbf{u})$. Since \mathbf{w} is connected, the factors $x\mathbf{u}x$ and $y\mathbf{v}'$ cannot be disjoint. It follows that \mathbf{u} and \mathbf{v}' share some common letter, whence $\mathbf{w} = x\mathbf{u}x\mathbf{y}\mathbf{v}' \stackrel{(5.2b)}{\approx} x\mathbf{u}y\mathbf{x}\mathbf{v}'$.

Case 2: $y \in \text{con}(\mathbf{u})$. Then clearly $\mathbf{w} = x\mathbf{u}x\mathbf{y}\mathbf{v}' \stackrel{(5.2c)}{\approx} x\mathbf{u}y\mathbf{x}\mathbf{v}'$.

In either case, the identities (5.2) can be used to interchange the last x in \mathbf{w} with the letter y that immediately follows it. It is easily seen that the resulting word is connected. The same argument can be repeated to move the last x in \mathbf{w} to the right until it becomes the last letter of the word. Hence $\mathbf{w} \stackrel{(5.2)}{\approx} x\mathbf{u}\mathbf{v}x$ and the result now holds by Lemma 5.3. \square

PROPOSITION 5.5. *The equality $\mathbf{X}_{(5.1)} = \mathbf{A}_0 \vee \mathbf{N}_2^1 \vee \mathbf{Z}_2$ holds.*

Proof. By Lemma 2.5, the semigroup $A_0 \times N_2^1 \times \mathbb{Z}_2$ is idempotent-separable. Therefore by Lemma 2.6(i), there exists some basis Σ for $A_0 \times N_2^1 \times \mathbb{Z}_2$ that consists of identities formed by connected words. It is routinely verified, either directly or by Lemmas 2.1 and 2.4, that A_0 , N_2^1 , and \mathbb{Z}_2 satisfy the identities (5.1). Hence the inclusion $\mathbf{A}_0 \vee \mathbf{N}_2^1 \vee \mathbf{Z}_2 \subseteq \mathbf{X}_{(5.1)}$ holds. To complete the proof, it remains to show that any identity $\mathbf{w} \approx \mathbf{w}'$ in Σ is implied by the identities (5.1).

By Lemma 5.4, the words \mathbf{w} and \mathbf{w}' can be chosen to be in canonical form. Since N_2^1 satisfies $\mathbf{w} \approx \mathbf{w}'$, it follows from Lemma 2.1(v) that $\text{con}(\mathbf{w}) = \text{con}(\mathbf{w}')$ and $\text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$, whence $\mathbf{w} = x_0^{e_0} x_1^{e_1} \dots x_r^{e_r} y_1 \dots y_s x_0$ and $\mathbf{w}' = x_0^{e'_0} x_1^{e'_1} \dots x_r^{e'_r} y_1 \dots y_s x_0$ for some $e_0, e'_0 \in \{1, 2\}$ and $e_1, e'_1, \dots, e_r, e'_r \in \{2, 3\}$. Since \mathbb{Z}_2 satisfies $\mathbf{w} \approx \mathbf{w}'$, it follows from Lemma 2.1(viii) that $e_i = e'_i$ for all i . Hence the identity $\mathbf{w} \approx \mathbf{w}'$ is trivial and is implied by the identities (5.1). \square

5.2. Proof of Proposition 5.1

Let \mathbf{V} be any proper subvariety of $\mathbf{X}_{(5.1)}$. Then by Proposition 5.5, the variety \mathbf{V} does not contain one of the following semigroups: A_0 , N_2^1 , and \mathbb{Z}_2 .

Case 1: $A_0 \notin \mathbf{V}$. Then it follows from Lemma 2.8 that the variety \mathbf{V} satisfies the identity $\alpha : (x^2y^2)^3 \approx x^2y^2$. Since

$$x^2y^2 \stackrel{\alpha}{\approx} x^2y^2x^2y^2x^2y^2 \stackrel{(5.1c)}{\approx} x^5y^5xy \stackrel{(5.1a)}{\approx} xyxy,$$

the variety \mathbf{V} satisfies the identity $xyxy \approx x^2y^2$. It is then routinely shown that \mathbf{V} satisfies the identities in Condition 15 and so is finitely based.

Case 2: $N_2^1 \notin \mathbf{V}$. Then it follows from Lemma 2.9 that the variety \mathbf{V} satisfies the identity $\beta : h^2(xh^2)^3 \approx h^2xh^2$. Since

$$x(yx)^3 \stackrel{(5.1a)}{\approx} x^3yx^3yxyx \stackrel{(5.2a)}{\approx} x^2yx^2yx^2yx^2 \stackrel{\beta}{\approx} x^2yx^2 \stackrel{(5.2a)}{\approx} x^3yx \stackrel{(5.1a)}{\approx} xyx$$

and

$$x^2yx^2 \stackrel{\beta}{\approx} x^2yx^2yx^2yx^2 \stackrel{(5.2a)}{\approx} x^6y^3x^2 \stackrel{(5.1a)}{\approx} x^2y^3x^2,$$

the variety \mathbf{V} satisfies the identities $x(yx)^3 \approx xyx$ and $x^2y^3x^2 \approx x^2yx^2$. It is then routinely shown that \mathbf{V} satisfies the identities in Condition 12 and so is finitely based.

Case 3: $\mathbb{Z}_2 \notin \mathbf{V}$. Then it follows from Lemma 2.1(viii) that the variety \mathbf{V} satisfies the identity $x^3 \approx x^2$. It is easily shown that \mathbf{V} satisfies the identities in Condition 6 and so is finitely based.

6. On Condition 8

Let $\mathbf{X}_{(6.1)}$ denote the variety defined by the identities

$$x^3yz \approx xyz, \quad x^3y \approx xy^3, \quad xyx^2 \approx x^3y, \quad xyxy \approx xy^2x. \tag{6.1}$$

Then Condition 8 is equivalent to the following result.

PROPOSITION 6.1. All subvarieties of $\mathbf{X}_{(6.1)}$ are finitely based.

Some preliminary results are established in §§ 6.1 and 6.2. The proof of Proposition 6.1 is then given in § 6.3.

6.1. Identities satisfied by $\mathbf{X}_{(6.1)}$

LEMMA 6.2. The identities (6.1) imply the identities

$$x^3yz \approx xy^3z \approx xyz^3 \approx xyz, \tag{6.2a}$$

$$xyx^2z \approx xyz, \quad xyz^2 \approx xyz, \tag{6.2b}$$

$$xHyKxy \approx xHyKyx. \tag{6.2c}$$

Proof. It is easily seen that the identities (6.1) imply the identities $\{(6.2a), (6.2b)\}$. Note that the fourth identity in (6.1) implies the identity $\alpha : y^2x^2y \approx y^2xyx$. Since

$$\begin{aligned} xHyKxy &\stackrel{(6.2a)}{\approx} xHyKx^2y \stackrel{(6.2b)}{\approx} xHyKx(y^2x^2y) \stackrel{\alpha}{\approx} xHyKxy^2xyx \\ &\stackrel{(6.2b)}{\approx} xHyKx^2yx \stackrel{(6.2b)}{\approx} xHyKyx, \end{aligned}$$

the identities (6.1) also imply the identities (6.2c). □

6.2. A canonical form

In this section, a word \mathbf{w} with $\text{ini}(\mathbf{w}) = x_0 \dots x_m$ is said to be in *canonical form* if the following are satisfied:

(I) if $m = 0$, then \mathbf{w} is any one of the words

$$x_0, x_0^2, x_0^3, x_0^4; \tag{6.3a}$$

(II) if $m = 1$, then \mathbf{w} is any one of the words

$$x_0x_1, x_0^2x_1, x_0^3x_1, x_0x_1^2, x_0^2x_1^2, x_0x_1x_0, x_0^2x_1x_0, x_0x_1^2x_0, x_0^2x_1^2x_0; \tag{6.3b}$$

(III) if $m \geq 2$, then

$$\mathbf{w} = x_0^{e_0} \prod_{i=1}^m (x_i^{e_i} \mathbf{w}_i) \tag{6.4}$$

where $e_i \in \{1, 2\}$ and $\mathbf{w}_i \in \{x_0^{f_0} \dots x_{i-1}^{f_{i-1}} \mid f_0, \dots, f_{i-1} \in \{0, 1\}\}$.

LEMMA 6.3. *Let \mathbf{w} be any word. Then there exists some word $\bar{\mathbf{w}}$ in canonical form such that the identities (6.1) imply the identity $\mathbf{w} \approx \bar{\mathbf{w}}$.*

Proof. Suppose $\text{ini}(\mathbf{w}) = x_0 \dots x_m$. Then by Lemma 6.2, it suffices to convert \mathbf{w} , using the identities $\{(6.1), (6.2)\}$, into a word in canonical form. This is routine if $m \in \{0, 1\}$. Hence assume $m \geq 2$. Then \mathbf{w} can be written as $\mathbf{w} = \prod_{i=0}^m (x_i \mathbf{u}_i)$ for some $\mathbf{u}_i \in \{x_0, \dots, x_i\}^*$. Since the letters in each \mathbf{u}_i are non-first occurrences in \mathbf{w} , the identities (6.2c) can be used to arrange them, within \mathbf{u}_i , in any order. Specifically, any occurrence of x_i in \mathbf{u}_i can be moved to the left and grouped with the singleton factor x_i that precedes \mathbf{u}_i , and any remaining letters in \mathbf{u}_i can be ordered according to their indices. Thus \mathbf{w} can be converted into the word in (6.4) with $e_i \geq 1$ and $\mathbf{w}_i \in \{x_0^{f_0} \dots x_{i-1}^{f_{i-1}} \mid f_0, \dots, f_{i-1} \geq 0\}$. The identities $\{(6.2a), (6.2b)\}$ can then be applied so that $e_i \in \{1, 2\}$ and $\mathbf{w}_i \in \{x_0^{f_0} \dots x_{i-1}^{f_{i-1}} \mid f_0, \dots, f_{i-1} \in \{0, 1\}\}$. \square

LEMMA 6.4. *Let \mathbf{V} be any subvariety of $\mathbf{X}_{(6.1)}$ that satisfies an identity $\mathbf{w} \approx \mathbf{w}'$.*

- (i) *If $\text{ini}(\mathbf{w}) \neq \text{ini}(\mathbf{w}')$, then \mathbf{V} is finitely based.*
- (ii) *If $\text{occ}(x, \mathbf{w}) \not\equiv \text{occ}(x, \mathbf{w}') \pmod{2}$ for some $x \in \mathcal{X}$, then \mathbf{V} is finitely based.*
- (iii) *If $\text{occ}(x, y, \mathbf{w}) \not\equiv \text{occ}(x, y, \mathbf{w}') \pmod{2}$ for some $x, y \in \mathcal{X}$, then \mathbf{V} is finitely based.*

Proof. It is shown in each of the following parts that \mathbf{V} satisfies the identity $xyx \approx x^2y$ and so is finitely based by Condition 4.

(i) Suppose $\text{ini}(\mathbf{w}) \neq \text{ini}(\mathbf{w}')$. Then $L_2^1 \notin \mathbf{V}$ by Lemma 2.1(iii). If $x^4, x^4y^4z^4$, and $x^4z^4y^4$ are distinct elements in the \mathbf{V} -free semigroup $F_{\mathbf{V}}$ over \mathcal{X} , then they form a subsemigroup of $F_{\mathbf{V}}$ isomorphic to L_2^1 , and this implies the contradiction $L_2^1 \in \mathbf{V}$. Hence these three elements of $F_{\mathbf{V}}$ cannot be distinct, so that \mathbf{V} satisfies either $x^4 \approx x^4y^4z^4$, $x^4 \approx x^4z^4y^4$, or $x^4y^4z^4 \approx x^4z^4y^4$; it is easily seen that any one of the first two identities implies the third identity, whence \mathbf{V} satisfies the third identity. It follows that \mathbf{V} satisfies the identity $\alpha : x^2y^2z^2 \approx x^2z^2y^2$. Since

$$xyx \stackrel{(6.2a)}{\approx} x^3yx \stackrel{(6.2b)}{\approx} (x^2(xy)^2y^2)y \stackrel{\alpha}{\approx} x^2y^2(xy)^2y \stackrel{(6.2)}{\approx} x^2y,$$

the variety \mathbf{V} satisfies the identity $xyx \approx x^2y$.

(ii) Suppose $\text{occ}(x, \mathbf{w}) \not\equiv \text{occ}(x, \mathbf{w}') \pmod{2}$ for some $x \in \mathcal{X}$. Then \mathbf{V} satisfies the identity $\beta : x^4 \approx x^3$. Since

$$xyx \stackrel{(6.2a)}{\approx} x^3yx^3 \stackrel{\beta}{\approx} x^4yx^4 \stackrel{(6.2b)}{\approx} x^4y \stackrel{(6.2a)}{\approx} x^2y,$$

the variety \mathbf{V} satisfies the identity $xyx \approx x^2y$.

(iii) Suppose $\text{occ}(x, y, \mathbf{w}) \not\equiv \text{occ}(x, y, \mathbf{w}') \pmod{2}$ for some $x, y \in \mathcal{X}$. Then by parts (i) and (ii), it suffices to further assume:

- (a) $\text{ini}(\mathbf{w}) = \text{ini}(\mathbf{w}')$;
- (b) $\text{occ}(x, \mathbf{w}) \equiv \text{occ}(x, \mathbf{w}') \pmod{2}$.

Write $\mathbf{w} = \mathbf{w}_1 y \mathbf{w}_2$ and $\mathbf{w}' = \mathbf{w}'_1 y \mathbf{w}'_2$ where $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}'_1, \mathbf{w}'_2 \in \mathcal{X}^*$ are such that $y \notin \text{con}(\mathbf{w}_1 \mathbf{w}'_1)$. Let $e_i = \text{occ}(x, \mathbf{w}_i)$ and $e'_i = \text{occ}(x, \mathbf{w}'_i)$. Then $e_1 = \text{occ}(x, y, \mathbf{w}) \not\equiv \text{occ}(x, y, \mathbf{w}') = e'_1 \pmod{2}$ by assumption, whence $(e_1, e'_1) \neq (0, 0)$. If either e_1 or e'_1 is zero, then (a) is violated. Thus $e_1, e'_1 \geq 1$. By symmetry, it suffices to assume:

- (c) $e_1 \in \{1, 3, 5, \dots\}$;
- (d) $e'_1 \in \{2, 4, 6, \dots\}$.

Since $e_1 + e_2 = \text{occ}(x, \mathbf{w}) \equiv \text{occ}(x, \mathbf{w}') = e'_1 + e'_2 \pmod{2}$, it follows that $e_2 \not\equiv e'_2 \pmod{2}$. Let $r \in \{0, 1\}$ be such that $e_2 + r \equiv 1 \pmod{2}$ and $e'_2 + r \equiv 0 \pmod{2}$. Let φ denote the substitution

$$t \mapsto \begin{cases} y^2 & \text{if } t = y, \\ x^2 & \text{if } t \in \mathcal{X} \setminus \{x, y\}. \end{cases}$$

Then $\mathbf{w}_1 \varphi = x^{e_1+2k}$ for some $k \geq 0$, and $\mathbf{w}_2 \varphi \in \{x, y^2\}^*$ with $\text{occ}(x, \mathbf{w}_2 \varphi) = e_2 + 2\ell$ for some $\ell \geq 0$. In view of (c),

$$\begin{aligned} (\mathbf{w}\varphi)yx^r &= \mathbf{w}_1\varphi \cdot y\varphi \cdot \mathbf{w}_2\varphi \cdot yx^r \stackrel{(6.2a)}{\approx} xy^2(\mathbf{w}_2\varphi)yx^r \stackrel{(6.2b)}{\approx} xy^2x^{e_2}yx^r \\ &\stackrel{(6.2c)}{\approx} xy^3x^{e_2+r} \stackrel{(6.2a)}{\approx} xyx. \end{aligned}$$

Similarly, $\mathbf{w}'_1 \varphi = x^{e'_1+2k'}$ for some $k' \geq 0$, and $\mathbf{w}'_2 \varphi \in \{x, y^2\}^*$ with $\text{occ}(x, \mathbf{w}'_2 \varphi) = e'_2 + 2\ell'$ for some $\ell' \geq 0$. In view of (d),

$$\begin{aligned} (\mathbf{w}'\varphi)yx^r &= \mathbf{w}'_1\varphi \cdot y\varphi \cdot \mathbf{w}'_2\varphi \cdot yx^r \stackrel{(6.2a)}{\approx} x^2y^2(\mathbf{w}'_2\varphi)yx^r \stackrel{(6.2b)}{\approx} x^2y^2x^{e'_2}yx^r \\ &\stackrel{(6.2c)}{\approx} x^2y^3x^{e'_2+r} \stackrel{(6.2b)}{\approx} x^2y^3 \stackrel{(6.2a)}{\approx} x^2y. \end{aligned}$$

Consequently, \mathbf{V} satisfies the identity $xyx \approx x^2y$. □

LEMMA 6.5. Let \mathbf{w} and \mathbf{w}' be any words in canonical form such that $|\text{con}(\mathbf{w})| \geq 3$. Suppose:

- (i) $\text{ini}(\mathbf{w}) = \text{ini}(\mathbf{w}')$;
- (ii) $\text{occ}(x, \mathbf{w}) \equiv \text{occ}(x, \mathbf{w}') \pmod{2}$ for all $x \in \mathcal{X}$;
- (iii) $\text{occ}(x, y, \mathbf{w}) \equiv \text{occ}(x, y, \mathbf{w}') \pmod{2}$ for all $x, y \in \mathcal{X}$.

Then the words \mathbf{w} and \mathbf{w}' are identical.

Proof. By assumption, $\text{ini}(\mathbf{w}) = x_0 \dots x_m$ for some $m \geq 2$. It follows from (i) and (III) that

$$\mathbf{w} = x_0^{e_0} \prod_{i=1}^m (x_i^{e_i} \mathbf{w}_i) \quad \text{and} \quad \mathbf{w}' = x_0^{e'_0} \prod_{i=1}^m (x_i^{e'_i} \mathbf{w}'_i)$$

for some $e_i, e'_i \in \{1, 2\}$ and $\mathbf{w}_i, \mathbf{w}'_i \in \{x_0^{f_0} \dots x_{i-1}^{f_{i-1}} \mid f_0, \dots, f_{i-1} \in \{0, 1\}\}$. It is obvious that $\text{occ}(x_m, \mathbf{w}) = e_m$ and $\text{occ}(x_m, \mathbf{w}') = e'_m$, so that $e_m = e'_m$ by (ii). For each $i < m$, since $\text{occ}(x_i, x_{i+1}, \mathbf{w}) = e_i$ and $\text{occ}(x_i, x_{i+1}, \mathbf{w}') = e'_i$, it follows from (iii) that $e_i = e'_i$. Therefore $(e_0, \dots, e_m) = (e'_0, \dots, e'_m)$.

If $(\mathbf{w}_1, \dots, \mathbf{w}_{m-1}) = (\mathbf{w}'_1, \dots, \mathbf{w}'_{m-1})$ and $\mathbf{w}_m \neq \mathbf{w}'_m$, say $x_j \in \text{con}(\mathbf{w}_m) \setminus \text{con}(\mathbf{w}'_m)$ for some $j < m$, then (ii) is violated because

$$\begin{aligned} \text{occ}(x_j, \mathbf{w}) &= \text{occ}(x_j, x_0^{e_0} x_1^{e_1} \mathbf{w}_1 x_2^{e_2} \mathbf{w}_2 \dots x_{m-1}^{e_{m-1}} \mathbf{w}_{m-1}) + \text{occ}(x_j, x_m^{e_m} \mathbf{w}_m) \\ &= \text{occ}(x_j, x_0^{e'_0} x_1^{e'_1} \mathbf{w}'_1 x_2^{e'_2} \mathbf{w}'_2 \dots x_{m-1}^{e'_{m-1}} \mathbf{w}'_{m-1}) + 1 \end{aligned}$$

and

$$\begin{aligned}\text{occ}(x_j, \mathbf{w}') &= \text{occ}(x_j, x_0^{e'_0} x_1^{e'_1} \mathbf{w}'_1 x_2^{e'_2} \mathbf{w}'_2 \dots x_{m-1}^{e'_{m-1}} \mathbf{w}'_{m-1}) + \text{occ}(x_j, x_m^{e'_m} \mathbf{w}'_m) \\ &= \text{occ}(x_j, x_0^{e'_0} x_1^{e'_1} \mathbf{w}'_1 x_2^{e'_2} \mathbf{w}'_2 \dots x_{m-1}^{e'_{m-1}} \mathbf{w}'_{m-1}) + 0.\end{aligned}$$

Hence $\mathbf{w}_m = \mathbf{w}'_m$. Suppose that $\ell < m$ is the least integer such that $\mathbf{w}_\ell \neq \mathbf{w}'_\ell$, say $k < \ell$ is such that $x_k \in \text{con}(\mathbf{w}_\ell) \setminus \text{con}(\mathbf{w}'_\ell)$. Then $(\mathbf{w}_1, \dots, \mathbf{w}_{\ell-1}) = (\mathbf{w}'_1, \dots, \mathbf{w}'_{\ell-1})$, whence (iii) is violated because

$$\begin{aligned}\text{occ}(x_k, x_{\ell+1}, \mathbf{w}) &= \text{occ}(x_k, x_0^{e_0} x_1^{e_1} \mathbf{w}_1 x_2^{e_2} \mathbf{w}_2 \dots x_{\ell-1}^{e_{\ell-1}} \mathbf{w}_{\ell-1}) + \text{occ}(x_k, x_\ell^{e_\ell} \mathbf{w}_\ell) \\ &= \text{occ}(x_k, x_0^{e_0} x_1^{e_1} \mathbf{w}'_1 x_2^{e_2} \mathbf{w}'_2 \dots x_{\ell-1}^{e_{\ell-1}} \mathbf{w}'_{\ell-1}) + 1\end{aligned}$$

and

$$\begin{aligned}\text{occ}(x_k, x_{\ell+1}, \mathbf{w}') &= \text{occ}(x_k, x_0^{e'_0} x_1^{e'_1} \mathbf{w}'_1 x_2^{e'_2} \mathbf{w}'_2 \dots x_{\ell-1}^{e'_{\ell-1}} \mathbf{w}'_{\ell-1}) + \text{occ}(x_k, x_\ell^{e'_\ell} \mathbf{w}'_\ell) \\ &= \text{occ}(x_k, x_0^{e'_0} x_1^{e'_1} \mathbf{w}'_1 x_2^{e'_2} \mathbf{w}'_2 \dots x_{\ell-1}^{e'_{\ell-1}} \mathbf{w}'_{\ell-1}) + 0.\end{aligned}$$

Thus $(\mathbf{w}_1, \dots, \mathbf{w}_m) = (\mathbf{w}'_1, \dots, \mathbf{w}'_m)$, so that $\mathbf{w} = \mathbf{w}'$. \square

6.3. Proof of Proposition 6.1

Let \mathbf{V} be any proper subvariety of $\mathbf{X}_{(6.1)}$. Then $\mathbf{V} = \mathbf{X}_{(6.1)}\Sigma$ for some set Σ of nontrivial identities. By Lemma 6.3, words that form the identities in Σ can be chosen to be in canonical form. Let $\mathbf{w} \approx \mathbf{w}'$ be any identity in Σ . By Lemma 6.4, the variety \mathbf{V} is finitely based if \mathbf{w} and \mathbf{w}' violate any of the following conditions:

- (a) $\text{ini}(\mathbf{w}) = \text{ini}(\mathbf{w}')$;
- (b) $\text{occ}(x, \mathbf{w}) \equiv \text{occ}(x, \mathbf{w}') \pmod{2}$ for all $x \in \mathcal{X}$;
- (c) $\text{occ}(x, y, \mathbf{w}) \equiv \text{occ}(x, y, \mathbf{w}') \pmod{2}$ for all $x, y \in \mathcal{X}$.

Hence assume that \mathbf{w} and \mathbf{w}' satisfy (a)–(c). If the identity $\mathbf{w} \approx \mathbf{w}'$ involves at least three distinct letters, then it is trivial by Lemma 6.5, contradicting the choice of identities for Σ . Therefore $\mathbf{w} \approx \mathbf{w}'$ involves at most two distinct letters, whence \mathbf{w} and \mathbf{w}' are from (6.3). Since $\mathbf{w} \approx \mathbf{w}'$ is arbitrary in Σ , it follows that Σ contains finitely many identities. (In fact, since the identities in Σ satisfy (a)–(c), it is easily shown that $\Sigma \subseteq \{x^3 \approx x, x^4 \approx x^2, x^3y \approx xy\}$.)

REMARK 6.6. It is routinely shown that N_3 and the orthodox completely regular semigroup

$$O = \langle a, b \mid a^2 = ab = a, b^2 = 1 \rangle$$

of order four belong to the variety $\mathbf{X}_{(6.1)}$. From the proofs in this section, a proper subvariety of $\mathbf{X}_{(6.1)}$ satisfies one of the following identities:

$$x^4 \approx x^2, \tag{6.5}$$

$$xyx \approx x^2y. \tag{6.6}$$

By Lemma 2.1(vi) or direct verification, the semigroup N_3 does not satisfy the identity (6.5). The semigroup O does not satisfy the identity (6.6) because $bab \neq b^2a$ in O . Consequently, the variety $\mathbf{X}_{(6.1)}$ is generated by N_3 and O .

7. On Condition 9

Let $\mathbf{X}_{(7.1)}$ denote the variety defined by the identities

$$x^3\mathbf{H}x \approx x\mathbf{H}x, \quad xyxy \approx x^2y^2, \quad xy^2z \approx xzy^2. \tag{7.1}$$

Then Condition 9 is equivalent to the following result.

PROPOSITION 7.1. All subvarieties of $\mathbf{X}_{(7.1)}$ are finitely based.

It follows from Lemma 2.1(i) that the variety $\mathbf{X}_{(7.1)}$ contains the semigroup L_2 . All subvarieties of $\mathbf{X}_{(7.1)}$ that contain L_2 are shown to be finitely based in §7.2, while those that do not contain L_2 are shown to be finitely based in §7.3. The proof of Proposition 7.1 is thus complete.

7.1. Identities satisfied by subvarieties of $\mathbf{X}_{(7.1)}$

LEMMA 7.2. The identities (7.1) imply the identities

$$x^4 \approx x^2, \quad x^3yx \approx xyx, \quad xyx^3 \approx xyx, \tag{7.2a}$$

$$xHxyx^2 \approx xHxy, \tag{7.2b}$$

$$hx^2y \approx hyx^2, \tag{7.2c}$$

$$hxyKxTy \approx hyxKxTy, \quad xHxyKy \approx xHyxKy, \quad xHyKxy \approx xHyKyx. \tag{7.2d}$$

Proof. It is clear that the identities (7.1) imply the identities (7.2a)–(7.2c). Since

$$\begin{aligned} hxyKxTy &\stackrel{(7.2a)}{\approx} hyy^2x^2xKxTy \stackrel{(7.2c)}{\approx} hx^2yxy^2KxTy \stackrel{(7.1)}{\approx} hx^3y^3KxTy \\ &\stackrel{(7.2a)}{\approx} hxyKxTy, \end{aligned}$$

the identities (7.1) imply $hxyKxTy \approx hyxKxTy$ in (7.2d). A similar argument shows that (7.1) also imply the other identities in (7.2d). \square

LEMMA 7.3. Let \mathbf{V} be any subvariety of $\mathbf{X}_{(7.1)}$ that satisfies an identity $\mathbf{w} \approx \mathbf{w}'$. Suppose that any of the following holds:

- (i) either $\text{con}(\mathbf{w}) \neq \text{con}(\mathbf{w}')$ or $\text{sim}(\mathbf{w}) \neq \text{sim}(\mathbf{w}')$;
- (ii) $\text{occ}(x, \mathbf{w}) \not\equiv \text{occ}(x, \mathbf{w}') \pmod{2}$ for some $x \in \mathcal{X}$;
- (iii) $\text{occ}(x, y, \mathbf{w}) \not\equiv \text{occ}(x, y, \mathbf{w}') \pmod{2}$ for some $x \in \text{con}(\mathbf{w})$ and $y \in \text{sim}(\mathbf{w})$.

Then \mathbf{V} is finitely based.

Proof. It is shown that each of (i)–(iii) implies that \mathbf{V} satisfies the identity $xyx \approx x^2y$ and so is finitely based by Condition 4.

Suppose that (i) holds. Then $N_2^1 \notin \mathbf{V}$ by Lemma 2.1(v). Hence by Lemma 2.9, the variety \mathbf{V} satisfies the identity $h^2(xh^2)^3 \approx h^2xh^2$; this identity can be converted into $\alpha : h^2x^3 \approx h^2x$ by the identities (7.2b). Then \mathbf{V} satisfies the identity $xyx \approx x^2y$ because

$$xyx \stackrel{(7.2a)}{\approx} x^3yx \stackrel{\alpha}{\approx} x^3y^3x \stackrel{(7.2a)}{\approx} xy^3x \stackrel{(7.2c)}{\approx} xyxy^2 \stackrel{(7.2d)}{\approx} x^2y^3 \stackrel{\alpha}{\approx} x^2y.$$

Suppose that (ii) holds. Then it is easily shown that \mathbf{V} satisfies the identity $\beta : x^3 \approx x^2$; it also satisfies the identity $xyx \approx x^2y$ because

$$xyx \stackrel{(7.2a)}{\approx} x^3yx^3 \stackrel{\beta}{\approx} x^2yx^2 \stackrel{(7.2b)}{\approx} x^2y.$$

Suppose that (iii) holds, that is, $\text{occ}(x, y, \mathbf{w}) \not\equiv \text{occ}(x, y, \mathbf{w}') \pmod{2}$ for some $x \in \text{con}(\mathbf{w})$ and $y \in \text{sim}(\mathbf{w})$. As established in the last two paragraphs, \mathbf{V} is finitely based if (i) or (ii) holds. Hence it suffices to further assume that (i) and (ii) do not hold, that is, $\text{con}(\mathbf{w}) = \text{con}(\mathbf{w}')$, $\text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$, and $\text{occ}(x, \mathbf{w}) \equiv \text{occ}(x, \mathbf{w}') \pmod{2}$. Since $y \in \text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$, it follows that $\mathbf{w} = \mathbf{w}_1y\mathbf{w}_2$ and $\mathbf{w}' = \mathbf{w}'_1y\mathbf{w}'_2$ for some $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}'_1, \mathbf{w}'_2 \in (\mathcal{X} \setminus \{y\})^*$. Let $e_i = \text{occ}(x, \mathbf{w}_i)$ and $e'_i = \text{occ}(x, \mathbf{w}'_i)$. Then $e_1 = \text{occ}(x, y, \mathbf{w})$, $e'_1 = \text{occ}(x, y, \mathbf{w}')$, and

$$e_1 + e_2 = \text{occ}(x, \mathbf{w}) \equiv \text{occ}(x, \mathbf{w}') = e'_1 + e'_2 \pmod{2}.$$

Since $e_1 \not\equiv e'_1 \pmod{2}$ by assumption, it follows that $e_2 \not\equiv e'_2 \pmod{2}$. By symmetry, it suffices to assume that e_1 is even and e'_1 is odd. Then there exists some $r \in \{0, 1\}$ such that $e_2 + r$ is even and $e'_2 + r$ is odd. Let φ denote the substitution $z \mapsto x^2$ for all $z \in \mathcal{X} \setminus \{x, y\}$. Then

$$x^2(\mathbf{w}\varphi)x^{r+2} = x^2(\mathbf{w}_1\varphi)y(\mathbf{w}_2\varphi)x^{r+2} \stackrel{(7.2a)}{\approx} x^2yx^2 \stackrel{(7.2b)}{\approx} x^2y$$

and

$$x^2(\mathbf{w}'\varphi)x^{r+2} = x^2(\mathbf{w}'_1\varphi)y(\mathbf{w}'_2\varphi)x^{r+2} \stackrel{(7.2a)}{\approx} xyx,$$

so that \mathbf{V} satisfies the identity $xyx \approx x^2y$. □

7.2. Subvarieties of $\mathbf{X}_{(7.1)}$ containing L_2

PROPOSITION 7.4. *Let \mathbf{V} be any subvariety of $\mathbf{X}_{(7.1)}$ such that $L_2 \in \mathbf{V}$. Then \mathbf{V} is finitely based.*

The proof of this result is given at the end of the subsection.

LEMMA 7.5. *Let $x \in \mathcal{X}$ and $\mathbf{w}, \mathbf{w}' \in \mathcal{X}^+$ be such that $\text{con}(\mathbf{w}) = \text{con}(\mathbf{w}')$, $\text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$, and $x \notin \text{con}(\mathbf{w}\mathbf{w}')$. Suppose that the words \mathbf{w} and \mathbf{w}' begin with the same non-simple letter. Then $\mathbf{X}_{(7.1)}\{\mathbf{w} \approx \mathbf{w}'\} = \mathbf{X}_{(7.1)}\{x\mathbf{w} \approx x\mathbf{w}'\}$.*

Proof. By assumption, $\mathbf{w}, \mathbf{w}' \in z\mathcal{X}^*z\mathcal{X}^*$ with $z = \text{h}(\mathbf{w}) = \text{h}(\mathbf{w}')$, so that the deductions $z^2\mathbf{w} \stackrel{(7.2a)}{\approx} \mathbf{w}$ and $z^2\mathbf{w}' \stackrel{(7.2a)}{\approx} \mathbf{w}'$ hold. Since the identity $x\mathbf{w} \approx x\mathbf{w}'$ clearly implies the identity $z^2\mathbf{w} \approx z^2\mathbf{w}'$, the inclusion $\mathbf{X}_{(7.1)}\{x\mathbf{w} \approx x\mathbf{w}'\} \subseteq \mathbf{X}_{(7.1)}\{\mathbf{w} \approx \mathbf{w}'\}$ holds. It is obvious that the inclusion $\mathbf{X}_{(7.1)}\{\mathbf{w} \approx \mathbf{w}'\} \subseteq \mathbf{X}_{(7.1)}\{x\mathbf{w} \approx x\mathbf{w}'\}$ also holds. □

Let \mathbf{w} be any non-simple word with $\text{h}(\mathbf{w}) \in \text{sim}(\mathbf{w})$ and let x_1, \dots, x_r be all its distinct non-simple letters listed in alphabetical order. In this section, such a word is said to be in *canonical form* if

$$\mathbf{w} = \prod_{i=1}^m (y_i \mathbf{w}_i) \tag{7.3}$$

for some $m \geq 1$ such that all of the following are satisfied:

- (I) the letters y_1, \dots, y_m are all simple in \mathbf{w} ;
- (II) the letters of $\mathbf{w}_1, \dots, \mathbf{w}_m \in \mathcal{X}^*$ are all non-simple in \mathbf{w} ;
- (III) $\mathbf{w}_1, \dots, \mathbf{w}_{m-1} \in \{x_1^{e_1} \dots x_r^{e_r} \mid e_1, \dots, e_r \in \{0, 1\}\}$;
- (IV) $\mathbf{w}_m = x_1^{f_1} \dots x_r^{f_r}$ for some $f_1, \dots, f_r \in \{0, 1, 2, 3\}$;
- (V) if $\text{occ}(x_i, \mathbf{w}_1 \dots \mathbf{w}_{m-1}) = 0$, then $f_i \in \{2, 3\}$;
- (VI) if $\text{occ}(x_i, \mathbf{w}_1 \dots \mathbf{w}_{m-1}) = 1$, then $f_i \in \{1, 2\}$;
- (VII) if $\text{occ}(x_i, \mathbf{w}_1 \dots \mathbf{w}_{m-1}) \geq 2$, then $f_i \in \{0, 1\}$.

LEMMA 7.6. *Let \mathbf{w} be any non-simple word with $\text{h}(\mathbf{w}) \in \text{sim}(\mathbf{w})$. Then there exists some word $\bar{\mathbf{w}}$ in canonical form such that the identities (7.1) imply the identity $\mathbf{w} \approx \bar{\mathbf{w}}$.*

Proof. Generality is not lost by assuming that x_1, \dots, x_r are all the distinct non-simple letters of \mathbf{w} listed in alphabetical order. By Lemma 7.2, it suffices to convert \mathbf{w} , using the identities (7.2), into a word in canonical form. It is clear that \mathbf{w} can be written in the form (7.3) with (I) and (II) satisfied. The identities (7.2d) can be used to order the letters in each \mathbf{w}_i alphabetically, resulting in $\mathbf{w}_i \in \{x_1^{e_1} \dots x_r^{e_r} \mid e_1, \dots, e_r \geq 0\}$. The identities (7.2a) can then be used to reduce the exponent of each x_j , so that $\mathbf{w}_i \in \{x_1^{e_1} \dots x_r^{e_r} \mid e_1, \dots, e_r \in \{0, 1, 2, 3\}\}$. If $i < m$ is such that \mathbf{w}_i contains the factor x_j^2 , then this factor can be moved by the identity (7.2c)

to the right and combined with the corresponding power of x_j in \mathbf{w}_m . Hence (III) is satisfied, and $\mathbf{w}_m = x_1^{f_1} \dots x_r^{f_r}$ for some $f_1, \dots, f_r \geq 0$. The identities (7.2a) can be used to reduce the exponents f_1, \dots, f_r in \mathbf{w}_m appropriately until (IV) is satisfied. Since x_i is non-simple in \mathbf{w} , it follows that $\text{occ}(x_i, \mathbf{w}_1 \dots \mathbf{w}_{m-1}) + f_i = \text{occ}(x_i, \mathbf{w}) \geq 2$, whence (V) is satisfied. If $\text{occ}(x_i, \mathbf{w}_1 \dots \mathbf{w}_{m-1}) = 1$ and $f_i = 3$, then the identities (7.2a) can be used to reduce the exponent f_i to 1. Thus (VI) is satisfied. Finally, if $\text{occ}(x_i, \mathbf{w}_1 \dots \mathbf{w}_{m-1}) \geq 2$ and $f_i \in \{2, 3\}$, then the identities (7.2b) can be used to reduce the exponent f_i to 0 or 1, so that (VII) is satisfied. \square

Proof of Proposition 7.4. By assumption, $\mathbf{V} = \mathbf{X}_{(7.1)}\Sigma$ for some set Σ of identities. The variety \mathbf{V} is finitely based if Σ contains some identity $\mathbf{w} \approx \mathbf{w}'$ that satisfies any of (i)–(iii) in Lemma 7.3. Therefore it suffices to assume that every identity $\mathbf{w} \approx \mathbf{w}'$ in Σ satisfies all of the following:

- (a) $\text{con}(\mathbf{w}) = \text{con}(\mathbf{w}')$ and $\text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$;
- (b) $\text{occ}(x, \mathbf{w}) \equiv \text{occ}(x, \mathbf{w}') \pmod{2}$ for all $x \in \mathcal{X}$;
- (c) $\text{occ}(x, y, \mathbf{w}) \equiv \text{occ}(x, y, \mathbf{w}') \pmod{2}$ for all $x \in \mathcal{X}$ and $y \in \text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$.

Since $L_2 \in \mathbf{V}$, it follows from Lemma 2.1(i) that any identity $\mathbf{w} \approx \mathbf{w}'$ in Σ satisfies the property that $\text{h}(\mathbf{w}) = \text{h}(\mathbf{w}')$; by Lemma 7.5, the letter $\text{h}(\mathbf{w}) = \text{h}(\mathbf{w}')$ can be assumed to be simple in \mathbf{w} and \mathbf{w}' . Hence by Lemma 7.6, the words \mathbf{w} and \mathbf{w}' can be chosen to be in canonical form.

Let $\mathbf{w} \approx \mathbf{w}'$ be any identity from Σ . By (a), the words \mathbf{w} and \mathbf{w}' share the same simple letters. By (c), the order of appearance of these simple letters is the same in \mathbf{w} and \mathbf{w}' . Hence

$$\mathbf{w} = \prod_{i=1}^m (y_i \mathbf{w}_i) \quad \text{and} \quad \mathbf{w}' = \prod_{i=1}^m (y_i \mathbf{w}'_i).$$

It follows from (a), (c), and (III) that

- (d) $\mathbf{w}_i = \mathbf{w}'_i$ for all $i < m$.

Now (IV) and (b) imply $\mathbf{w}_m = x_1^{f_1} \dots x_r^{f_r}$ and $\mathbf{w}'_m = x_1^{f'_1} \dots x_r^{f'_r}$ for some $f_i, f'_i \in \{0, 1, 2, 3\}$ with $f_i \equiv f'_i \pmod{2}$. Suppose $f_i < f'_i$ for some i . Then there are two cases: $(f_i, f'_i) = (0, 2)$ and $(f_i, f'_i) = (1, 3)$. If $(f_i, f'_i) = (0, 2)$, then

$$\text{occ}(x_i, \mathbf{w}_1 \dots \mathbf{w}_{m-1}) \geq 2 \quad \text{by (VII)} \quad \text{and} \quad \text{occ}(x_i, \mathbf{w}'_1 \dots \mathbf{w}'_{m-1}) \leq 1 \quad \text{by (V) and (VI)}.$$

If $(f_i, f'_i) = (1, 3)$, then

$$\text{occ}(x_i, \mathbf{w}_1 \dots \mathbf{w}_{m-1}) \geq 1 \quad \text{by (VI) and (VII)} \quad \text{and} \quad \text{occ}(x_i, \mathbf{w}'_1 \dots \mathbf{w}'_{m-1}) = 0 \quad \text{by (V)}.$$

Since $\mathbf{w}_1 \dots \mathbf{w}_{m-1} = \mathbf{w}'_1 \dots \mathbf{w}'_{m-1}$ by (d), both cases are impossible. Therefore $f_i = f'_i$ for all i , whence $\mathbf{w} = \mathbf{w}'$. Since the identity $\mathbf{w} \approx \mathbf{w}'$ is arbitrary in Σ , every identity in Σ is trivial. Consequently, the variety \mathbf{V} coincides with $\mathbf{X}_{(7.1)}$ and is finitely based. \square

7.3. Subvarieties of $\mathbf{X}_{(7.1)}$ not containing L_2

LEMMA 7.7. *Let \mathbf{V} be any subvariety of $\mathbf{X}_{(7.1)}$ that satisfies an identity $\mathbf{w} \approx \mathbf{w}'$. Suppose $\text{h}(\mathbf{w}) \neq \text{h}(\mathbf{w}')$ with either $\text{h}(\mathbf{w}) \in \text{sim}(\mathbf{w})$ or $\text{h}(\mathbf{w}') \in \text{sim}(\mathbf{w}')$. Then \mathbf{V} is finitely based.*

Proof. If either $\text{con}(\mathbf{w}) \neq \text{con}(\mathbf{w}')$ or $\text{sim}(\mathbf{w}) \neq \text{sim}(\mathbf{w}')$, then the variety \mathbf{V} is finitely based by Lemma 7.3. Therefore assume $\text{con}(\mathbf{w}) = \text{con}(\mathbf{w}')$ and $\text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$. Let $x = \text{h}(\mathbf{w})$ and $y = \text{h}(\mathbf{w}')$. By symmetry, it suffices to assume $x \in \text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$. Let φ denote the substitution $z \mapsto y^2$ for all $z \in \mathcal{X} \setminus \{x\}$. Then the deductions $\mathbf{w}\varphi \stackrel{(7.2)}{\approx} xy^2$ and $\mathbf{w}'\varphi \stackrel{(7.2)}{\approx} y^2x$ hold, so that \mathbf{V} satisfies the identity $xy^2 \approx y^2x$; any subvariety of $\mathbf{X}_{(7.1)}$ that satisfies this identity is finitely based [9, Theorem 7.2]. \square

PROPOSITION 7.8. *Let \mathbf{V} be any subvariety of $\mathbf{X}_{(7.1)}$ such that $L_2 \notin \mathbf{V}$. Then \mathbf{V} is finitely based.*

Proof. By assumption, $\mathbf{V} = \mathbf{X}_{(7.1)}\Sigma$ for some set Σ of identities. Let

$$\Sigma_{\oplus} = \{\mathbf{w} \approx \mathbf{w}' \in \Sigma \mid \mathbf{h}(\mathbf{w}) = \mathbf{h}(\mathbf{w}')\} \quad \text{and} \quad \Sigma_{\ominus} = \{\mathbf{w} \approx \mathbf{w}' \in \Sigma \mid \mathbf{h}(\mathbf{w}) \neq \mathbf{h}(\mathbf{w}')\},$$

so that $\Sigma = \Sigma_{\oplus} \cup \Sigma_{\ominus}$. By Lemma 2.1(i), the variety $\mathbf{X}_{(7.1)}\Sigma_{\oplus}$ contains L_2 and so is finitely based by Proposition 7.4. It remains to show that $\mathbf{X}_{(7.1)}\Sigma_{\ominus}$ is finitely based, so that the variety $\mathbf{V} = \mathbf{X}_{(7.1)}\Sigma_{\oplus} \cap \mathbf{X}_{(7.1)}\Sigma_{\ominus}$ is also finitely based.

By Lemmas 7.3 and 7.7, the variety \mathbf{V} is finitely based if some identity $\mathbf{w} \approx \mathbf{w}'$ from Σ_{\ominus} violates any of the following conditions:

- (a) $\text{con}(\mathbf{w}) = \text{con}(\mathbf{w}')$ and $\text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$;
- (b) $\mathbf{h}(\mathbf{w})$ is non-simple in \mathbf{w} and $\mathbf{h}(\mathbf{w}')$ is non-simple in \mathbf{w}' .

Hence it suffices to assume that any identity $\mathbf{w} \approx \mathbf{w}'$ in Σ_{\ominus} satisfies (a) and (b).

Let $\sigma : \mathbf{w} \approx \mathbf{w}'$ be any identity from Σ_{\ominus} , so that $\mathbf{h}(\mathbf{w}) = x \neq y = \mathbf{h}(\mathbf{w}')$. Let $\varphi : \mathcal{X} \rightarrow \mathcal{X}^+$ denote the substitution

$$z \mapsto \begin{cases} x^2 & \text{if } z = x, \\ y^2 & \text{otherwise.} \end{cases}$$

Then the deductions $\mathbf{w}\varphi \stackrel{(7.2)}{\approx} x^2y^2$ and $\mathbf{w}'\varphi \stackrel{(7.2)}{\approx} y^2x^2$ hold, so that the identities $\{(7.2), \sigma\}$ imply the identity

$$x^2y^2 \approx y^2x^2. \quad (7.4)$$

Thus $\mathbf{X}_{(7.1)}\{\sigma\} = \mathbf{X}_{(7.1)}\{(7.4), \sigma\}$. It follows from (a) and (b) that $\mathbf{w} = x\mathbf{h}\mathbf{y}\mathbf{t}$ for some $\mathbf{h}, \mathbf{t} \in \mathcal{X}^*$ such that $x, y \in \text{con}(\mathbf{h}\mathbf{t})$. Since

$$\mathbf{w} \stackrel{(7.2a)}{\approx} x^3\mathbf{h}\mathbf{y}^3\mathbf{t} \stackrel{(7.2c)}{\approx} x^2y^2x\mathbf{h}\mathbf{y}\mathbf{t} \stackrel{(7.4)}{\approx} y^2x^2x\mathbf{h}\mathbf{y}\mathbf{t} \stackrel{(7.2a)}{\approx} y^2\mathbf{w},$$

the equality $\mathbf{X}_{(7.1)}\{\sigma\} = \mathbf{X}_{(7.1)}\{(7.4), \tilde{\sigma}\}$ holds, where $\tilde{\sigma}$ is the identity $y^2\mathbf{w} \approx \mathbf{w}'$ that satisfies the property $\mathbf{h}(y^2\mathbf{w}) = \mathbf{h}(\mathbf{w}')$.

Since the identity σ is arbitrary in Σ_{\ominus} , the construction of $\tilde{\sigma}$ from σ in the preceding paragraph can be repeated on every identity in Σ_{\ominus} to obtain $\tilde{\Sigma}_{\ominus} = \{\tilde{\sigma} \mid \sigma \in \Sigma_{\ominus}\}$. It follows that $\mathbf{X}_{(7.1)}\Sigma_{\ominus} = \mathbf{X}_{(7.1)}\{(7.4)\} \cap \mathbf{X}_{(7.1)}\tilde{\Sigma}_{\ominus}$. By Lemma 2.1(i), the semigroup L_2 satisfies the identities in $\tilde{\Sigma}_{\ominus}$ and so belongs to $\mathbf{X}_{(7.1)}\tilde{\Sigma}_{\ominus}$. Therefore $\mathbf{X}_{(7.1)}\tilde{\Sigma}_{\ominus}$ is finitely based by Proposition 7.4. Consequently, $\mathbf{X}_{(7.1)}\Sigma_{\ominus}$ is also finitely based. \square

8. On Condition 10

Let $\mathbf{X}_{(8.1)}$ denote the variety defined by the identities

$$x^3 \approx x^2, \quad x^2yx \approx xyx, \quad xyx^2 \approx xyx, \quad (8.1a)$$

$$x^2y^2 \approx y^2x^2, \quad (8.1b)$$

$$x^2yz^2 \approx xyz^2x. \quad (8.1c)$$

Then Condition 10 is equivalent to the following result.

PROPOSITION 8.1. *All subvarieties of $\mathbf{X}_{(8.1)}$ are finitely based.*

Some preliminary results are established in §§ 8.1 and 8.2. The proof of Proposition 8.1 is then given in § 8.3.

8.1. *A canonical form*

Let \mathbf{w} be any non-simple word and let x_1, \dots, x_r be all its distinct non-simple letters listed in alphabetical order. In this section, such a word is said to be in *canonical form* if

$$\mathbf{w} = \mathbf{s}_0 \prod_{i=1}^m (\mathbf{w}_i \mathbf{s}_i) \tag{8.2}$$

for some $m \geq 1$ such that all of the following are satisfied:

- (I) the letters of $\mathbf{s}_0, \mathbf{s}_m \in \mathcal{X}^*$ and $\mathbf{s}_1, \dots, \mathbf{s}_{m-1} \in \mathcal{X}^+$ are all simple in \mathbf{w} ;
 - (II) $\mathbf{w}_1, \dots, \mathbf{w}_m \in \{x_1^{e_1} \dots x_r^{e_r} \mid e_1, \dots, e_r \in \{0, 2\}\}$ with $\mathbf{w}_1, \dots, \mathbf{w}_m \neq \emptyset$;
 - (III) $\text{con}(\mathbf{w}_1) \subseteq \dots \subseteq \text{con}(\mathbf{w}_m)$.
- Note that (II) and (III) imply:
- (IV) $\mathbf{w}_m = x_1^2 \dots x_r^2$.

LEMMA 8.2. *Let \mathbf{w} be any non-simple word. Then there exists some word $\bar{\mathbf{w}}$ in canonical form such that the identities (8.1) imply the identity $\mathbf{w} \approx \bar{\mathbf{w}}$.*

Proof. It suffices to convert \mathbf{w} , using the identities (8.1), into a word in canonical form. Let x_1, \dots, x_r be all the distinct non-simple letters of \mathbf{w} in alphabetical order. Clearly \mathbf{w} can be written in the form (8.2) such that (I) is satisfied and $\mathbf{w}_1, \dots, \mathbf{w}_m \in \{x_1, \dots, x_r\}^+$. Suppose $x \in \text{con}(\mathbf{w}_i) \setminus \text{con}(\mathbf{w}_{i+1})$, say $\mathbf{w}_i = \mathbf{a}x\mathbf{b}$ for some $\mathbf{a}, \mathbf{b} \in \mathcal{X}^*$. Then since the letters x and $h = h(\mathbf{w}_{i+1})$ are non-simple in \mathbf{w} ,

$$\begin{aligned} \mathbf{w} &= \dots \mathbf{w}_i \mathbf{s}_i \cdot \mathbf{w}_{i+1} \mathbf{s}_{i+1} \dots \stackrel{(8.1a)}{\approx} \dots \mathbf{a}x^3 \mathbf{b} \mathbf{s}_i \cdot h^2 \mathbf{w}_{i+1} \mathbf{s}_{i+1} \dots \\ &\stackrel{(8.1c)}{\approx} \dots \mathbf{a}x \mathbf{b} \mathbf{s}_i \cdot h^2 x^2 \mathbf{w}_{i+1} \mathbf{s}_{i+1} \dots \stackrel{(8.1b)}{\approx} \dots \mathbf{w}_i \mathbf{s}_i \cdot x^2 h^2 \mathbf{w}_{i+1} \mathbf{s}_{i+1} \dots \\ &\stackrel{(8.1a)}{\approx} \dots \mathbf{w}_i \mathbf{s}_i \cdot (x \mathbf{w}_{i+1}) \mathbf{s}_{i+1} \dots, \end{aligned}$$

that is, the identities (8.1) can be used to convert \mathbf{w}_{i+1} into $x\mathbf{w}_{i+1}$. It is easily seen how this procedure can be repeated until \mathbf{w} is converted into a word of the form (8.2) with (I) and (III) satisfied. Since the letters of each \mathbf{w}_i are non-simple in \mathbf{w} , apply the identities (8.1a) to replace each letter x_j in \mathbf{w}_i by its square x_j^2 , apply the identity (8.1b) to alphabetically order these squares in \mathbf{w}_i , and apply the identities (8.1a) to eliminate any repeated squares. The resulting factor is of the form $x_1^{e_1} \dots x_r^{e_r}$ with $e_1, \dots, e_r \in \{0, 2\}$. Therefore (II) is satisfied. \square

8.2. *Some finitely based subvarieties of $\mathbf{X}_{(8.1)}$*

LEMMA 8.3. *Suppose that \mathbf{V} is any subvariety of $\mathbf{X}_{(8.1)}$ that satisfies an identity $\mathbf{w} \approx \mathbf{w}'$ with either $\text{con}(\mathbf{w}) \neq \text{con}(\mathbf{w}')$ or $\text{sim}(\mathbf{w}) \neq \text{sim}(\mathbf{w}')$. Then \mathbf{V} is finitely based.*

Proof. It follows from the assumption and Lemma 2.1(v) that $N_2^1 \notin \mathbf{V}$. Then by Lemma 2.9, the variety \mathbf{V} satisfies the identity $\alpha : h^2(xh^2)^3 \approx h^2xh^2$. Since

$$xyx \stackrel{(8.1a)}{\approx} x^2yx^2 \stackrel{\alpha}{\approx} x^2(yx^2)^3 \stackrel{(8.1a)}{\approx} x^2(y^2x^2)^3 \stackrel{(8.1b)}{\approx} x^8y^6 \stackrel{(8.1a)}{\approx} x^2y^2,$$

the variety \mathbf{V} satisfies the identity $xyx \approx x^2y^2$; any variety that satisfies this identity is finitely based [34]. \square

LEMMA 8.4. *Suppose that \mathbf{V} is any subvariety of $\mathbf{X}_{(8.1)}$ that satisfies an identity $\mathbf{w} \approx \mathbf{w}'$ with $F_{SS}(\mathbf{w}) \neq F_{SS}(\mathbf{w}')$. Then \mathbf{V} is finitely based.*

Proof. If either $\text{con}(\mathbf{w}) \neq \text{con}(\mathbf{w}')$ or $\text{sim}(\mathbf{w}) \neq \text{sim}(\mathbf{w}')$, then the variety \mathbf{V} is finitely based by Lemma 8.3. Hence assume $\text{con}(\mathbf{w}) = \text{con}(\mathbf{w}')$ and $\text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$. By Lemma 2.10, the variety \mathbf{V} satisfies the identity (2.3) with $n = 2$. It follows that \mathbf{V} satisfies the identities in Condition 6 and so is finitely based. \square

LEMMA 8.5. *Suppose that \mathbf{V} is any subvariety of $\mathbf{X}_{(8.1)}$ that satisfies an identity $\mathbf{w} \approx \mathbf{w}'$ with $\mathbf{w}_{\text{sim}} \neq \mathbf{w}'_{\text{sim}}$. Then \mathbf{V} is finitely based.*

Proof. If either $\text{con}(\mathbf{w}) \neq \text{con}(\mathbf{w}')$ or $\text{sim}(\mathbf{w}) \neq \text{sim}(\mathbf{w}')$, then the variety \mathbf{V} is finitely based by Lemma 8.3. Hence assume $\text{con}(\mathbf{w}) = \text{con}(\mathbf{w}')$ and $\text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$. Since $\mathbf{w}_{\text{sim}} \neq \mathbf{w}'_{\text{sim}}$ by assumption, there exist some distinct letters $y, z \in \text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$ such that $\mathbf{w} \in \mathcal{X}^*y\mathcal{X}^*z\mathcal{X}^*$ and $\mathbf{w}' \in \mathcal{X}^*z\mathcal{X}^*y\mathcal{X}^*$. Let $\varphi : \mathcal{X} \rightarrow \mathcal{X}^+$ denote the substitution

$$t \mapsto \begin{cases} txt & \text{if } t \in \{y, z\}, \\ x & \text{otherwise.} \end{cases}$$

Then the deductions $\mathbf{w}\varphi \stackrel{(8.1a)}{\approx} xyxzx$ and $\mathbf{w}'\varphi \stackrel{(8.1a)}{\approx} xzxyx$ hold so that \mathbf{V} satisfies the identity $xyxzx \approx xzxyx$. Hence \mathbf{V} satisfies the identities in Condition 11 and so is finitely based. \square

8.3. *Proof of Proposition 8.1*

LEMMA 8.6. *Let \mathbf{w} and \mathbf{w}' be words and x and y be letters such that $x \in \text{con}(\mathbf{w}) = \text{con}(\mathbf{w}')$ and $y \in \text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$. Suppose that the first occurrence of x precedes y in \mathbf{w} , but y precedes the first occurrence of x in \mathbf{w}' . Then the identities $\{(8.1), \mathbf{w} \approx \mathbf{w}'\}$ imply the identity*

$$h^2yx^2h^2 \approx x^2h^2yx^2h^2. \tag{8.3}$$

Proof. By assumption, $\mathbf{w} = \mathbf{w}_1y\mathbf{w}_2$ and $\mathbf{w}' = \mathbf{w}'_1y\mathbf{w}'_2$ for some $\mathbf{w}_i, \mathbf{w}'_i \in (\mathcal{X} \setminus \{y\})^*$ such that $x \in \text{con}(\mathbf{w}_1) \setminus \text{con}(\mathbf{w}'_1)$. Let $\varphi : \mathcal{X} \rightarrow \mathcal{X}^+$ denote the substitution

$$t \mapsto \begin{cases} x^2h^2 & \text{if } t = x, \\ yx^2h^2 & \text{if } t = y, \\ h^2 & \text{otherwise.} \end{cases}$$

Then since the deductions $h^2(\mathbf{w}\varphi)x^2h^2 \stackrel{(8.1)}{\approx} x^2h^2yx^2h^2$ and $h^2(\mathbf{w}'\varphi)x^2h^2 \stackrel{(8.1)}{\approx} h^2yx^2h^2$ hold, the identities $\{(8.1), \mathbf{w} \approx \mathbf{w}'\}$ imply the identity (8.3). \square

Let \mathbf{V} be any proper subvariety of $\mathbf{X}_{(8.1)}$. Then $\mathbf{V} = \mathbf{X}_{(8.1)}\Sigma$ for some nonempty set Σ of nontrivial identities. By Lemmas 8.3–8.5, the variety \mathbf{V} is finitely based if some identity $\mathbf{w} \approx \mathbf{w}'$ in Σ does not satisfy any of the following conditions:

- (a) $\text{con}(\mathbf{w}) = \text{con}(\mathbf{w}')$ and $\text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$;
- (b) $F_{SS}(\mathbf{w}) = F_{SS}(\mathbf{w}')$;
- (c) $\mathbf{w}_{\text{sim}} = \mathbf{w}'_{\text{sim}}$.

Hence assume that (a)–(c) hold for every identity $\mathbf{w} \approx \mathbf{w}'$ in Σ .

Let $\mathbf{w} \approx \mathbf{w}'$ be any identity in Σ . By (a), the words \mathbf{w} and \mathbf{w}' are either both simple or both non-simple. If the words \mathbf{w} and \mathbf{w}' are both simple, then by (c), the identity $\mathbf{w} \approx \mathbf{w}'$ is contradictorily trivial. Therefore the words \mathbf{w} and \mathbf{w}' are both non-simple, say with non-simple letters x_1, \dots, x_r . By Lemma 8.2, the words \mathbf{w} and \mathbf{w}' can be chosen to be in canonical form. It then follows from (a)–(c) that

$$\mathbf{w} = s_0 \prod_{i=1}^m (\mathbf{w}_i s_i) \quad \text{and} \quad \mathbf{w}' = s_0 \prod_{i=1}^m (\mathbf{w}'_i s_i),$$

where

$$(d) \mathbf{w}_m = x_1^2 \dots x_r^2 = \mathbf{w}'_m$$

by (IV). But since the identity $\mathbf{w} \approx \mathbf{w}'$ is nontrivial, $\text{con}(\mathbf{w}_j) \neq \text{con}(\mathbf{w}'_j)$ for some $j < m$, say $x_i \in \text{con}(\mathbf{w}_j) \setminus \text{con}(\mathbf{w}'_j)$. Then $x_i \notin \text{con}(\mathbf{w}'_1 \dots \mathbf{w}'_j)$ by (III), so that $x_i \in \text{con}(\mathbf{w}'_{j+1} \dots \mathbf{w}'_m)$. The letter $y = h(\mathbf{s}_j) \in \text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$ is preceded in \mathbf{w} by the first occurrence of x_i while y precedes the first occurrence of x_i in \mathbf{w}' . Therefore by Lemma 8.6,

$$(e) \mathbf{X}_{(8.1)}\{\mathbf{w} \approx \mathbf{w}'\} = \mathbf{X}_{(8.1)}\{\mathbf{w} \approx \mathbf{w}', (8.3)\}.$$

Choose any letter $x_\ell \in \text{con}(\mathbf{w}_1)$. Then $x_\ell \in \bigcap_{i=1}^m \text{con}(\mathbf{w}_i)$ by (III). For any $i < m$, the factor \mathbf{w}_i can be written in the form $\mathbf{w}_i = \mathbf{a}x_\ell^2\mathbf{b}$, where $\mathbf{a} = x_1^{e_1} \dots x_{\ell-1}^{e_{\ell-1}}$ and $\mathbf{b} = x_{\ell+1}^{e_{\ell+1}} \dots x_r^{e_r}$ for some $e_1, \dots, e_{\ell-1}, e_{\ell+1}, \dots, e_r \in \{0, 2\}$. Then by (d),

$$\begin{aligned} \mathbf{w} &= \dots \mathbf{w}_i \mathbf{s}_i \dots \mathbf{w}_m \mathbf{s}_m \\ (8.1b) \quad &\approx \dots (\mathbf{a} \mathbf{b} x_\ell^2) \mathbf{s}_i \dots (x_1^2 \dots x_{\ell-1}^2 x_{\ell+1}^2 \dots x_r^2 x_\ell^2) \mathbf{s}_m \\ (8.3) \quad &\approx \dots (\mathbf{a} \mathbf{b} x_r^2 x_\ell^2) \mathbf{s}_i \dots (x_1^2 \dots x_{\ell-1}^2 x_{\ell+1}^2 \dots x_r^2 x_\ell^2) \mathbf{s}_m \\ &\vdots \\ (8.3) \quad &\approx \dots (\mathbf{a} \mathbf{b} x_1^2 \dots x_{\ell-1}^2 x_{\ell+1}^2 \dots x_r^2 x_\ell^2) \mathbf{s}_i \dots (x_1^2 \dots x_{\ell-1}^2 x_{\ell+1}^2 \dots x_r^2 x_\ell^2) \mathbf{s}_m \\ (8.1b) \quad &\approx \dots (x_1^{2+e_1} \dots x_{\ell-1}^{2+e_{\ell-1}} x_\ell^2 x_{\ell+1}^{2+e_{\ell+1}} \dots x_r^{2+e_r}) \mathbf{s}_i \dots (x_1^2 \dots x_{\ell-1}^2 x_\ell^2 x_{\ell+1}^2 \dots x_r^2) \mathbf{s}_m \\ (8.1a) \quad &\approx \dots \mathbf{w}_m \mathbf{s}_i \dots \mathbf{w}_m \mathbf{s}_m, \end{aligned}$$

that is, \mathbf{w}_i is converted into \mathbf{w}_m . This argument can be repeated on \mathbf{w}_j for every $j < m$, so that the identities $\{(8.1), (8.3)\}$ imply the identity $\mathbf{w} \approx \mathbf{s}_0 \prod_{i=1}^m (\mathbf{w}_m \mathbf{s}_i)$. Similarly, the identities $\{(8.1), (8.3)\}$ imply the identity $\mathbf{w}' \approx \mathbf{s}_0 \prod_{i=1}^m (\mathbf{w}'_m \mathbf{s}_i)$. Therefore since

$$\mathbf{w} \stackrel{(8.1),(8.3)}{\approx} \mathbf{s}_0 \prod_{i=1}^m (\mathbf{w}_m \mathbf{s}_i) \stackrel{(d)}{=} \mathbf{s}_0 \prod_{i=1}^m (\mathbf{w}'_m \mathbf{s}_i) \stackrel{(8.1),(8.3)}{\approx} \mathbf{w}',$$

the identities $\{(8.1), (8.3)\}$ imply the identity $\mathbf{w} \approx \mathbf{w}'$. It thus follows from (e) that the equality $\mathbf{X}_{(8.1)}\{\mathbf{w} \approx \mathbf{w}'\} = \mathbf{X}_{(8.1)}\{(8.3)\}$ holds. But the identity $\mathbf{w} \approx \mathbf{w}'$ is arbitrary in Σ . Therefore $\mathbf{X}_{(8.1)}\Sigma = \mathbf{X}_{(8.1)}\{(8.3)\}$. Consequently, \mathbf{V} is finitely based.

9. On Condition 14

Let $\mathbf{X}_{(9.1)}$ denote the variety defined by the identities

$$x^4 \approx x^2, \quad xy^3x \approx xyx, \quad xyx^3 \approx xyx, \tag{9.1a}$$

$$x^2yx \approx xyx^2, \quad xyxzx \approx x^2yzx, \tag{9.1b}$$

$$xyxy \approx xy^2x. \tag{9.1c}$$

Then Condition 14 is equivalent to the following result.

PROPOSITION 9.1. *All subvarieties of $\mathbf{X}_{(9.1)}$ are finitely based.*

It is shown in § 9.1 that $\mathbf{X}_{(9.1)} = \mathbf{A}_0 \vee \mathbf{L}_2^1 \vee \mathbf{Z}_2$. The proof of Proposition 9.1 is then given in § 9.2.

9.1. Identities satisfied by $\mathbf{X}_{(9.1)}$

LEMMA 9.2. The identities (9.1) imply the identities

$$xHx^2Kx \approx xHKx, \quad (9.2a)$$

$$xHy^3Kx \approx xHyKx, \quad (9.2b)$$

$$xHyKxy \approx xHyKyx, \quad (9.2c)$$

$$xHyzyKx \approx xHy^2zKx. \quad (9.2d)$$

Proof. It is easily seen that the identities (9.1) imply the identities (9.2a). Since

$$xHy^3Kx \stackrel{(9.2a)}{\approx} xHx^2y^3x^2Kx \stackrel{(9.1a)}{\approx} xHx^2yx^2Kx \stackrel{(9.2a)}{\approx} xHyKx,$$

the identities (9.1) imply the identities (9.2b). Since

$$\begin{aligned} xHyKxy &\stackrel{(9.2a)}{\approx} xHyKy^2xy \stackrel{(9.2a)}{\approx} xHyKyx^2yxy \stackrel{(9.1c)}{\approx} xHyKyx^2y^2x \\ &\stackrel{(9.2a)}{\approx} xHyKy^3x \stackrel{(9.2b)}{\approx} xHyKyx, \end{aligned}$$

the identities (9.1) imply the identities (9.2c). Since

$$\begin{aligned} xHy^2zKx &\stackrel{(9.2b)}{\approx} xHy(yz)^3Kx \stackrel{(9.2c)}{\approx} xHy^2z^3y^2Kx \stackrel{(9.1b)}{\approx} xHy^3z^3yKx \\ &\stackrel{(9.2b)}{\approx} xHyzyKx, \end{aligned}$$

the identities (9.1) imply the identities (9.2d). \square

In this section, a non-simple word \mathbf{w} is said to be in *canonical form* if

$$\mathbf{w} = x_0^{e_0} x_1^{e_1} \dots x_m^{e_m} x_0 \quad (9.3)$$

for some distinct letters x_0, \dots, x_m with $e_0, \dots, e_m \in \{1, 2\}$. Note that if $m = 0$, then the word \mathbf{w} in (9.3) is $x_0^{e_0+1}$.

LEMMA 9.3. Let \mathbf{w} be any connected word. Then there exists some word $\bar{\mathbf{w}}$ in canonical form such that the identities (9.1) imply the identity $\mathbf{w} \approx \bar{\mathbf{w}}$.

Proof. By Lemma 9.2, it suffices to convert \mathbf{w} , using the identities $\{(9.1), (9.2)\}$, into a word in canonical form. Since \mathbf{w} is connected, the letter $x_0 = h(\mathbf{w})$ occurs at least twice in \mathbf{w} . There are two cases.

Case 1: $h(\mathbf{w}) = t(\mathbf{w})$. Then $\mathbf{w} = x_0 \mathbf{a} x_0$ for some $\mathbf{a} \in \mathcal{X}^*$. The identities (9.1b) can be used to group any x_0 in \mathbf{a} with the first x_0 in \mathbf{w} , resulting in a word of the form $x_0^{e_0} \mathbf{b} x_0$, where $e_0 \geq 1$ and $\mathbf{b} \in (\mathcal{X} \setminus \{x_0\})^*$. The identities (9.2d) can be used to group any non-first occurrence of any letter x in \mathbf{b} with its first occurrence: $x_0 c x d x e x_0 \stackrel{(9.2d)}{\approx} x_0 c x^2 d e x_0$. The resulting word is of the form (9.3) with $e_i \geq 1$. The identities $\{(9.2a), (9.2b)\}$ can be used to reduce each e_i to a number in $\{1, 2\}$, resulting in a word in canonical form.

Case 2: $h(\mathbf{w}) \neq t(\mathbf{w})$. Then $\mathbf{w} = x_0 \mathbf{a} x_0 y \mathbf{b}$ for some $\mathbf{a}, \mathbf{b} \in \mathcal{X}^*$ and $y \in \mathcal{X}$ with $x_0 \notin \text{con}(y\mathbf{b})$. Since \mathbf{w} is connected and its suffix $y\mathbf{b}$ does not contain the letter x_0 , the factor \mathbf{a} is nonempty.

If $y \in \text{con}(\mathbf{a})$, then $\mathbf{w} = x_0\mathbf{a}x_0y\mathbf{b} \stackrel{(9.2c)}{\approx} x_0\mathbf{a}yx_0\mathbf{b}$. If $y \notin \text{con}(\mathbf{a})$, then the connectedness of \mathbf{w} implies that \mathbf{a} and \mathbf{b} share some common letter, whence

$$\mathbf{w} \stackrel{(9.2a)}{\approx} x_0\mathbf{a}x_0^3y\mathbf{b} \stackrel{(9.2d)}{\approx} x_0\mathbf{a}x_0^2yx_0\mathbf{b} \stackrel{(9.2a)}{\approx} x_0\mathbf{a}yx_0\mathbf{b}.$$

In any case, the identities (9.2) can be used to interchange the last x_0 in \mathbf{w} with the letter y that immediately follows it. This procedure can be repeated until the last x_0 in \mathbf{w} is the last letter of the word. Hence $\mathbf{w} \stackrel{(9.2)}{\approx} x_0\mathbf{c}x_0$ for some $\mathbf{c} \in \mathcal{X}^+$. By Case 1, the identities $\{(9.1), (9.2)\}$ can be used to convert $x_0\mathbf{c}x_0$ into a word in canonical form. \square

PROPOSITION 9.4. *The equality $\mathbf{X}_{(9.1)} = \mathbf{A}_0 \vee \mathbf{L}_2^1 \vee \mathbf{Z}_2$ holds.*

Proof. By Lemma 2.5, the semigroup $A_0 \times L_2^1 \times \mathbb{Z}_2$ is idempotent-separable. Therefore by Lemma 2.6(i), there exists some basis Σ for $\mathbf{A}_0 \vee \mathbf{L}_2^1 \vee \mathbf{Z}_2$ that consists of identities formed by connected words. It is routinely verified, either directly or by Lemmas 2.1 and 2.4, that A_0 , L_2^1 , and \mathbb{Z}_2 satisfy the identities (9.1). Hence the inclusion $\mathbf{A}_0 \vee \mathbf{L}_2^1 \vee \mathbf{Z}_2 \subseteq \mathbf{X}_{(9.1)}$ holds. To complete the proof, it remains to show that any identity $\mathbf{w} \approx \mathbf{w}'$ in Σ is implied by the identities (9.1).

By Lemma 9.3, there exist words $\bar{\mathbf{w}}$ and $\bar{\mathbf{w}}'$ in canonical form such that the identities (9.1) imply $\mathbf{w} \approx \bar{\mathbf{w}}$ and $\mathbf{w}' \approx \bar{\mathbf{w}}'$. Since $\mathbf{A}_0 \vee \mathbf{L}_2^1 \vee \mathbf{Z}_2$ satisfies the identity $\bar{\mathbf{w}} \approx \bar{\mathbf{w}}'$, it follows from Lemma 2.1 parts (iii) and (viii) that $\text{ini}(\bar{\mathbf{w}}) = \text{ini}(\bar{\mathbf{w}}')$ and $\text{occ}(x, \bar{\mathbf{w}}) \equiv \text{occ}(x, \bar{\mathbf{w}}') \pmod{2}$ for all $x \in \mathcal{X}$. It is then easily shown that the words $\bar{\mathbf{w}}$ and $\bar{\mathbf{w}}'$ are identical. Consequently, the identities (9.1) imply the identity $\mathbf{w} \approx \mathbf{w}'$. \square

9.2. *Proof of Proposition 9.1*

Let \mathbf{V} be any proper subvariety of $\mathbf{X}_{(9.1)}$. Then by Proposition 9.4, the variety \mathbf{V} does not contain one of the semigroups A_0 , L_2^1 , and \mathbb{Z}_2 .

Case 1: $A_0 \notin \mathbf{V}$. Then by Lemma 2.8, the variety \mathbf{V} satisfies the identity $\alpha : (x^2y^2)^3 \approx x^2y^2$. Since

$$x^2y^2 \stackrel{\alpha}{\approx} x^2y^2x^2y^2x^2y^2 \stackrel{(9.1b)}{\approx} xyx^5y^5 \stackrel{(9.1a)}{\approx} xyxy,$$

the variety \mathbf{V} satisfies the identity $xyxy \approx x^2y^2$. It is then routinely shown that \mathbf{V} satisfies the identities in Condition 15 and so is finitely based.

Case 2: $L_2^1 \notin \mathbf{V}$. If x^2 , $x^2y^2z^2x^2$, and $x^2z^2y^2x^2$ are distinct elements in the \mathbf{V} -free semigroup $F_{\mathbf{V}}$ over \mathcal{X} , then they form a subsemigroup of $F_{\mathbf{V}}$ isomorphic to L_2^1 , and this implies the contradiction $L_2^1 \in \mathbf{V}$. Therefore these three elements of $F_{\mathbf{V}}$ cannot be distinct, whence \mathbf{V} satisfies either $x^2 \approx x^2y^2z^2x^2$, $x^2 \approx x^2z^2y^2x^2$, or $x^2y^2z^2x^2 \approx x^2z^2y^2x^2$; it is easily seen that any one of the first two identities implies the third identity, so that \mathbf{V} satisfies the third identity. It follows that \mathbf{V} satisfies the identity $\beta : xy^2z^2x \approx xz^2y^2x$. Since

$$\begin{aligned} xyxzx &\stackrel{(9.2b)}{\approx} xy^3xz^3x \stackrel{(9.2d)}{\approx} xy^2xz^2yzx \stackrel{(9.1b)}{\approx} xy^2z^2xyzx \stackrel{\beta}{\approx} xz^2y^2xyzx \\ &\stackrel{(9.2d)}{\approx} xz^3y^3x^2 \stackrel{(9.2b)}{\approx} xzyx^2 \stackrel{(9.1b)}{\approx} xzxyx, \end{aligned}$$

the variety \mathbf{V} satisfies the identity $xyxzx \approx xzxyx$. It is routinely verified that \mathbf{V} satisfies the identities in Condition 12 and so is finitely based.

Case 3: $\mathbb{Z}_2 \notin \mathbf{V}$. Then by Lemma 2.1(viii), the variety \mathbf{V} satisfies the identity $x^3 \approx x^2$. It follows that \mathbf{V} satisfies the identities in Condition 6 and so is finitely based.

10. On Condition 20

Let $\mathbf{X}_{(10.1)}$ denote the variety defined by the identities

$$xHx^2 \approx xHx, \quad (10.1a)$$

$$x^2yxz \approx x^2yz, \quad (10.1b)$$

$$xHyKxy \approx xHyKyx. \quad (10.1c)$$

Then Condition 20 is equivalent to the following result.

PROPOSITION 10.1. *Any subvariety of $\mathbf{X}_{(10.1)}$ that does not satisfy the identity*

$$h^2xyxy \approx h^2x^2y^2 \quad (10.2)$$

is finitely based.

In this section, the following semigroup of order four is required:

$$P_2 = \langle a, b \mid a^2 = ab = a, b^2a = b^2 \rangle.$$

Let \mathbf{P}_2^1 denote the semigroup variety generated by the monoid P_2^1 . It is shown in § 10.3 that $\mathbf{X}_{(10.1)} = \mathbf{P}_2^1 \vee \mathbf{J}$. The proof of Proposition 10.1 is then given in § 10.4.

10.1. Some identities satisfied by subvarieties of $\mathbf{X}_{(10.1)}$

LEMMA 10.2 [19, Corollary 6.6]. *The variety \mathbf{P}_2^1 is defined by the identities (10.1) and*

$$x^2yx \approx x^2y. \quad (10.3)$$

LEMMA 10.3. *The identity (10.2) is an exclusion identity for P_2^1 in $\mathbf{X}_{(10.1)}$.*

Proof. Let \mathbf{V} be any subvariety of $\mathbf{X}_{(10.1)}$ such that $P_2 \notin \mathbf{V}$. Let \overleftarrow{P}_2 denote the dual semigroup of P_2 . It is routinely shown that A_0 , B_2 , and \overleftarrow{P}_2 do not satisfy the identity (10.1b), so that $A_0, B_2, \overleftarrow{P}_2 \notin \mathbf{V}$. It follows from Volkov [48, Theorem 2.1] that \mathbf{V} satisfies the identity

$$xyxy \approx x^2y^2. \quad (10.4)$$

The semigroup P_2 does not satisfy the identity (10.4) because $baba \neq b^2a^2$. Therefore the identity (10.4) is an exclusion identity for P_2 in $\mathbf{X}_{(10.1)}$.

Recall that $\psi_2 : \mathcal{X} \rightarrow \mathcal{X}^+$ denotes the substitution $z \mapsto h^2zh^2$ for all $z \in \mathcal{X}$. By Lemma 2.7, the identity $(xyxy)\psi_2 \approx (x^2y^2)\psi_2$ is an exclusion identity for P_2^1 in $\mathbf{X}_{(10.1)}$. Since

$$(xyxy)\psi_2 = h^2xh^4yh^4xh^4yh^2 \stackrel{(10.1c)}{\approx} h^2xh^4yh^4xh^6y \stackrel{(10.1b)}{\approx} h^2xyxy$$

and

$$(x^2y^2)\psi_2 = h^2xh^4xh^4yh^4yh^2 \stackrel{(10.1c)}{\approx} h^2xh^4xh^4yh^6y \stackrel{(10.1b)}{\approx} h^2x^2y^2,$$

the identity (10.2) is an exclusion identity for P_2^1 in $\mathbf{X}_{(10.1)}$. \square

LEMMA 10.4. *Let \mathbf{V} be any subvariety of $\mathbf{X}_{(10.1)}$ such that $N_2^1 \in \mathbf{V}$ and $J \notin \mathbf{V}$. Then \mathbf{V} satisfies the identity (10.3).*

Proof. Consider the elements $a = x^2y^2$, $b = x^2y$, $c = x^2$, and $d = bc = x^2yx^2$ in the \mathbf{V} -free semigroup $F_{\mathbf{V}}$. It follows from Lemma 2.1(v) that among these four elements, only b and d are possibly equal in $F_{\mathbf{V}}$ while any other pair of distinct elements are unequal in $F_{\mathbf{V}}$. Suppose $b \neq d$ in $F_{\mathbf{V}}$. Then $\{a, b, c, d\}$ constitutes the following subsemigroup S of $F_{\mathbf{V}}$.

$$\begin{array}{c|cccc}
 S & a & b & c & d \\
 \hline
 a & a & a & a & a \\
 b & a & a & d & a \\
 c & a & b & c & d \\
 d & a & a & d & a
 \end{array}$$

The quotient $S/\{a, d\}$ is isomorphic to J , so that the contradiction $J \in \mathbf{V}$ is deduced. Thus the elements b and d must be equal in $F_{\mathbf{V}}$, whence \mathbf{V} satisfies the identity $x^2y \approx x^2yx^2$. It follows that \mathbf{V} satisfies the identity (10.3). \square

10.2. A canonical form

In this section, any word \mathbf{w} with $t = t(\mathbf{w}) \in \text{sim}(\mathbf{w})$ is said to be in *canonical form* if

$$\mathbf{w} = \left(\prod_{i=1}^m (x_i \mathbf{w}_i) \right) t, \tag{10.5}$$

where $x_1, \dots, x_m \in \mathcal{X}$ and $\mathbf{w}_1, \dots, \mathbf{w}_m \in \mathcal{X}^*$ are such that:

- (I) $\text{ini}(\mathbf{w}) = x_1 \dots x_m t$;
- (II) $\mathbf{w}_i \in \{x_1^{e_1} \dots x_i^{e_i} \mid e_1, \dots, e_i \in \{0, 1\}\}$;
- (III) $\text{con}(\mathbf{w}_i) \cap \text{con}(\mathbf{w}_j) = \emptyset$ whenever $i \neq j$.

It follows that

- (IV) $\text{occ}(x, \mathbf{w}) \leq 2$ for any $x \in \mathcal{X}$.

LEMMA 10.5. *Let \mathbf{w} be any word with $t = t(\mathbf{w}) \in \text{sim}(\mathbf{w})$. Then there exists some word $\overline{\mathbf{w}}$ in canonical form with $t = t(\overline{\mathbf{w}}) \in \text{sim}(\overline{\mathbf{w}})$ such that the identities (10.1) imply the identity $\mathbf{w} \approx \overline{\mathbf{w}}$.*

Proof. It suffices to convert \mathbf{w} , using the identities (10.1), into a word in canonical form. Note that if $\mathbf{w} = \mathbf{a}\mathbf{b}\mathbf{x}\mathbf{c}\mathbf{x}\mathbf{d}\mathbf{t}$ for some $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathcal{X}^*$, then

$$\mathbf{w} \stackrel{(10.1a)}{\approx} \begin{cases} \mathbf{a}\mathbf{b}\mathbf{x}\mathbf{c}\mathbf{d}\mathbf{t} & \text{if } \mathbf{c} = \emptyset, \\ \mathbf{a}\mathbf{b}\mathbf{x}^2\mathbf{c}\mathbf{x}\mathbf{d}\mathbf{t} \stackrel{(10.1b)}{\approx} \mathbf{a}\mathbf{b}\mathbf{x}^2\mathbf{c}\mathbf{d}\mathbf{t} \stackrel{(10.1a)}{\approx} \mathbf{a}\mathbf{b}\mathbf{x}\mathbf{c}\mathbf{d}\mathbf{t} & \text{if } \mathbf{c} \neq \emptyset. \end{cases}$$

Therefore any third or later occurrence of a letter in \mathbf{w} can be removed by the identities (10.1). Hence (IV) is satisfied.

Now the letter t is simple in \mathbf{w} , so that $\text{ini}(\mathbf{w}) = x_1 \dots x_m t$ for some $x_1, \dots, x_m \in \mathcal{X}$. It is clear that \mathbf{w} can be written in the form (10.5) with (I) satisfied and $\mathbf{w}_i \in \{x_1, \dots, x_i\}^*$ for all i . It follows from (IV) that (III) is satisfied, and that within each \mathbf{w}_i , no letter occurs more than once. Since the letters in each \mathbf{w}_i are non-first occurrences in \mathbf{w} , the identities (10.1c) can be used to order them according to their indices. Hence (II) is satisfied. \square

10.3. A basis for $\mathbf{P}_2^1 \vee \mathbf{J}$

LEMMA 10.6. *Let $\mathbf{w} \approx \mathbf{w}'$ be any identity satisfied by N_2^1 and let $t \in \mathcal{X} \setminus \text{con}(\mathbf{w}\mathbf{w}')$. Suppose $t(\mathbf{w}) \notin \text{sim}(\mathbf{w})$ and $t(\mathbf{w}') \notin \text{sim}(\mathbf{w}')$. Then $\mathbf{X}_{(10.1)}\{\mathbf{w} \approx \mathbf{w}'\} = \mathbf{X}_{(10.1)}\{\mathbf{w}t \approx \mathbf{w}'t\}$.*

Proof. Let σ_1 and σ_2 denote the identities $\mathbf{w} \approx \mathbf{w}'$ and $\mathbf{w}t \approx \mathbf{w}'t$, respectively. The inclusion $\mathbf{X}_{(10.1)}\{\sigma_1\} \subseteq \mathbf{X}_{(10.1)}\{\sigma_2\}$ is obvious. Let $x = t(\mathbf{w})$ and $y = t(\mathbf{w}')$. (Note that x and y need not be distinct.) The assumptions $x \notin \text{sim}(\mathbf{w})$ and $y \notin \text{sim}(\mathbf{w}')$ imply the deductions $\mathbf{w} \stackrel{(10.1a)}{\approx} \mathbf{w}x$ and $\mathbf{w}' \stackrel{(10.1a)}{\approx} \mathbf{w}'y$. It follows from Lemma 2.1(v) that $\text{con}(\mathbf{w}) = \text{con}(\mathbf{w}')$ and $\text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$, whence y is non-simple in \mathbf{w} . Therefore $\mathbf{w} = \mathbf{a}y\mathbf{b}y\mathbf{c}$ for some $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathcal{X}^*$. If $\mathbf{c} = \emptyset$, then

$$\mathbf{w}x = \mathbf{a}y\mathbf{b}yx \stackrel{(10.1a)}{\approx} \mathbf{a}y\mathbf{b}yyx = \mathbf{w}yx.$$

If $\mathbf{c} \neq \emptyset$, then

$$\mathbf{w}x \stackrel{(10.1a)}{\approx} \mathbf{a}y\mathbf{b}y^2\mathbf{c}x \stackrel{(10.1b)}{\approx} \mathbf{a}y\mathbf{b}y^2\mathbf{c}yx \stackrel{(10.1a)}{\approx} \mathbf{a}y\mathbf{b}y\mathbf{c}yx = \mathbf{w}yx.$$

Therefore the deduction $\mathbf{w}x \stackrel{(10.1)}{\approx} \mathbf{w}yx$ holds in any case. Now since

$$\mathbf{w} \stackrel{(10.1a)}{\approx} \mathbf{w}x \stackrel{(10.1)}{\approx} \mathbf{w}yx \stackrel{(10.1c)}{\approx} \mathbf{w}xy \stackrel{(10.1a)}{\approx} \mathbf{w}y \stackrel{\sigma_2}{\approx} \mathbf{w}'y \stackrel{(10.1a)}{\approx} \mathbf{w}',$$

the identities $\{(10.1), \sigma_2\}$ imply σ_1 . Hence the inclusion $\mathbf{X}_{(10.1)}\{\sigma_2\} \subseteq \mathbf{X}_{(10.1)}\{\sigma_1\}$ holds. \square

PROPOSITION 10.7. *The equality $\mathbf{X}_{(10.1)} = \mathbf{P}_2^1 \vee \mathbf{J}$ holds.*

Proof. It is routinely verified that P_2^1 and J satisfy the identities (10.1), so that the inclusion $\mathbf{P}_2^1 \vee \mathbf{J} \subseteq \mathbf{X}_{(10.1)}$ holds. Hence $\mathbf{P}_2^1 \vee \mathbf{J} = \mathbf{X}_{(10.1)}\Sigma$ for some set Σ of identities. Let $\mathbf{w} \approx \mathbf{w}'$ be any identity from Σ . Since the subsemigroups $\{1, ba, b^2\}$ and $\{1, b, b^2\}$ of P_2^1 are isomorphic to L_2^1 and N_2^1 , respectively, it follows from Lemma 2.1 parts (iii) and (v) that

(a) $\text{ini}(\mathbf{w}) = \text{ini}(\mathbf{w}')$ and $\text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$.

By Lemma 2.2 and since J satisfies the identity $\mathbf{w} \approx \mathbf{w}'$, one of the following holds:

(b) $t(\mathbf{w}) = t(\mathbf{w}') \in \text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$;

(c) $t(\mathbf{w}) \notin \text{sim}(\mathbf{w})$ and $t(\mathbf{w}') \notin \text{sim}(\mathbf{w}')$.

If (c) holds, then it follows from Lemma 10.6 that the identity $\mathbf{w} \approx \mathbf{w}'$ in Σ can be replaced by one that satisfies (b) without changing the definition of $\mathbf{X}_{(10.1)}\Sigma$. Therefore the identity $\mathbf{w} \approx \mathbf{w}'$ can be assumed to satisfy both (a) and (b). By Lemma 10.5, the words \mathbf{w} and \mathbf{w}' can be chosen to be in canonical form. It then follows from (a) and (b) that

$$\mathbf{w} = \left(\prod_{i=1}^m (x_i \mathbf{w}_i) \right) t \quad \text{and} \quad \mathbf{w}' = \left(\prod_{i=1}^m (x_i \mathbf{w}'_i) \right) t,$$

where $t = t(\mathbf{w}) = t(\mathbf{w}')$. Let ℓ be the least integer such that $\text{con}(\mathbf{w}_\ell) \neq \text{con}(\mathbf{w}'_\ell)$, say $k \leq \ell$ is such that $x_k \in \text{con}(\mathbf{w}_\ell) \setminus \text{con}(\mathbf{w}'_\ell)$. Then $\text{occ}(x_k, \mathbf{w}) = 2$ by (IV), and \mathbf{w} is of the form

$$\mathbf{w} = \dots x_k \mathbf{w}_k \dots x_\ell \underbrace{(\dots x_k \dots)}_{\mathbf{w}_\ell} \dots$$

It follows from (a) and (IV) that $\text{occ}(x_k, \mathbf{w}') = 2$. But since $\text{con}(\mathbf{w}_i) = \text{con}(\mathbf{w}'_i)$ for all $i < \ell$, the second x_k in \mathbf{w}' occurs in \mathbf{w}'_r for some $r > \ell$, whence \mathbf{w}' is of the form

$$\mathbf{w}' = \dots x_k \mathbf{w}'_k \dots x_\ell \mathbf{w}'_\ell \dots x_r \underbrace{(\dots x_k \dots)}_{\mathbf{w}'_r} \dots$$

Let $\varphi : \mathcal{X} \rightarrow P_2^1$ denote the substitution

$$z \mapsto \begin{cases} b & \text{if } z = x_k, \\ a & \text{if } z = x_r, \\ 1 & \text{otherwise.} \end{cases}$$

Then $\mathbf{w}\varphi = b^2$ and $\mathbf{w}'\varphi = ba$, which is impossible. Therefore the integer ℓ does not exist, whence $\text{con}(\mathbf{w}_i) = \text{con}(\mathbf{w}'_i)$ for all i . By (II), the identity $\mathbf{w} \approx \mathbf{w}'$ is trivial and so is satisfied by $\mathbf{X}_{(10.1)}$. Since $\mathbf{w} \approx \mathbf{w}'$ is arbitrary in Σ , it follows that $\mathbf{P}_2^1 \vee \mathbf{J} = \mathbf{X}_{(10.1)}\Sigma = \mathbf{X}_{(10.1)}$. \square

10.4. Proof of Proposition 10.1

Let S be any semigroup in $\mathbf{X}_{(10.1)}$ that does not satisfy the identity (10.2). By Proposition 10.7, the variety \mathbf{V} generated by S is a subvariety of $\mathbf{P}_2^1 \vee \mathbf{J}$. Since S is finitely based if $\mathbf{V} = \mathbf{P}_2^1 \vee \mathbf{J}$, it suffices to assume $\mathbf{V} \neq \mathbf{P}_2^1 \vee \mathbf{J}$, so that either $P_2^1 \notin \mathbf{V}$ or $J \notin \mathbf{V}$. If $P_2^1 \notin \mathbf{V}$, then it follows from Lemma 10.3 that S satisfies the identity (10.2), contradicting the assumption. Hence $P_2^1 \in \mathbf{V}$ and $J \notin \mathbf{V}$. By Lemma 10.4, the variety \mathbf{V} satisfies the identity (10.3). It then follows from Lemma 10.2 that $\mathbf{V} = \mathbf{P}_2^1$, whence \mathbf{V} is finitely based.

PART III. FINITE BASIS PROPERTY FOR THE SPORADIC CASES

11. $\mathcal{A}_1, \mathcal{B}_1, \mathcal{B}_3, \mathcal{D}_{10}$

This section establishes the finite basis property of the following semigroups.

\mathcal{A}_1	1 2 3 4 5 6	\mathcal{B}_1	1 2 3 4 5 6	\mathcal{B}_3	1 2 3 4 5 6	\mathcal{D}_{10}	1 2 3 4 5 6
1	1 1 1 1 1 1	1	1 1 1 1 1 1	1	1 1 1 1 1 1	1	1 1 1 1 1 1
2	1 1 1 1 1 1	2	1 1 1 1 1 1	2	1 1 1 1 1 2	2	1 1 1 1 1 2
3	1 1 1 1 1 3	3	1 2 3 4 5 6	3	1 1 1 1 1 2	3	1 1 2 1 2 3
4	1 1 1 3 1 4	4	4 4 4 4 4 4	4	1 1 1 2 1 4	4	4 4 4 4 4 4
5	5 5 5 5 5 5	5	4 4 5 4 4 4	5	5 5 5 5 5 5	5	1 1 2 1 2 3
6	1 2 3 4 5 6	6	4 4 6 4 4 5	6	1 2 3 4 5 6	6	1 2 3 4 5 6

PROPOSITION 11.1. The variety $\mathbf{J} \vee \mathbf{L}_2^1 \vee \mathbf{N}_3^1$ is defined by the identities

$$x^3Hx \approx x^2Hx, \tag{11.1a}$$

$$xyxz \approx x^2yz, \tag{11.1b}$$

$$xHyKxy \approx xHyKyx. \tag{11.1c}$$

The proof of Proposition 11.1 is given in §11.2.

Let $S \in \{\mathcal{A}_1, \mathcal{B}_1, \mathcal{B}_3, \mathcal{D}_{10}\}$. By Proposition 11.1, it is routinely verified that $S \in \mathbf{J} \vee \mathbf{L}_2^1 \vee \mathbf{N}_3^1$. Conversely, the following are isomorphic to J, L_2^1 , and N_3^1 :

- the subsemigroups $\{1, 2, 6\}, \{1, 5, 6\}$, and $\{1, 3, 4, 6\}$ of \mathcal{A}_1 ;
- the subsemigroups $\{1, 2, 3\}, \{1, 3, 4\}$, and $\{3, 4, 5, 6\}$ of \mathcal{B}_1 ;
- the divisor $\{1, 2, 3, 6\}/\{1, 2\}$ and the subsemigroups $\{1, 5, 6\}$ and $\{1, 2, 4, 6\}$ of \mathcal{B}_3 ;
- the divisor $\{1, 2, 3, 5, 6\}/\{1, 2, 3\}$ and the subsemigroups $\{1, 4, 6\}$ and $\{1, 2, 3, 6\}$ of \mathcal{D}_{10} .

Hence the variety generated by S contains J, L_2^1 , and N_3^1 , and so coincides with $\mathbf{J} \vee \mathbf{L}_2^1 \vee \mathbf{N}_3^1$. Consequently, S is finitely based by Proposition 11.1.

11.1. Compact and pseudo-compact words

A compact word is a word of the form $x_1^{e_1} \dots x_m^{e_m}$, where $x_1, \dots, x_m \in \mathcal{X}$ are distinct and $e_1, \dots, e_m \in \{1, 2, 3\}$. A pseudo-compact word is a word of the form $\mathbf{c}x^e\mathbf{s}x$, where the following are satisfied:

- (I) $\mathbf{c} \in \mathcal{X}^*$ is a compact word;
- (II) $x \in \mathcal{X}$ with $e \in \{1, 2\}$;

- (III) $\mathbf{s} \in \mathcal{X}^+$ is a simple word;
 (IV) \mathbf{c} , x , and \mathbf{s} are pairwise disjoint.

Note that when the very last letter is removed from a pseudo-compact word $\mathbf{c}x^e\mathbf{s}x$, then the resulting word $\mathbf{c}x^e\mathbf{s}$ is compact.

LEMMA 11.2. *Let \mathbf{w} be any word. Then there exists some word $\bar{\mathbf{w}}$ that is compact or pseudo-compact such that the identities (11.1) imply the identity $\mathbf{w} \approx \bar{\mathbf{w}}$.*

Proof. Generality is not lost by assuming $\text{ini}(\mathbf{w}) = x_1 \dots x_m$. It suffices to convert \mathbf{w} , using the identities (11.1), into a word that is either compact or pseudo-compact. With the exception of the very last letter of \mathbf{w} , any non-first occurrence of a letter x in \mathbf{w} can be gathered, using the identity (11.1b), with the first occurrence of x in \mathbf{w} . The resulting word \mathbf{w}' is of one of the following forms:

- (a) $x_1^{e_1} \dots x_m^{e_m}$ for some $e_1, \dots, e_m \geq 1$;
 (b) $x_1^{e_1} \dots x_m^{e_m} x_k$ for some $k < m$ and $e_1, \dots, e_m \geq 1$.

If \mathbf{w}' is of the form (a), then the identity $x^4 \approx x^3$ from (11.1a) can be used to convert it into a compact word. Thus assume that \mathbf{w}' is of the form (b), that is,

$$\mathbf{w}' = \mathbf{c} \cdot x_k^{e_k} \cdot x_{k+1}^{e_{k+1}} \dots x_m^{e_m} \cdot x_k$$

where $\mathbf{c} = x_1^{e_1} \dots x_{k-1}^{e_{k-1}}$. If $e_{k+1} = \dots = e_m = 1$, then it is clear that the identities (11.1a) can be used to convert \mathbf{w}' into a pseudo-compact word. Hence assume that $\ell \in \{k+1, \dots, m\}$ is the greatest such that $e_\ell \geq 2$. Then

$$\begin{aligned} \mathbf{w}' &= \mathbf{c} \cdot x_k^{e_k} \cdot x_{k+1}^{e_{k+1}} \dots x_\ell^{e_\ell} x_{\ell+1} \dots x_m \cdot x_k \\ (11.1b) \quad &\approx \mathbf{c} \cdot x_k^{e_k} \cdot x_{k+1}^{e_{k+1}} \dots x_\ell^{e_\ell-1} x_{\ell+1} \dots x_m \cdot x_\ell x_k \\ (11.1c) \quad &\approx \mathbf{c} \cdot x_k^{e_k} \cdot x_{k+1}^{e_{k+1}} \dots x_\ell^{e_\ell-1} x_{\ell+1} \dots x_m \cdot x_k x_\ell \\ (11.1b) \quad &\approx \underbrace{\mathbf{c} \cdot x_k^{e_k+1} \cdot x_{k+1}^{e_{k+1}} \dots x_\ell^{e_\ell-1} x_{\ell+1} \dots x_m \cdot x_\ell}_{\mathbf{w}''}, \end{aligned}$$

and the identities (11.1a) can be used to convert \mathbf{w}'' into a pseudo-compact word. \square

11.2. Proof of Proposition 11.1

It is routinely checked that J , L_2^1 , and N_3^1 satisfy the identities (11.1). Thus it remains to show that any identity $\mathbf{w} \approx \mathbf{w}'$ satisfied by J , L_2^1 , and N_3^1 is deducible from the identities (11.1). By Lemma 11.2, the words \mathbf{w} and \mathbf{w}' can be chosen to be compact or pseudo-compact. Hence the following holds by Lemma 2.1 parts (iii) and (vii):

- (a) $\text{ini}(\mathbf{w}) = \text{ini}(\mathbf{w}')$;
 (b) for each $x \in \mathcal{X}$, either $\text{occ}(x, \mathbf{w}) = \text{occ}(x, \mathbf{w}') \leq 2$ or $\text{occ}(x, \mathbf{w}), \text{occ}(x, \mathbf{w}') \geq 3$.

It is then easily shown that if \mathbf{w} and \mathbf{w}' are both compact, then they are identical. Therefore it suffices to assume that \mathbf{w} is pseudo-compact, say

$$\mathbf{w} = x_1^{e_1} \dots x_m^{e_m} x^e y_1 \dots y_n x$$

for some distinct letters $x_1, \dots, x_m, x, y_1, \dots, y_n$ with $m \geq 0$ and $n \geq 1$, and some exponents $e_1, \dots, e_m \in \{1, 2, 3\}$ and $e \in \{1, 2\}$.

Suppose that \mathbf{w}' is compact, so that by (a),

$$\mathbf{w}' = x_1^{e'_1} \dots x_m^{e'_m} x^{e'} y_1^{f_1} \dots y_n^{f_n}$$

for some $e'_1, \dots, e'_m, e', f_1, \dots, f_n \in \{1, 2, 3\}$. Then $f_1 = \dots = f_n = 1$ by (b). But now \mathbf{w} ends with the non-simple letter x while \mathbf{w}' ends with the simple letter y_n , contradicting Lemma 2.2. Therefore \mathbf{w}' must be pseudo-compact, so that by (a),

$$\mathbf{w}' = x_1^{e'_1} \dots x_m^{e'_m} x^{e'} y_1^{f_1} \dots y_n^{f_n} z$$

for some $z \in \{x_1, \dots, x_m, x, y_1, \dots, y_n\}$. Since $\text{occ}(y_1, \mathbf{w}) = \dots = \text{occ}(y_n, \mathbf{w}) = 1$, it follows from (b) that $f_1 = \dots = f_n = 1$ and $z \notin \{y_1, \dots, y_n\}$. Now x is non-simple in \mathbf{w} . Therefore, if $z = x_i$ for some i , then the definition of a pseudo-compact word forces the letter x to be simple in \mathbf{w}' , contradicting (b). Hence $z = x$ is the only possibility. It is then easily shown by (b) that the words \mathbf{w} and \mathbf{w}' are identical.

12. $\mathcal{A}_2, \mathcal{B}_7, \mathcal{B}_8$

This section establishes the finite basis property of the following semigroups.

\mathcal{A}_2	1 2 3 4 5 6	\mathcal{B}_7	1 2 3 4 5 6	\mathcal{B}_8	1 2 3 4 5 6
1	1 1 1 1 1 1	1	1 1 1 1 1 1	1	1 1 1 1 1 1
2	1 1 1 1 1 1	2	1 1 1 1 1 2	2	1 1 1 1 1 2
3	1 1 1 1 2 3	3	1 1 1 1 1 3	3	1 1 1 1 1 3
4	4 4 4 4 4 4	4	1 1 1 2 1 4	4	1 1 1 2 1 4
5	1 1 1 1 2 3	5	1 1 3 1 5 1	5	1 1 3 1 5 3
6	1 2 3 4 5 6	6	1 2 1 4 1 6	6	1 2 1 4 1 6

12.1. Finite basis property of \mathcal{A}_2

PROPOSITION 12.1. The variety generated by \mathcal{A}_2 is defined by the identities

$$x^3y \approx x^2y, \tag{12.1a}$$

$$xyxz \approx x^2yz, \tag{12.1b}$$

$$x^2Hy^2Kx \approx x^2Hy^2Ky. \tag{12.1c}$$

LEMMA 12.2. Suppose that $\mathbf{w} \approx \mathbf{w}'$ is any identity satisfied by \mathcal{A}_2 . Then:

- (i) $\text{ini}(\mathbf{w}) = \text{ini}(\mathbf{w}')$;
- (ii) $\text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$;
- (iii) either $|\mathbf{w}|, |\mathbf{w}'| \geq 3$ or $\text{occ}(x, \mathbf{w}) = \text{occ}(x, \mathbf{w}')$ for any $x \in \mathcal{X}$;
- (iv) $\text{t}(\mathbf{w}) \in \text{sim}(\mathbf{w})$ if and only if $\text{t}(\mathbf{w}') \in \text{sim}(\mathbf{w}')$.

Proof. This follows from Lemmas 2.1 and 2.2 since the subsemigroups $\{1, 4, 6\}$, $\{1, 3, 6\}$, $\{1, 2, 5\}$, and $\{1, 2, 6\}$ of \mathcal{A}_2 are isomorphic to L_2^1 , N_2^1 , N_3 , and J , respectively. \square

In this subsection, a word \mathbf{w} that contains at least two distinct letters is said to be in canonical form if

$$\mathbf{w} = x_1^{e_1} \dots x_m^{e_m} y, \tag{12.2}$$

where $x_1, \dots, x_m, y \in \mathcal{X}$ and $e_1, \dots, e_m \in \{1, 2\}$ satisfy the following:

- (I) the letters x_1, \dots, x_m are distinct while y can be any letter;
- (II) if $y = x_i$ and $e_i = 2$ for some $i \geq 2$, then $e_1 = \dots = e_{i-1} = 1$.

LEMMA 12.3. Let \mathbf{w} be any word that contains at least two distinct letters. Then there exists some word $\bar{\mathbf{w}}$ in canonical form such that the identities (12.1) imply the identity $\mathbf{w} \approx \bar{\mathbf{w}}$.

Proof. It suffices to convert \mathbf{w} , using the identities (12.1), into a word in canonical form. Since \mathbf{w} contains at least two distinct letters, $\mathbf{w} = \mathbf{w}'y$ for some $\mathbf{w}' \in \mathcal{X}^+$ and $y \in \mathcal{X}$. Then the identities $\{(12.1a), (12.1b)\}$ can be used to convert \mathbf{w} into a word of the form (12.2) with (I) satisfied. Suppose $y = x_i$ and $e_i = 2$ for some $i \geq 2$. Then $\mathbf{w} = x_1^{e_1} \dots x_{i-1}^{e_{i-1}} x_i^2 x_{i+1}^{e_{i+1}} \dots x_m^{e_m} x_i$. If $e_1 = \dots = e_{i-1} = 1$, then \mathbf{w} satisfies (II). Therefore suppose $e_\ell = 2$ for some least integer $\ell \in \{1, \dots, i - 1\}$. Then

$$\begin{aligned} \mathbf{w} &= x_1 \dots x_{\ell-1} x_\ell^2 x_{\ell+1}^{e_{\ell+1}} \dots x_{i-1}^{e_{i-1}} x_i^2 x_{i+1}^{e_{i+1}} \dots x_m^{e_m} x_i \\ &\stackrel{(12.1c)}{\approx} x_1 \dots x_{\ell-1} x_\ell^2 x_{\ell+1}^{e_{\ell+1}} \dots x_{i-1}^{e_{i-1}} x_i^2 x_{i+1}^{e_{i+1}} \dots x_m^{e_m} x_\ell, \end{aligned}$$

so that (II) is satisfied. □

Proof of Proposition 12.1. It is routinely checked that \mathcal{A}_2 satisfies the identities (12.1). Therefore it suffices to show that any identity $\mathbf{w} \approx \mathbf{w}'$ satisfied by \mathcal{A}_2 is implied by the identities (12.1). If either \mathbf{w} or \mathbf{w}' involves only one letter, say $\text{con}(\mathbf{w}) = \{x\}$, then it is easily shown by Lemma 12.2 parts (i) and (iii) that the identity $\mathbf{w} \approx \mathbf{w}'$ is implied by the identity (12.1a). Hence assume that \mathbf{w} and \mathbf{w}' each contains at least two distinct letters and so, by Lemma 12.3, can be chosen to be in canonical form. Since $\text{ini}(\mathbf{w}) = \text{ini}(\mathbf{w}')$ by Lemma 12.2(i),

$$\mathbf{w} = x_1^{e_1} \dots x_m^{e_m} y \quad \text{and} \quad \mathbf{w}' = x_1^{e'_1} \dots x_m^{e'_m} y'.$$

Case 1: $y \in \text{sim}(\mathbf{w})$. Then $y' \in \text{sim}(\mathbf{w}')$ by Lemma 12.2(iv), and $y = y'$ by part (i) of the same lemma. Therefore $y, y' \notin \{x_1, \dots, x_m\}$, whence $e_i = \text{occ}(x_i, \mathbf{w})$ and $e'_i = \text{occ}(x_i, \mathbf{w}')$ for all i . Now Lemma 12.2(ii) implies $e_i = e'_i$ for all i . Hence the identity $\mathbf{w} \approx \mathbf{w}'$ is trivial and is implied by the identities (12.1).

Case 2: $y \notin \text{sim}(\mathbf{w})$. Then $y' \notin \text{sim}(\mathbf{w}')$ by Lemma 12.2(iv). Seeking a contradiction, suppose $y \neq y'$. By symmetry, it suffices to assume $y = x_i$ and $y' = x_j$ with $i < j$. Then

$$\mathbf{w} = x_1^{e_1} \dots x_{i-1}^{e_{i-1}} \cdot x_i^{e_i} \cdot x_{i+1}^{e_{i+1}} \dots x_{j-1}^{e_{j-1}} \cdot x_j^{e_j} \cdot x_{j+1}^{e_{j+1}} \dots x_m^{e_m} \cdot x_i$$

and

$$\mathbf{w}' = x_1^{e'_1} \dots x_{i-1}^{e'_{i-1}} \cdot x_i^{e'_i} \cdot x_{i+1}^{e'_{i+1}} \dots x_{j-1}^{e'_{j-1}} \cdot x_j^{e'_j} \cdot x_{j+1}^{e'_{j+1}} \dots x_m^{e'_m} \cdot x_j.$$

Since $x_i \notin \text{sim}(\mathbf{w}')$, it follows from Lemma 12.2(ii) that $x_i \notin \text{sim}(\mathbf{w}')$. Therefore $e'_i = 2$, whence $e'_j = 1$ by (II). Since $x_j \notin \text{sim}(\mathbf{w})$, it follows from Lemma 12.2(ii) that $x_j \notin \text{sim}(\mathbf{w})$, whence $e_j = 2$. Thus

$$\mathbf{w} = x_1^{e_1} \dots x_{i-1}^{e_{i-1}} \cdot x_i^{e_i} \cdot x_{i+1}^{e_{i+1}} \dots x_{j-1}^{e_{j-1}} \cdot x_j^2 \cdot x_{j+1}^{e_{j+1}} \dots x_m^{e_m} \cdot x_i$$

and

$$\mathbf{w}' = x_1^{e'_1} \dots x_{i-1}^{e'_{i-1}} \cdot x_i^2 \cdot x_{i+1}^{e'_{i+1}} \dots x_{j-1}^{e'_{j-1}} \cdot x_j \cdot x_{j+1}^{e'_{j+1}} \dots x_m^{e'_m} \cdot x_j.$$

Let $\varphi : \mathcal{X} \rightarrow \mathcal{A}_2$ denote the substitution

$$z \mapsto \begin{cases} 5 & \text{if } z = x_j, \\ 6 & \text{otherwise.} \end{cases}$$

Then $\mathbf{w}\varphi \in \{6\}^* \cdot 5^2 \cdot \{6\}^+ = \{1\}$ and $\mathbf{w}'\varphi \in \{6\}^* \cdot 5 \cdot \{6\}^* \cdot 5 = \{2\}$, which is the required contradiction.

Therefore $y = y'$ is the only possibility, whence $y = y' = x_j$ for some j , so that

$$\mathbf{w} = x_1^{e_1} \dots x_{j-1}^{e_{j-1}} \cdot x_j^{e_j} \cdot x_{j+1}^{e_{j+1}} \dots x_m^{e_m} \cdot x_j$$

and

$$\mathbf{w}' = x_1^{e'_1} \dots x_{j-1}^{e'_{j-1}} \cdot x_j^{e'_j} \cdot x_{j+1}^{e'_{j+1}} \dots x_m^{e'_m} \cdot x_j.$$

It follows from Lemma 12.2(ii) that $(e_1, \dots, e_{j-1}, e_{j+1}, \dots, e_m) = (e'_1, \dots, e'_{j-1}, e'_{j+1}, \dots, e'_m)$. If $e_j \neq e'_j$, say $(e_j, e'_j) = (2, 1)$, then

$$\mathbf{w}\varphi \in \{6\}^* \cdot 5^2 \cdot \{6\}^* \cdot 5 = \{1\} \quad \text{and} \quad \mathbf{w}'\varphi \in \{6\}^* \cdot 5 \cdot \{6\}^* \cdot 5 = \{2\},$$

which is impossible. Therefore $e_j = e'_j$, whence the identity $\mathbf{w} \approx \mathbf{w}'$ is trivial and is implied by the identities (12.1). □

12.2. Finite basis property of \mathcal{B}_8

PROPOSITION 12.4. *The variety generated by \mathcal{B}_8 is defined by the identities*

$$x^3\mathbf{H}x \approx x^2\mathbf{H}x, \tag{12.3a}$$

$$x\mathbf{H}y\mathbf{z}\mathbf{K}x \approx x\mathbf{H}\mathbf{z}y\mathbf{K}x, \tag{12.3b}$$

$$x\mathbf{H}y\mathbf{K}x\mathbf{y} \approx x\mathbf{H}y\mathbf{K}y\mathbf{x}, \quad x\mathbf{y}\mathbf{H}x\mathbf{K}y \approx y\mathbf{x}\mathbf{H}x\mathbf{K}y. \tag{12.3c}$$

LEMMA 12.5. *Let $S \in \{\mathcal{B}_7, \mathcal{B}_8\}$. Suppose that $\mathbf{w} \approx \mathbf{w}'$ is any identity satisfied by S . Then for any $x \in \mathcal{X}$, either $\text{occ}(x, \mathbf{w}) = \text{occ}(x, \mathbf{w}') \leq 2$ or $\text{occ}(x, \mathbf{w}), \text{occ}(x, \mathbf{w}') \geq 3$.*

Proof. This follows from Lemma 2.1(vii) since the subsemigroup $\{1, 2, 4, 6\}$ of S is isomorphic to N_3^1 . □

For the remainder of this section, a non-simple word \mathbf{w} is said to be in *canonical form* if

$$\mathbf{w} = x_0^{e_0} x_1^{e_1} \dots x_m^{e_m} y_1 \dots y_n x_0,$$

where all of the following are satisfied:

- (I) x_0, x_1, \dots, x_m are distinct non-simple letters of \mathbf{w} in alphabetical order;
 - (II) $e_0 \in \{1, 2\}$ and $e_1, \dots, e_m \in \{2, 3\}$;
 - (III) y_1, \dots, y_n are distinct simple letters of \mathbf{w} in alphabetical order.
- Note that if all letters of the word \mathbf{w} are non-simple, then $n = 0$ and $y_1 \dots y_n = \emptyset$.

LEMMA 12.6. *Let \mathbf{w} be any non-simple word with $h(\mathbf{w}) = t(\mathbf{w})$. Then there exists some word $\bar{\mathbf{w}}$ in canonical form such that the identities (12.3) imply the identity $\mathbf{w} \approx \bar{\mathbf{w}}$.*

Proof. It suffices to convert \mathbf{w} , using the identities (12.3), into a word in canonical form. By assumption, $\mathbf{w} = x\mathbf{w}'x$ for some $\mathbf{w}' \in \mathcal{X}^*$. Suppose that x, x_1, \dots, x_m are the non-simple letters of \mathbf{w} and y_1, \dots, y_n are the simple letters of \mathbf{w} . Generality is not lost by assuming that x_1, \dots, x_m are in alphabetical order and y_1, \dots, y_n are in alphabetical order. Since the letters in the factor \mathbf{w}' of \mathbf{w} are sandwiched between two occurrences of x , they can be rearranged by the identities (12.3b) to produce the word $x^e x_1^{e_1} \dots x_m^{e_m} y_1 \dots y_n x$, where $e + 1 = \text{occ}(x, \mathbf{w}) \geq 2$ and $e_i = \text{occ}(x_i, \mathbf{w}) \geq 2$. The identities (12.3a) can then be used to reduce the exponents, so that $e \in \{1, 2\}$ and $e_1, \dots, e_m \in \{2, 3\}$. The resulting word is in canonical form if x alphabetically precedes x_1 . If x does not alphabetically precede x_1 , then

$$\begin{aligned} x^e x_1^{e_1} x_2^{e_2} \dots x_m^{e_m} y_1 \dots y_n x &\stackrel{(12.3b)}{\approx} x^e x_1^{e_1-1} x_2^{e_2} \dots x_m^{e_m} y_1 \dots y_n x_1 x \\ &\stackrel{(12.3c)}{\approx} x_1^{e_1-1} x^e x_2^{e_2} \dots x_m^{e_m} y_1 \dots y_n x x_1 \\ &\stackrel{(12.3b)}{\approx} x_1^{e_1-1} x^{e+1} x_2^{e_2} \dots x_m^{e_m} y_1 \dots y_n x_1, \end{aligned}$$

and the factors $x^{e+1}, x_2^{e_2}, \dots, x_m^{e_m}$ in the latter word can be alphabetically ordered by the identities (12.3b), resulting in a word in canonical form. \square

LEMMA 12.7. *Let \mathbf{w} be any connected word. Then there exists some word $\bar{\mathbf{w}}$ in canonical form such that the identities (12.3) imply the identity $\mathbf{w} \approx \bar{\mathbf{w}}$.*

Proof. It suffices to convert \mathbf{w} , using the identities (12.3), into a word in canonical form. Since \mathbf{w} is connected, the letter $x = h(\mathbf{w})$ occurs at least twice in \mathbf{w} . Hence $\mathbf{w} = x\mathbf{u}x\mathbf{v}$ for some $\mathbf{u}, \mathbf{v} \in \mathcal{X}^*$ such that $x \notin \text{con}(\mathbf{v})$. If $\mathbf{v} = \emptyset$, then the result follows from Lemma 12.6. Therefore assume $\mathbf{v} \neq \emptyset$, whence $\mathbf{w} = x\mathbf{u}x\mathbf{y}\mathbf{v}'$ for some $y \in \mathcal{X}$ and $\mathbf{v}' \in \mathcal{X}^*$. There are two cases.

Case 1: $y \notin \text{con}(\mathbf{u})$. Since \mathbf{w} is connected, the factors $x\mathbf{u}x$ and $y\mathbf{v}'$ cannot be disjoint. It follows that \mathbf{u} and \mathbf{v}' share some common letter, whence $\mathbf{w} = x\mathbf{u}x\mathbf{y}\mathbf{v}' \stackrel{(12.3b)}{\approx} x\mathbf{u}y\mathbf{x}\mathbf{v}'$.

Case 2: $y \in \text{con}(\mathbf{u})$. Then clearly $\mathbf{w} = x\mathbf{u}x\mathbf{y}\mathbf{v}' \stackrel{(12.3c)}{\approx} x\mathbf{u}y\mathbf{x}\mathbf{v}'$.

In any case, the identities (12.3) can be used to interchange the last x in \mathbf{w} with the letter y that immediately follows it. It is easily seen that the resulting word $\mathbf{w}' = x\mathbf{u}y\mathbf{x}\mathbf{v}'$ is connected. The same procedure can be repeated to move the last x in \mathbf{w}' to the right until it becomes the last letter of the word. Hence $\mathbf{w} \stackrel{(12.3)}{\approx} x\mathbf{u}\mathbf{v}x$ and the result now holds by Lemma 12.6. \square

Proof of Proposition 12.4. It is easily verified that \mathcal{B}_8 satisfies the identities (12.3) and is idempotent-separable. Since the subsemigroup $\{1, 3, 5, 6\}$ of \mathcal{B}_8 is isomorphic to A_0 , the variety generated by \mathcal{B}_8 contains A_0 . It follows from Lemma 2.6(i) that \mathcal{B}_8 has a basis Σ that consists of identities formed by connected words. Hence $\{(12.3)\} \cup \Sigma$ is also a basis for \mathcal{B}_8 . By Lemma 12.7, the words that form the identities in Σ can be chosen to be in canonical form. Let $\mathbf{w} \approx \mathbf{w}'$ be any identity from Σ . Then

$$\mathbf{w} = x_0^{e_0} x_1^{e_1} \dots x_m^{e_m} y_1 \dots y_n x_0 \quad \text{and} \quad \mathbf{w}' = z_0^{f_0} z_1^{f_1} \dots z_p^{f_p} t_1 \dots t_q z_0$$

where $e_0, f_0 \in \{1, 2\}$ and $e_1, \dots, e_m, f_1, \dots, f_p \in \{2, 3\}$. By Lemma 12.5:

- (a) \mathbf{w} and \mathbf{w}' share the same simple letters, so that $(y_1, \dots, y_n) = (t_1, \dots, t_q)$ by (III);
 - (b) \mathbf{w} and \mathbf{w}' share the same non-simple letters, so that $(x_0, \dots, x_m) = (z_1, \dots, z_p)$ by (I).
- It then follows from (II) and Lemma 12.5 that $e_i = f_i$ for all i , whence the identity $\mathbf{w} \approx \mathbf{w}'$ is trivial. Since $\mathbf{w} \approx \mathbf{w}'$ is arbitrary in Σ , every identity in Σ is trivial. Consequently, the identities (12.3) form a basis for \mathcal{B}_8 . \square

12.3. Finite basis property of \mathcal{B}_7

PROPOSITION 12.8. *The variety generated by \mathcal{B}_7 is defined by the identities (12.3) and*

$$xHxyKy \approx xHyxKy. \quad (12.4)$$

LEMMA 12.9. *Let \mathbf{w} be any product of pairwise disjoint connected words. Then there exists some word $\bar{\mathbf{w}}$ in canonical form such that the identities $\{(12.3), (12.4)\}$ imply the identity $\mathbf{w} \approx \bar{\mathbf{w}}$.*

Proof. By assumption, $\mathbf{w} = \mathbf{w}_1 \dots \mathbf{w}_k$ for some connected words $\mathbf{w}_1, \dots, \mathbf{w}_k$ that are pairwise disjoint. It suffices to convert \mathbf{w} , using the identities $\{(12.3), (12.4)\}$, into a word in canonical form. By Lemma 12.7, the identities (12.3) can be used to convert each \mathbf{w}_i into

a word in canonical form. Therefore $\mathbf{w}_i \stackrel{(12.3)}{\approx} x_i \mathbf{w}'_i x_i$ for some $x_i \in \mathcal{X}$ and $\mathbf{w}'_i \in \mathcal{X}^*$. Now

$$\begin{aligned} \mathbf{w} &\stackrel{(12.3)}{\approx} x_1 \mathbf{w}'_1 x_1 \cdot x_2 \mathbf{w}'_2 x_2 \cdot x_3 \mathbf{w}'_3 x_3 \dots x_{k-1} \mathbf{w}'_{k-1} x_{k-1} \cdot x_k \mathbf{w}'_k x_k \\ &\stackrel{(12.4)}{\approx} \underbrace{x_1 \mathbf{w}'_1 x_2 \cdot x_1 \mathbf{w}'_2 x_3 \cdot x_2 \mathbf{w}'_3 x_4 \dots x_{k-2} \mathbf{w}'_{k-1} x_k \cdot x_{k-1} \mathbf{w}'_k x_k}_{\mathbf{w}'}, \end{aligned}$$

where the word \mathbf{w}' is connected. Hence by Lemma 12.7, the identities (12.3) can be used to convert \mathbf{w}' into a word in canonical form. \square

Proof of Proposition 12.8. It is easily verified that the semigroup \mathcal{B}_7 satisfies the identities $\{(12.3), (12.4)\}$ and is idempotent-separable. Let \mathbf{V} denote the variety generated by \mathcal{B}_7 . Then $B_0 \in \mathbf{V}$ because the subsemigroup $\{1, 3, 5, 6\}$ of \mathcal{B}_7 is isomorphic to B_0 . The semigroup A_0 does not satisfy the identities (12.4) because $aabb \neq abab$ in A_0 . Therefore $A_0 \notin \mathbf{V}$, and it follows from Lemma 2.6(ii) that \mathcal{B}_7 has a basis Σ that consists of identities formed by words that are products of pairwise disjoint connected words. Hence $\{(12.3), (12.4)\} \cup \Sigma$ is also a basis for \mathcal{B}_7 . By Lemma 12.9, the words that form the identities in Σ can be chosen to be in canonical form. Following the arguments in the proof of Proposition 12.4, every identity in Σ is trivial. Consequently, the identities $\{(12.3), (12.4)\}$ form a basis for \mathcal{B}_7 . \square

13. $\mathcal{A}_3, \mathcal{A}_5, \mathcal{A}_6, \mathcal{A}_8, \mathcal{A}_9, \mathcal{A}_{10}, \mathcal{G}_4, \mathcal{G}_5$

This section establishes the finite basis property of the following semigroups.

\mathcal{A}_3	1 2 3 4 5 6	\mathcal{A}_5	1 2 3 4 5 6	\mathcal{A}_6	1 2 3 4 5 6	\mathcal{A}_8	1 2 3 4 5 6
1	1 1 1 1 1 1	1	1 1 1 1 1 1	1	1 1 1 1 1 1	1	1 1 1 1 1 1
2	1 1 1 1 1 1	2	1 1 1 1 1 1	2	1 1 1 1 1 1	2	1 1 1 1 1 1
3	1 1 3 1 5 6	3	1 1 3 3 3 3	3	1 1 3 3 5 6	3	1 2 3 3 3 3
4	1 2 1 4 1 1	4	1 1 4 4 4 4	4	1 2 3 4 5 6	4	1 2 3 4 5 6
5	5 5 5 5 5 5	5	1 2 3 4 5 6	5	5 5 5 5 5 5	5	1 2 5 5 5 5
6	5 5 6 5 1 3	6	1 2 4 3 6 5	6	5 5 6 6 1 3	6	1 2 5 6 3 4

\mathcal{A}_9	1 2 3 4 5 6	\mathcal{A}_{10}	1 2 3 4 5 6	\mathcal{G}_4	1 2 3 4 5 6	\mathcal{G}_5	1 2 3 4 5 6
1	1 1 1 1 1 1	1	1 1 1 1 1 1	1	1 1 1 1 1 1	1	1 1 1 1 1 1
2	1 1 1 1 1 1	2	1 1 1 1 1 1	2	1 2 2 2 5 6	2	1 2 2 4 4 6
3	1 2 3 4 5 6	3	1 2 3 4 5 6	3	1 2 2 2 5 6	3	1 2 3 4 5 6
4	1 2 4 3 6 5	4	4 4 4 4 4 4	4	1 2 3 4 5 6	4	6 4 4 2 2 1
5	5 5 5 5 5 5	5	4 4 4 4 4 4	5	5 5 5 5 5 5	5	6 4 4 2 2 1
6	6 6 6 6 6 6	6	4 5 6 1 2 3	6	5 6 6 6 1 2	6	6 6 6 6 6 6

Let \mathbf{O} denote the variety generated by the semigroup O in Remark 6.6.

PROPOSITION 13.1. *The variety $\mathbf{J} \vee \mathbf{O}$ is defined by the identities*

$$xHx^3 \approx xHx, \tag{13.1a}$$

$$xHx^2y \approx xHy, \tag{13.1b}$$

$$xHyKxy \approx xHyKyx. \tag{13.1c}$$

The proof of Proposition 13.1 is given in § 13.3.

Let $S \in \{\mathcal{A}_3, \mathcal{A}_5, \mathcal{A}_6, \mathcal{A}_8, \mathcal{A}_9, \mathcal{A}_{10}, \mathcal{G}_4, \mathcal{G}_5\}$. By Proposition 13.1, it is routinely verified that $S \in \mathbf{J} \vee \mathbf{O}$. Conversely, the following are isomorphic to J and O :

- the subsemigroups $\{1, 2, 4\}$ and $\{1, 3, 5, 6\}$ of $S' \in \{\mathcal{A}_3, \mathcal{A}_6\}$;
- the subsemigroups $\{1, 2, 5\}$ and $\{3, 4, 5, 6\}$ of \mathcal{A}_5 ;
- the subsemigroups $\{1, 2, 3\}$ and $\{3, 4, 5, 6\}$ of $S'' \in \{\mathcal{A}_8, \mathcal{A}_9\}$;
- the subsemigroups $\{1, 2, 3\}$ and $\{1, 3, 4, 6\}$ of \mathcal{A}_{10} ;
- the subsemigroups $\{2, 3, 4\}$ and $\{1, 2, 5, 6\}$ of \mathcal{G}_4 ;
- the divisor $\{2, 3, 4, 5\}/\{2, 4\}$ and the subsemigroup $\{1, 2, 4, 6\}$ of \mathcal{G}_5 .

Hence the variety generated by S contains J and O , and so coincides with $\mathbf{J} \vee \mathbf{O}$. Consequently, S is finitely based by Proposition 13.1.

13.1. Identities satisfied by O

LEMMA 13.2. *Let $\mathbf{w} \approx \mathbf{w}'$ be any identity satisfied by O . Then $\text{occ}(x, y, \mathbf{w}) \equiv \text{occ}(x, y, \mathbf{w}') \pmod{2}$ for any $x, y \in \mathcal{X}$.*

Proof. Seeking a contradiction, suppose $\text{occ}(x, y, \mathbf{w}) \not\equiv \text{occ}(x, y, \mathbf{w}') \pmod{2}$ for some letters $x, y \in \mathcal{X}$, say $\text{occ}(x, y, \mathbf{w}) = 2p$ and $\text{occ}(x, y, \mathbf{w}') = 2q + 1$ for some $p, q \geq 0$. Let $\varphi : \mathcal{X} \rightarrow O$ denote the substitution

$$z \mapsto \begin{cases} b & \text{if } z = x, \\ a & \text{if } z = y, \\ 1 & \text{otherwise.} \end{cases}$$

Then $\mathbf{w}\varphi = b^{2p} \cdot a \dots = a$ and $\mathbf{w}'\varphi = b^{2q+1} \cdot a \dots = ba$ give the required contradiction. \square

LEMMA 13.3. *Let $\mathbf{w} \approx \mathbf{w}'$ be any identity satisfied by O and let $t \in \mathcal{X} \setminus \text{con}(\mathbf{w}\mathbf{w}')$. Suppose $\mathbf{t}(\mathbf{w}) \notin \text{sim}(\mathbf{w})$ and $\mathbf{t}(\mathbf{w}') \notin \text{sim}(\mathbf{w}')$. Then $(\mathbf{J} \vee \mathbf{O})\{\mathbf{w} \approx \mathbf{w}'\} = (\mathbf{J} \vee \mathbf{O})\{\mathbf{w}t \approx \mathbf{w}'t\}$.*

Proof. Let σ_1 and σ_2 denote the identities $\mathbf{w} \approx \mathbf{w}'$ and $\mathbf{w}t \approx \mathbf{w}'t$, respectively. Then the inclusion $(\mathbf{J} \vee \mathbf{O})\{\sigma_1\} \subseteq (\mathbf{J} \vee \mathbf{O})\{\sigma_2\}$ is obvious. Let $x = \mathbf{t}(\mathbf{w})$ and $y = \mathbf{t}(\mathbf{w}')$. (Note that x and y need not be distinct.) Since $x \notin \text{sim}(\mathbf{w})$ and $y \notin \text{sim}(\mathbf{w}')$ by assumption, the deductions $\mathbf{w} \stackrel{(13.1a)}{\approx} \mathbf{w}x^2$ and $\mathbf{w}' \stackrel{(13.1a)}{\approx} \mathbf{w}'y^2$ hold. The subsemigroup $\{a, ba, 1\}$ of O is isomorphic to L_2^1 , so that $\text{ini}(\mathbf{w}) = \text{ini}(\mathbf{w}')$ by Lemma 2.1(iii). Therefore $y \in \text{con}(\mathbf{w})$ and $\mathbf{w}x^2 \stackrel{(13.1b)}{\approx} \mathbf{w}y^2x^2$. Since

$$\mathbf{w} \stackrel{(13.1a)}{\approx} \mathbf{w}x^2 \stackrel{(13.1b)}{\approx} \mathbf{w}y^2x^2 \stackrel{(13.1c)}{\approx} \mathbf{w}x^2y^2 \stackrel{(13.1a)}{\approx} \mathbf{w}y^2 \stackrel{(13.1a)}{\approx} \mathbf{w}'y^2 \stackrel{(13.1a)}{\approx} \mathbf{w}',$$

the identities $\{(13.1), \sigma_2\}$ imply σ_1 , so the inclusion $(\mathbf{J} \vee \mathbf{O})\{\sigma_2\} \subseteq (\mathbf{J} \vee \mathbf{O})\{\sigma_1\}$ follows. \square

13.2. A canonical form

In this section, any word \mathbf{w} with $t = \mathbf{t}(\mathbf{w}) \in \text{sim}(\mathbf{w})$ is said to be in *canonical form* if

$$\mathbf{w} = \left(\prod_{i=1}^m (x_i \mathbf{w}_i) \right) t, \tag{13.2}$$

where $x_1, \dots, x_m \in \mathcal{X}$ and $\mathbf{w}_1, \dots, \mathbf{w}_m \in \mathcal{X}^*$ are such that:

- (I) $\text{ini}(\mathbf{w}) = x_1 \dots x_m t$;
- (II) $\mathbf{w}_i \in \{x_1^{e_1} \dots x_i^{e_i} \mid e_1, \dots, e_i \in \{0, 1\}\}$.

LEMMA 13.4. *Let \mathbf{w} be any word with $t = \mathbf{t}(\mathbf{w}) \in \text{sim}(\mathbf{w})$. Then there exists some word $\bar{\mathbf{w}}$ in canonical form with $t = \mathbf{t}(\bar{\mathbf{w}}) \in \text{sim}(\bar{\mathbf{w}})$ such that the identities (13.1) imply the identity $\mathbf{w} \approx \bar{\mathbf{w}}$.*

Proof. The letter t is simple in \mathbf{w} , so that $\text{ini}(\mathbf{w}) = x_1 \dots x_m t$ for some $x_1, \dots, x_m \in \mathcal{X}$. It is clear that \mathbf{w} can be written in the form (13.2) with (I) satisfied and $\mathbf{w}_i \in \{x_1, \dots, x_i\}^*$ for all i . Since the letters in each \mathbf{w}_i are non-first occurrences in \mathbf{w} , the identities (13.1c) can be used to order them so that $\mathbf{w}_i \in \{x_1^{e_1} \dots x_i^{e_i} \mid e_1, \dots, e_i \geq 0\}$. The identities $\{(13.1a), (13.1b)\}$ can then be used to reduce each exponent e_i to a number in $\{0, 1\}$. Hence (II) is satisfied. \square

13.3. *Proof of Proposition 13.1*

Let $\mathbf{X}_{(13.1)}$ denote the variety defined by the identities (13.1). It is routinely checked that $J, O \in \mathbf{X}_{(13.1)}$, so that $\mathbf{J} \vee \mathbf{O} = \mathbf{X}_{(13.1)} \Sigma$ for some set Σ of identities. Let $\mathbf{w} \approx \mathbf{w}'$ be any identity in Σ . The subsemigroup $\{a, ba, 1\}$ of O is isomorphic to L_2^1 , hence by Lemma 2.1(iii),

(a) $\text{ini}(\mathbf{w}) = \text{ini}(\mathbf{w}')$.

Since J satisfies the identity $\mathbf{w} \approx \mathbf{w}'$, it follows from Lemma 2.2 that one of the following holds:

- (b) $\text{t}(\mathbf{w}) \in \text{sim}(\mathbf{w}), \text{t}(\mathbf{w}') \in \text{sim}(\mathbf{w}')$, and $\text{t}(\mathbf{w}) = \text{t}(\mathbf{w}')$;
- (c) $\text{t}(\mathbf{w}) \notin \text{sim}(\mathbf{w})$ and $\text{t}(\mathbf{w}') \notin \text{sim}(\mathbf{w}')$.

If (c) holds, then Lemma 13.3 implies that the identity $\mathbf{w} \approx \mathbf{w}'$ in Σ can be replaced by one that satisfies (b) without changing the definition of $\mathbf{X}_{(13.1)} \Sigma$. Therefore the identity $\mathbf{w} \approx \mathbf{w}'$ can be assumed to satisfy both (a) and (b). By Lemma 13.4, the words \mathbf{w} and \mathbf{w}' can be chosen to be in canonical form. It then follows from (a) and (b) that

$$\mathbf{w} = \left(\prod_{i=1}^m (x_i \mathbf{w}_i) \right) t \quad \text{and} \quad \mathbf{w}' = \left(\prod_{i=1}^m (x_i \mathbf{w}'_i) \right) t.$$

Let ℓ be the least integer such that $\text{con}(\mathbf{w}_\ell) \neq \text{con}(\mathbf{w}'_\ell)$, say $x_k \in \text{con}(\mathbf{w}_\ell) \setminus \text{con}(\mathbf{w}'_\ell)$ for some $k \leq \ell$. Then it follows from (II) that $\mathbf{w}_i = \mathbf{w}'_i$ for all $i < \ell$. But now Lemma 13.2 is violated since $\text{occ}(x_k, t, \mathbf{w}) \not\equiv \text{occ}(x_k, t, \mathbf{w}') \pmod{2}$ if $\ell = m$, and $\text{occ}(x_k, x_{\ell+1}, \mathbf{w}) \not\equiv \text{occ}(x_k, x_{\ell+1}, \mathbf{w}') \pmod{2}$ if $\ell < m$. Therefore the integer ℓ does not exist, whence $\mathbf{w}_i = \mathbf{w}'_i$ for all i . The identity $\mathbf{w} \approx \mathbf{w}'$ is thus trivial and is satisfied by $\mathbf{X}_{(13.1)}$. Since $\mathbf{w} \approx \mathbf{w}'$ is arbitrary in Σ , it follows that $\mathbf{X}_{(13.1)} = \mathbf{X}_{(13.1)} \Sigma = \mathbf{J} \vee \mathbf{O}$.

14. $\mathcal{A}_4, \mathcal{A}_7, \mathcal{B}_2, \mathcal{G}_3$

This section establishes the finite basis property of the following semigroups.

\mathcal{A}_4	1	2	3	4	5	6	\mathcal{A}_7	1	2	3	4	5	6	\mathcal{B}_2	1	2	3	4	5	6	\mathcal{G}_3	1	2	3	4	5	6
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
2	1	1	1	1	1	1	2	1	1	1	1	1	1	2	1	1	1	1	1	1	2	1	2	1	2	5	6
3	1	1	3	1	5	6	3	1	2	3	1	5	6	3	3	3	3	3	3	3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	3	3	3	4	5	6	4	1	2	1	2	5	6
5	1	1	3	6	5	6	5	1	2	3	6	5	6	5	5	5	5	5	5	5	5	1	2	6	2	5	6
6	6	6	6	6	6	6	6	6	6	6	6	6	6	6	5	5	3	4	5	6	6	6	6	6	6	6	6

14.1. *Finite basis property of \mathcal{A}_7*

PROPOSITION 14.1. *The variety generated by \mathcal{A}_7 is defined by the identities*

$$x^2y \approx xy, \tag{14.1a}$$

$$xyx^2 \approx xyx, \tag{14.1b}$$

$$xHyKxy \approx xHyKy, \tag{14.1c}$$

$$xHyKyx \approx xHyKx. \tag{14.1d}$$

Let \mathbf{w} be any word such that $\text{ini}(\mathbf{w}) = x_0 \dots x_m$ with $m \geq 1$. In this section, such a word \mathbf{w} is said to be in α -canonical form if

$$\mathbf{w} = x_0 \prod_{i=1}^m (x_i \mathbf{w}_i), \tag{14.2}$$

where the following are satisfied:

- (I) $\mathbf{w}_m \in \{\emptyset, x_0, \dots, x_m\}$;
- (II) $\mathbf{w}_i \in \{\emptyset, x_0, \dots, x_{i-1}\}$ for each i such that $1 \leq i < m$.

LEMMA 14.2. *Let \mathbf{w} be any word that contains at least two distinct letters. Then there exists some word $\bar{\mathbf{w}}$ in α -canonical form such that the identities (14.1) imply the identity $\mathbf{w} \approx \bar{\mathbf{w}}$.*

Proof. It suffices to convert \mathbf{w} , using the identities (14.1), into a word in α -canonical form. Since \mathbf{w} contains at least two distinct letters, generality is not lost by assuming $\text{ini}(\mathbf{w}) = x_0 \dots x_m$ with $m \geq 1$. Then \mathbf{w} can be written in the form $\mathbf{w} = \prod_{i=0}^m (x_i \mathbf{w}_i)$ with $\mathbf{w}_i \in \{x_0, \dots, x_i\}^*$. If \mathbf{w}_0 is nonempty, then $\mathbf{w}_0 = x_0$ and the identity (14.1a) can be used to eliminate $\mathbf{w}_0 = x_0$ from \mathbf{w} . Hence \mathbf{w} is of the form (14.2).

Suppose $\mathbf{w}_m \neq \emptyset$. Then $\mathbf{w}_m = z_1 \dots z_s$ for some $z_1, \dots, z_s \in \{x_0, \dots, x_m\}$ with $s \geq 1$. Hence $\mathbf{w} = (x_0 \prod_{i=1}^{m-1} (x_i \mathbf{w}_i)) x_m z_1 \dots z_s$. The letters z_1, \dots, z_s are non-first occurrences in \mathbf{w} . Therefore if $s \geq 2$, then the identities (14.1b)–(14.1d) can be used to eliminate z_1, \dots, z_{s-1} from \mathbf{w}_m . Hence (I) is satisfied.

Suppose $\mathbf{w}_i \neq \emptyset$ for some $i < m$. Then $\mathbf{w}_i = y_1 \dots y_r$ for some $y_1, \dots, y_r \in \{x_0, \dots, x_i\}$ with $r \geq 1$. Hence

$$\mathbf{w} = \mathbf{p} x_i \underbrace{y_1 \dots y_r}_{\mathbf{w}_i} \mathbf{q}$$

where $\mathbf{p} = x_0 \prod_{j=1}^{i-1} (x_j \mathbf{w}_j)$ and $\mathbf{q} = \prod_{j=i+1}^m (x_j \mathbf{w}_j)$. The letters y_1, \dots, y_r are non-first occurrences in \mathbf{w} . Therefore if $r \geq 2$, then the identities (14.1b)–(14.1d) can be used to eliminate the letters y_1, \dots, y_{r-1} from \mathbf{w}_i . Hence $\mathbf{w} \stackrel{(14.1)}{\approx} \mathbf{p} x_i y_r \mathbf{q}$. If $y_r = x_i$, then the identity (14.1a) can be used to further eliminate y_r from \mathbf{w}_i . Therefore \mathbf{w}_i is converted into a word in $\{\emptyset, x_1, \dots, x_{i-1}\}$, whence (II) is satisfied. □

LEMMA 14.3. *Suppose that $\mathbf{w} \approx \mathbf{w}'$ is any identity satisfied by \mathcal{A}_7 . Then:*

- (i) $\text{ini}(\mathbf{w}) = \text{ini}(\mathbf{w}')$ and $\text{t}(\mathbf{w}) = \text{t}(\mathbf{w}')$;
- (ii) $\text{t}(\mathbf{w}) \in \text{sim}(\mathbf{w})$ if and only if $\text{t}(\mathbf{w}') \in \text{sim}(\mathbf{w}')$.

Proof. This follows from Lemmas 2.1 and 2.2 because the subsemigroups $\{1, 3, 6\}$, $\{3, 5\}$, and $\{1, 2, 5\}$ of \mathcal{A}_7 are isomorphic to L_2^1 , R_2 , and J , respectively. □

Proof of Proposition 14.1. It is routinely checked that \mathcal{A}_7 satisfies the identities (14.1). Hence it suffices to show that any identity $\mathbf{w} \approx \mathbf{w}'$ satisfied by \mathcal{A}_7 is implied by the identities (14.1). If either \mathbf{w} or \mathbf{w}' involves only one letter, say $\text{con}(\mathbf{w}) = \{x\}$, then it is easily shown by Lemma 14.3 that the identity $\mathbf{w} \approx \mathbf{w}'$ is implied by the identity (14.1a). Therefore assume that \mathbf{w} and \mathbf{w}' each contains at least two distinct letters and so by Lemma 14.2, can be chosen to be in α -canonical form. Since $\text{ini}(\mathbf{w}) = \text{ini}(\mathbf{w}')$ by Lemma 14.3(i), generality is not lost by assuming

$$\mathbf{w} = x_0 \prod_{i=1}^m (x_i \mathbf{w}_i) \quad \text{and} \quad \mathbf{w}' = x_0 \prod_{i=1}^m (x_i \mathbf{w}'_i) \quad \text{for some } m \geq 1.$$

Suppose $\mathbf{w}_m \neq \mathbf{w}'_m$. Then since $\mathbf{t}(\mathbf{w}) = \mathbf{t}(\mathbf{w}')$ by Lemma 14.3(i), it follows from (I) that $(\mathbf{w}_m, \mathbf{w}'_m) \in \{(x_m, \emptyset), (\emptyset, x_m)\}$, say $(\mathbf{w}_m, \mathbf{w}'_m) = (x_m, \emptyset)$. Hence $\mathbf{w} = (x_0 \prod_{i=1}^{m-1} (x_i \mathbf{w}_i)) x_m^2$ and $\mathbf{w}' = (x_0 \prod_{i=1}^{m-1} (x_i \mathbf{w}'_i)) x_m$. But now Lemma 14.3(ii) is violated because $\mathbf{t}(\mathbf{w}) \notin \text{sim}(\mathbf{w})$ and $\mathbf{t}(\mathbf{w}') \in \text{sim}(\mathbf{w}')$. Therefore $\mathbf{w}_m = \mathbf{w}'_m$.

Suppose $\mathbf{w}_\ell \neq \mathbf{w}'_\ell$ for some $\ell < m$, so that $\mathbf{w}_\ell, \mathbf{w}'_\ell \in \{\emptyset, x_0, \dots, x_{\ell-1}\}$ by (II). Then there are two cases.

Case 1: $\mathbf{w}_\ell \neq \emptyset \neq \mathbf{w}'_\ell$. Then $\mathbf{w}_\ell = x_j$ and $\mathbf{w}'_\ell = x_k$ for some $j, k \in \{0, \dots, \ell - 1\}$ with $j \neq k$. By symmetry, it suffices to assume $0 \leq j < k < \ell$. Then

$$\mathbf{w} = \mathbf{p}x_\ell x_j \mathbf{q} \quad \text{and} \quad \mathbf{w}' = \mathbf{p}'x_\ell x_k \mathbf{q}',$$

where $\mathbf{p} = x_0 \prod_{i=1}^{\ell-1} (x_i \mathbf{w}_i)$, $\mathbf{q} = \prod_{i=\ell+1}^m (x_i \mathbf{w}_i)$, $\mathbf{p}' = x_0 \prod_{i=1}^{\ell-1} (x_i \mathbf{w}'_i)$, and $\mathbf{q}' = \prod_{i=\ell+1}^m (x_i \mathbf{w}'_i)$. Let $\varphi : \mathcal{X} \rightarrow \mathcal{A}_7$ denote the substitution

$$z \mapsto \begin{cases} 5 & \text{if } z = x_j, \\ 4 & \text{if } z = x_{\ell+1}, \\ 3 & \text{otherwise.} \end{cases}$$

Then $\mathbf{w}\varphi \neq \mathbf{w}'\varphi$ because

$$\mathbf{w}\varphi = \mathbf{p}\varphi \cdot x_\ell\varphi \cdot x_j\varphi \cdot \mathbf{q}\varphi \in \{3, 5\}^* \cdot 3 \cdot 5 \cdot 4 \dots = \{6\}$$

and

$$\mathbf{w}'\varphi = \mathbf{p}'\varphi \cdot x_\ell\varphi \cdot x_k\varphi \cdot \mathbf{q}'\varphi \in \{3, 5\}^* \cdot 3 \cdot 3 \cdot 4 \dots = \{1\}.$$

Case 2: $\mathbf{w}_\ell = \emptyset \neq \mathbf{w}'_\ell$ or $\mathbf{w}_\ell \neq \emptyset = \mathbf{w}'_\ell$. Then by symmetry, it suffices to assume $\mathbf{w}_\ell = \emptyset$ and $\mathbf{w}'_\ell = x_j$ for some $j \in \{0, \dots, \ell - 1\}$. Hence

$$\mathbf{w} = \mathbf{p}x_\ell \mathbf{q} \quad \text{and} \quad \mathbf{w}' = \mathbf{p}'x_\ell x_j \mathbf{q}',$$

where $\mathbf{p}, \mathbf{q}, \mathbf{p}'$, and \mathbf{q}' are as defined in Case 1. Therefore $\mathbf{w}\varphi \neq \mathbf{w}'\varphi$ because

$$\mathbf{w}\varphi = \mathbf{p}\varphi \cdot x_\ell\varphi \cdot \mathbf{q}\varphi \in \{3, 5\}^* \cdot 3 \cdot 4 \dots = \{1\}$$

and

$$\mathbf{w}'\varphi = \mathbf{p}'\varphi \cdot x_\ell\varphi \cdot x_j\varphi \cdot \mathbf{q}'\varphi \in \{3, 5\}^* \cdot 3 \cdot 5 \cdot 4 \dots = \{6\}.$$

Since both cases are impossible, the integer ℓ does not exist. Therefore the identity $\mathbf{w} \approx \mathbf{w}'$ is trivial and so is implied by the identities (14.1). □

14.2. Finite basis property of $\mathcal{A}_4, \mathcal{B}_2$, and \mathcal{G}_3

PROPOSITION 14.4. *The variety generated by any $S \in \{\mathcal{A}_4, \mathcal{B}_2, \mathcal{G}_3\}$ is defined by the identities*

$$x^2y \approx xy, \tag{14.3a}$$

$$xy^2 \approx xy, \tag{14.3b}$$

$$xHyKxy \approx xHyKy, \tag{14.3c}$$

$$xHyKyx \approx xHyKx. \tag{14.3d}$$

Let \mathbf{w} be any word such that $\text{ini}(\mathbf{w}) = x_0 \dots x_m$ with $m \geq 1$. In this section, such a word \mathbf{w} is said to be in β -canonical form if

$$\mathbf{w} = x_0 \prod_{i=1}^m (x_i \mathbf{w}_i), \quad (14.4)$$

where $\mathbf{w}_i \in \{\emptyset, x_1, \dots, x_{i-1}\}$ for all i .

LEMMA 14.5. *Let \mathbf{w} be any word that contains at least two distinct letters. Then there exists some word $\bar{\mathbf{w}}$ in β -canonical form such that the identities (14.3) imply the identity $\mathbf{w} \approx \bar{\mathbf{w}}$.*

Proof. It suffices to convert \mathbf{w} , using the identities (14.3), into a word in β -canonical form. It is easily seen that the identities (14.3) imply the identities (14.1). Therefore by Lemma 14.2, the identities (14.3) can be used to convert \mathbf{w} into a word in α -canonical form, that is, a word of the form (14.4) such that $\mathbf{w}_m \in \{\emptyset, x_0, \dots, x_m\}$ and $\mathbf{w}_i \in \{\emptyset, x_0, \dots, x_{i-1}\}$ for all i with $1 \leq i < m$. If $\mathbf{w}_m = x_m$, then the identity (14.3b) can be used to eliminate $\mathbf{w}_m = x_m$ from \mathbf{w} . The resulting word is in β -canonical form. \square

LEMMA 14.6. *Let $S \in \{\mathcal{A}_4, \mathcal{B}_2, \mathcal{G}_3\}$. Suppose that $\mathbf{w} \approx \mathbf{w}'$ is any identity satisfied by S . Then $\text{ini}(\mathbf{w}) = \text{ini}(\mathbf{w}')$ and $\mathbf{t}(\mathbf{w}) = \mathbf{t}(\mathbf{w}')$.*

Proof. The subsemigroups $\{3, 5\} \subset \mathcal{A}_4$, $\{4, 6\} \subset \mathcal{B}_2$, and $\{2, 5\} \subset \mathcal{G}_3$ are isomorphic to R_2 , while the subsemigroups $\{1, 3, 6\} \subset \mathcal{A}_4$, $\{3, 4, 5\} \subset \mathcal{B}_2$, and $\{1, 2, 6\} \subset \mathcal{G}_3$ are isomorphic to L_2^1 . Hence the result follows from Lemma 2.1 parts (ii) and (iii). \square

Proof of Proposition 14.4. Let $S \in \{\mathcal{A}_4, \mathcal{B}_2, \mathcal{G}_3\}$. It is routinely checked that S satisfies the identities (14.3). Hence it suffices to show that any identity $\mathbf{w} \approx \mathbf{w}'$ satisfied by S is implied by the identities (14.3). If either \mathbf{w} or \mathbf{w}' involves only one letter, say $\text{con}(\mathbf{w}) = \{x\}$, then it is easily shown by Lemma 14.6 that the identity $\mathbf{w} \approx \mathbf{w}'$ is implied by the identity (14.3a). Therefore assume that \mathbf{w} and \mathbf{w}' each contains at least two distinct letters and so, by Lemma 14.5, can be chosen to be in β -canonical form. Since $\text{ini}(\mathbf{w}) = \text{ini}(\mathbf{w}')$ by Lemma 14.6, generality is not lost by assuming

$$\mathbf{w} = x_0 \prod_{i=1}^m (x_i \mathbf{w}_i) \quad \text{and} \quad \mathbf{w}' = x_0 \prod_{i=1}^m (x_i \mathbf{w}'_i) \quad \text{for some } m \geq 1.$$

Since $\mathbf{t}(\mathbf{w}) = \mathbf{t}(\mathbf{w}')$ by Lemma 14.6 and $\mathbf{w}_m, \mathbf{w}'_m \in \{\emptyset, x_0, \dots, x_{m-1}\}$, it follows that $\mathbf{w}_m = \mathbf{w}'_m$. Suppose $\mathbf{w}_\ell \neq \mathbf{w}'_\ell$ for some $\ell < m$, so that $\mathbf{w}_\ell, \mathbf{w}'_\ell \in \{\emptyset, x_1, \dots, x_{\ell-1}\}$. Then there are two cases.

Case 1: $\mathbf{w}_\ell \neq \emptyset \neq \mathbf{w}'_\ell$. Then $\mathbf{w}_\ell = x_j$ and $\mathbf{w}'_\ell = x_k$ for some $j, k \in \{0, \dots, \ell-1\}$ with $j \neq k$. By symmetry, it suffices to assume $0 \leq j < k < \ell$. Then

$$\mathbf{w} = \mathbf{p}x_\ell x_j \mathbf{q} \quad \text{and} \quad \mathbf{w}' = \mathbf{p}'x_\ell x_k \mathbf{q}'$$

where $\mathbf{p} = x_0 \prod_{i=1}^{\ell-1} (x_i \mathbf{w}_i)$, $\mathbf{q} = \prod_{i=\ell+1}^m (x_i \mathbf{w}_i)$, $\mathbf{p}' = x_0 \prod_{i=1}^{\ell-1} (x_i \mathbf{w}'_i)$, and $\mathbf{q}' = \prod_{i=\ell+1}^m (x_i \mathbf{w}'_i)$. Let φ_1 , φ_2 , and φ_3 denote the following substitutions into \mathcal{A}_4 , \mathcal{B}_2 , and \mathcal{G}_3 , respectively:

$$z \mapsto \begin{cases} 5 & \text{if } z = x_j, \\ 4 & \text{if } z = x_{\ell+1}, \\ 3 & \text{otherwise;} \end{cases} \quad z \mapsto \begin{cases} 6 & \text{if } z = x_j, \\ 1 & \text{if } z = x_{\ell+1}, \\ 4 & \text{otherwise;} \end{cases} \quad z \mapsto \begin{cases} 5 & \text{if } z = x_j, \\ 3 & \text{if } z = x_{\ell+1}, \\ 2 & \text{otherwise.} \end{cases}$$

Then $\mathbf{w}\varphi_1 \neq \mathbf{w}'\varphi_1$, $\mathbf{w}\varphi_2 \neq \mathbf{w}'\varphi_2$, and $\mathbf{w}\varphi_3 \neq \mathbf{w}'\varphi_3$ because

$$\mathbf{w}\varphi_1 = \mathbf{p}\varphi_1 \cdot x_{\ell}\varphi_1 \cdot x_j\varphi_1 \cdot \mathbf{q}\varphi_1 \in \{3, 5\}^* \cdot 3 \cdot 5 \cdot 4 \dots = \{6\}$$

and

$$\mathbf{w}'\varphi_1 = \mathbf{p}'\varphi_1 \cdot x_{\ell}\varphi_1 \cdot x_k\varphi_1 \cdot \mathbf{q}'\varphi_1 \in \{3, 5\}^* \cdot 3 \cdot 3 \cdot 4 \dots = \{1\};$$

$$\mathbf{w}\varphi_2 = \mathbf{p}\varphi_2 \cdot x_{\ell}\varphi_2 \cdot x_j\varphi_2 \cdot \mathbf{q}\varphi_2 \in \{4, 6\}^* \cdot 4 \cdot 6 \cdot 1 \dots = \{5\}$$

and

$$\mathbf{w}'\varphi_2 = \mathbf{p}'\varphi_2 \cdot x_{\ell}\varphi_2 \cdot x_k\varphi_2 \cdot \mathbf{q}'\varphi_2 \in \{4, 6\}^* \cdot 4 \cdot 4 \cdot 1 \dots = \{3\};$$

$$\mathbf{w}\varphi_3 = \mathbf{p}\varphi_3 \cdot x_{\ell}\varphi_3 \cdot x_j\varphi_3 \cdot \mathbf{q}\varphi_3 \in \{2, 5\}^* \cdot 2 \cdot 5 \cdot 3 \dots = \{6\}$$

and

$$\mathbf{w}'\varphi_3 = \mathbf{p}'\varphi_3 \cdot x_{\ell}\varphi_3 \cdot x_k\varphi_3 \cdot \mathbf{q}'\varphi_3 \in \{2, 5\}^* \cdot 2 \cdot 2 \cdot 3 \dots = \{1\}.$$

Case 2: $\mathbf{w}_{\ell} = \emptyset \neq \mathbf{w}'_{\ell}$ or $\mathbf{w}_{\ell} \neq \emptyset = \mathbf{w}'_{\ell}$. Then by symmetry, it suffices to assume $\mathbf{w}_{\ell} = \emptyset$ and $\mathbf{w}'_{\ell} = x_j$ for some $j \in \{0, \dots, \ell - 1\}$. Hence

$$\mathbf{w} = \mathbf{p}x_{\ell}\mathbf{q} \quad \text{and} \quad \mathbf{w}' = \mathbf{p}'x_{\ell}x_j\mathbf{q}'$$

where \mathbf{p} , \mathbf{q} , \mathbf{p}' , and \mathbf{q}' are as defined in Case 1. It follows that

$$\mathbf{w}\varphi_1 = \mathbf{p}\varphi_1 \cdot x_{\ell}\varphi_1 \cdot \mathbf{q}\varphi_1 \in \{3, 5\}^* \cdot 3 \cdot 4 \dots = \{1\}$$

and

$$\mathbf{w}'\varphi_1 = \mathbf{p}'\varphi_1 \cdot x_{\ell}\varphi_1 \cdot x_j\varphi_1 \cdot \mathbf{q}'\varphi_1 \in \{3, 5\}^* \cdot 3 \cdot 5 \cdot 4 \dots = \{6\};$$

$$\mathbf{w}\varphi_2 = \mathbf{p}\varphi_2 \cdot x_{\ell}\varphi_2 \cdot \mathbf{q}\varphi_2 \in \{4, 6\}^* \cdot 4 \cdot 1 \dots = \{3\}$$

and

$$\mathbf{w}'\varphi_2 = \mathbf{p}'\varphi_2 \cdot x_{\ell}\varphi_2 \cdot x_j\varphi_2 \cdot \mathbf{q}'\varphi_2 \in \{4, 6\}^* \cdot 4 \cdot 6 \cdot 1 \dots = \{5\};$$

$$\mathbf{w}\varphi_3 = \mathbf{p}\varphi_3 \cdot x_{\ell}\varphi_3 \cdot \mathbf{q}\varphi_3 \in \{2, 5\}^* \cdot 2 \cdot 3 \dots = \{1\}$$

and

$$\mathbf{w}'\varphi_3 = \mathbf{p}'\varphi_3 \cdot x_{\ell}\varphi_3 \cdot x_j\varphi_3 \cdot \mathbf{q}'\varphi_3 \in \{2, 5\}^* \cdot 2 \cdot 5 \cdot 3 \dots = \{6\}.$$

Therefore $\mathbf{w}\varphi_1 \neq \mathbf{w}'\varphi_1$, $\mathbf{w}\varphi_2 \neq \mathbf{w}'\varphi_2$, and $\mathbf{w}\varphi_3 \neq \mathbf{w}'\varphi_3$.

Since both cases are impossible, the integer ℓ does not exist. Therefore the identity $\mathbf{w} \approx \mathbf{w}'$ is trivial and so is implied by the identities (14.3). □

15. $\mathcal{B}_4, \mathcal{B}_5, \mathcal{B}_6, \mathcal{B}_9, \mathcal{B}_{10}, \mathcal{E}_1$

This section establishes the finite basis property of the following semigroups.

\mathcal{B}_4	1 2 3 4 5 6	\mathcal{B}_5	1 2 3 4 5 6	\mathcal{B}_6	1 2 3 4 5 6
1	1 1 1 1 1 1	1	1 1 1 1 1 1	1	1 1 1 1 1 1
2	1 1 1 1 1 2	2	1 1 1 1 1 2	2	1 1 1 1 1 2
3	1 1 1 1 1 3	3	1 1 1 1 1 3	3	1 1 1 1 1 3
4	1 1 1 2 1 4	4	1 1 1 2 1 4	4	1 1 1 2 1 4
5	1 1 2 1 1 1	5	1 1 2 1 1 1	5	1 1 2 1 1 2
6	1 2 1 4 5 6	6	1 2 2 4 5 6	6	1 2 2 4 5 6
\mathcal{B}_9	1 2 3 4 5 6	\mathcal{B}_{10}	1 2 3 4 5 6	\mathcal{E}_1	1 2 3 4 5 6
1	1 1 1 1 1 1	1	1 1 1 1 1 1	1	1 1 1 1 1 1
2	1 1 1 1 1 2	2	1 1 1 1 1 2	2	1 1 1 1 1 2
3	1 1 1 1 1 3	3	1 1 1 1 1 3	3	1 1 2 1 2 5
4	1 1 1 2 2 4	4	1 1 1 2 2 4	4	1 1 2 2 2 4
5	1 1 2 2 2 4	5	1 1 2 2 2 4	5	1 1 2 2 2 5
6	1 2 1 4 5 6	6	1 2 2 4 5 6	6	1 2 3 5 5 6

PROPOSITION 15.1. *The variety generated by any $S \in \{\mathcal{B}_4, \mathcal{B}_5, \mathcal{B}_6, \mathcal{B}_9, \mathcal{B}_{10}, \mathcal{E}_1\}$ is defined by the identities*

$$x^2HxKx \approx xHxKx, \quad xHx^2Kx \approx xHxKx, \quad xHxKx^2 \approx xHxKx, \tag{15.1a}$$

$$xHyKxTy \approx yHxKxTy, \quad xHyKxTy \approx xHxKyTy, \quad xHyKxTy \approx xHyKyTx, \tag{15.1b}$$

$$x^2hyKy \approx xhxyKy, \quad x^2HyKy \approx xHyKxy, \quad yHx^2ky \approx yHxkxy, \tag{15.1c}$$

$$xHxky^2 \approx xHxyky, \quad xHxKy^2 \approx xyHxKy, \quad xhy^2Kx \approx xyhyKx, \tag{15.1d}$$

$$hxykTk \approx hyhxtk, \quad kTkxyhy \approx kTkyhxx, \tag{15.1e}$$

$$hxyhy \approx hyhxx. \tag{15.1f}$$

The proof of Proposition 15.1 is given in § 15.4.

15.1. *Identities satisfied by $\mathcal{B}_4, \mathcal{B}_5, \mathcal{B}_6, \mathcal{B}_9, \mathcal{B}_{10}$, and \mathcal{E}_1*

For any word \mathbf{w} , let $\text{con}_k(\mathbf{w})$ denote the set of letters that occur exactly k times in \mathbf{w} :

$$\text{con}_k(\mathbf{w}) = \{x \in \mathcal{X} \mid \text{occ}(x, \mathbf{w}) = k\}.$$

Let $\text{con}_3^\infty(\mathbf{w}) = \bigcup_{k \geq 3} \text{con}_k(\mathbf{w})$. Note that $\text{con}(\mathbf{w}) = \text{sim}(\mathbf{w}) \cup \text{con}_2(\mathbf{w}) \cup \text{con}_3^\infty(\mathbf{w})$ is a disjoint union.

LEMMA 15.2. *Let $S \in \{\mathcal{B}_4, \mathcal{B}_5, \mathcal{B}_6, \mathcal{B}_9, \mathcal{B}_{10}, \mathcal{E}_1\}$. Suppose that $\mathbf{w} \approx \mathbf{w}'$ is any identity satisfied by S . Then:*

- (i) $\text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$, $\text{con}_2(\mathbf{w}) = \text{con}_2(\mathbf{w}')$, and $\text{con}_3^\infty(\mathbf{w}) = \text{con}_3^\infty(\mathbf{w}')$;
- (ii) either (a) $\text{t}(\mathbf{w}) \in \text{sim}(\mathbf{w})$, $\text{t}(\mathbf{w}') \in \text{sim}(\mathbf{w}')$, and $\text{t}(\mathbf{w}) = \text{t}(\mathbf{w}')$, or (b) $\text{t}(\mathbf{w}) \notin \text{sim}(\mathbf{w})$ and $\text{t}(\mathbf{w}') \notin \text{sim}(\mathbf{w}')$;

- (iii) either (a) $h(\mathbf{w}) \in \text{sim}(\mathbf{w})$, $h(\mathbf{w}') \in \text{sim}(\mathbf{w}')$, and $h(\mathbf{w}) = h(\mathbf{w}')$, or (b) $h(\mathbf{w}) \notin \text{sim}(\mathbf{w})$ and $h(\mathbf{w}') \notin \text{sim}(\mathbf{w}')$;
- (iv) $F_{SS}(\mathbf{w}) = F_{SS}(\mathbf{w}')$.

Proof. (i) The subsemigroup $\{1, 2, 4, 6\}$ of $S' \in \{\mathcal{B}_4, \mathcal{B}_5, \mathcal{B}_6, \mathcal{B}_9, \mathcal{B}_{10}\}$ and the divisor $\{1, 2, 3, 5, 6\}/\{3, 5\}$ of \mathcal{E}_1 are isomorphic to N_3^1 . Hence the result follows from Lemma 2.1(vii).

(ii) The subsemigroups $\{1, 5, 6\} \subset \mathcal{B}_4$ and $\{1, 5, 6\} \subset \mathcal{B}_5$ and the quotients $\mathcal{B}_6/\{1, 2, 3, 4\}$, $\mathcal{B}_9/\{1, 2, 3, 4\}$, $\mathcal{B}_{10}/\{1, 2, 3, 4\}$, and $\mathcal{E}_1/\{1, 2, 4, 5\}$ are isomorphic to J . Therefore the result follows from Lemma 2.2.

(iii) The subsemigroup $\{1, 3, 6\} \subset \mathcal{B}_4$ and the quotients $\mathcal{B}_5/\{1, 2, 4, 5\}$, $\mathcal{B}_6/\{1, 2, 4, 5\}$, $\mathcal{B}_9/\{1, 2, 4, 5\}$, $\mathcal{B}_{10}/\{1, 2, 4, 5\}$, and $\mathcal{E}_1/\{1, 2, 3, 5\}$ are isomorphic to the dual semigroup of J . Therefore the result follows from the dual result of Lemma 2.2.

(iv) The semigroup S does not satisfy the identity (2.3) with $n = 3$ because

$$6^3 \cdot 5 \cdot 6^3 \cdot 3 \cdot 6^3 \neq 6^3 \cdot 5 \cdot 3 \cdot 6^3 \quad \text{in } S' \in \{\mathcal{B}_4, \mathcal{B}_5, \mathcal{B}_6, \mathcal{B}_9, \mathcal{B}_{10}\}$$

and

$$6^3 \cdot 3 \cdot 6^3 \cdot 4 \cdot 6^3 \neq 6^3 \cdot 3 \cdot 4 \cdot 6^3 \quad \text{in } \mathcal{E}_1.$$

Therefore $F_{SS}(\mathbf{w}) = F_{SS}(\mathbf{w}')$ by Lemma 2.10 and part (i). □

15.2. α -canonical form

Let \mathbf{w} be any word with $\text{con}_2(\mathbf{w}) = \{x_1, \dots, x_m\}$ for some $m \geq 1$ and $\text{con}_3^\infty(\mathbf{w}) = \emptyset$. Suppose that the letters x_1, \dots, x_m are in alphabetical order. In this section, such a word \mathbf{w} is said to be in α -canonical form if

$$\begin{aligned} \mathbf{w} &= \mathbf{p} \left(\prod_{i=1}^r (x_i \mathbf{s}_i x_i \mathbf{t}_i) \right) \left(\prod_{i=r+1}^m x_i^2 \right) \mathbf{q} \\ &= \mathbf{p} \cdot x_1 \mathbf{s}_1 x_1 \mathbf{t}_1 \dots x_r \mathbf{s}_r x_r \mathbf{t}_r \cdot x_{r+1}^2 \dots x_m^2 \cdot \mathbf{q}, \end{aligned}$$

where all of the following are satisfied:

- (I) $r \in \{0, \dots, m\}$;
- (II) the letters of $\mathbf{p}, \mathbf{q}, \mathbf{t}_r \in \mathcal{X}^*$ and $\mathbf{s}_1, \dots, \mathbf{s}_r, \mathbf{t}_1, \dots, \mathbf{t}_{r-1} \in \mathcal{X}^+$ are simple in \mathbf{w} ;
- (III) if $r = m$, then $\mathbf{q} = \emptyset$.

Note the extreme cases:

$$\mathbf{w} = \begin{cases} \mathbf{p} \cdot x_1^2 \dots x_m^2 \cdot \mathbf{q} & \text{if } r = 0, \\ \mathbf{p} \cdot x_1 \mathbf{s}_1 x_1 \mathbf{t}_1 \dots x_m \mathbf{s}_m x_m \mathbf{t}_m & \text{if } r = m. \end{cases}$$

LEMMA 15.3. *Let \mathbf{w} be any non-simple word such that $\text{con}_3^\infty(\mathbf{w}) = \emptyset$. Then there exists some word $\overline{\mathbf{w}}$ in α -canonical form such that the identities (15.1) imply the identity $\mathbf{w} \approx \overline{\mathbf{w}}$.*

Proof. It suffices to convert \mathbf{w} , using the identities (15.1), into a word in α -canonical form. Since \mathbf{w} is non-simple, generality is not lost by assuming $\text{con}_2(\mathbf{w}) = \{x_1, \dots, x_m\}$ with x_1, \dots, x_m in alphabetical order. The identities (15.1b) can be used to interchange any two occurrences of non-simple letters in \mathbf{w} . In particular, the identities (15.1b) can be used to arrange the non-simple letters of \mathbf{w} so that they occur in alphabetical order, that is,

$$\mathbf{w} \stackrel{(15.1b)}{\approx} \underbrace{(\dots x_1 \dots x_1 \dots)(\dots x_2 \dots x_2 \dots) \dots (\dots x_m \dots x_m \dots)}_{\mathbf{w}'}$$

Consider the factorization of \mathbf{w}' that displays all occurrences of x_i and x_{i+1} individually:

$$\mathbf{w}' = \mathbf{a}x_i\mathbf{b}x_ix_{i+1}\mathbf{d}x_{i+1}\mathbf{e}$$

where $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e} \in \mathcal{X}^*$. There are two cases.

Case 1: $\mathbf{b} = \emptyset \neq \mathbf{c}$, that is, the two occurrences of x_i are adjacent but the second x_i is not adjacent with the first x_{i+1} . Then the two occurrences of x_i can be separated by the identities (15.1c):

$$\mathbf{w}' = \mathbf{a}x_i^2\mathbf{c}x_{i+1}\mathbf{d}x_{i+1}\mathbf{e} \stackrel{(15.1c)}{\approx} \mathbf{a}x_ix_{i+1}\mathbf{c}x_{i+1}\mathbf{d}x_{i+1}\mathbf{e}.$$

Case 2: $\mathbf{c} = \emptyset \neq \mathbf{d}$, that is, the second x_i is adjacent with the first x_{i+1} but the two occurrences of x_{i+1} are not adjacent. Then the second x_i and the first x_{i+1} can be separated by the identities (15.1d):

$$\mathbf{w}' = \mathbf{a}x_ix_{i+1}\mathbf{b}x_{i+1}\mathbf{d}x_{i+1}\mathbf{e} \stackrel{(15.1d)}{\approx} \mathbf{a}x_ix_{i+1}\mathbf{d}x_{i+1}^2\mathbf{e}.$$

It is easily seen how the procedures in Cases 1 and 2 can be repeated until \mathbf{w}' is converted by the identities $\{(15.1c), (15.1d)\}$ into a word in α -canonical form. □

15.3. β -canonical form

Let \mathbf{w} be any word with $\text{con}_2(\mathbf{w}) = \{x_1, \dots, x_m\}$ for some $m \geq 0$ and $\text{con}_3^\infty(\mathbf{w}) = \{y_1, \dots, y_n\}$ for some $n \geq 1$. Suppose that the letters x_1, \dots, x_m are in alphabetical order and the letters y_1, \dots, y_n are in alphabetical order. In this section, such a word \mathbf{w} is said to be in β -canonical form if

$$\mathbf{w} = \mathbf{p}\mathbf{x} \prod_{i=1}^r (\mathbf{y}\mathbf{s}_i) = \mathbf{p}\mathbf{x} \cdot \mathbf{y}\mathbf{s}_1 \dots \mathbf{y}\mathbf{s}_r,$$

where all of the following are satisfied:

- (I) $r \geq 1$;
- (II) the letters of $\mathbf{p}, \mathbf{s}_r \in \mathcal{X}^*$ and $\mathbf{s}_1, \dots, \mathbf{s}_{r-1} \in \mathcal{X}^+$ are simple in \mathbf{w} ;
- (III) $\mathbf{x} = x_1^2 \dots x_m^2$;
- (IV) $\mathbf{y} = y_1^3 \dots y_n^3$.

Note that \mathbf{x} is empty if $m = 0$, but \mathbf{y} is nonempty because $\text{con}_3^\infty(\mathbf{w}) \neq \emptyset$.

LEMMA 15.4. *Let \mathbf{w} be any non-simple word such that $\text{con}_3^\infty(\mathbf{w}) \neq \emptyset$. Then there exists some word $\overline{\mathbf{w}}$ in β -canonical form such that the identities (15.1) imply the identity $\mathbf{w} \approx \overline{\mathbf{w}}$.*

Proof. It suffices to convert \mathbf{w} , using the identities (15.1), into a word in β -canonical form. Since \mathbf{w} is non-simple, generality is not lost by assuming $\text{con}_2(\mathbf{w}) = \{x_1, \dots, x_m\}$ with x_1, \dots, x_m in alphabetical order, and $\text{con}_3^\infty(\mathbf{w}) = \{y_1, \dots, y_n\}$ with y_1, \dots, y_n in alphabetical order. Then \mathbf{w} can be written in the form

$$\mathbf{w} = \mathbf{p}\mathbf{w}_1\mathbf{s}_1 \dots \mathbf{w}_r\mathbf{s}_r,$$

where (I) and (II) are satisfied and the letters of $\mathbf{w}_1, \dots, \mathbf{w}_r \in \mathcal{X}^+$ are all non-simple in \mathbf{w} . Let $k \in \{1, \dots, n\}$. By assumption, $y_k \in \text{con}(\mathbf{w}_i)$ for some i . Suppose $y_k \notin \text{con}(\mathbf{w}_{i+1})$. Then

$$\begin{aligned} \mathbf{w} &= \dots \overbrace{\mathbf{a}y_k\mathbf{b}\mathbf{s}_i}^{\mathbf{w}_i} \cdot \mathbf{w}_{i+1}\mathbf{s}_{i+1} \dots && \text{for some } \mathbf{a}, \mathbf{b} \in \{x_1, \dots, x_m, y_1, \dots, y_n\}^* \\ \stackrel{(15.1a)}{\approx} &\dots \mathbf{a}y_k^2\mathbf{b}\mathbf{s}_i \cdot \mathbf{w}_{i+1}\mathbf{s}_{i+1} \dots && \text{since } \text{occ}(y_k, \mathbf{w}) \geq 3 \\ \stackrel{(15.1c)}{\approx} &\dots \mathbf{a}y_k\mathbf{b}\mathbf{s}_i \cdot y_k\mathbf{w}_{i+1}\mathbf{s}_{i+1} \dots && \text{since } h(\mathbf{w}_{i+1}) \text{ is non-simple in } \mathbf{w} \\ &= \dots \mathbf{w}_i\mathbf{s}_i \cdot (y_k\mathbf{w}_{i+1})\mathbf{s}_{i+1} \dots, \end{aligned}$$

that is, \mathbf{w}_{i+1} is converted by the identities $\{(15.1a), (15.1c)\}$ into $y_k \mathbf{w}_{i+1}$. If $y_k \notin \text{con}(\mathbf{w}_{i-1})$, then by a symmetrical argument, \mathbf{w}_{i-1} can be converted by the identities $\{(15.1a), (15.1d)\}$ into $\mathbf{w}_{i-1} y_k$. Since k is arbitrary in $\{1, \dots, n\}$, it is easily seen how this procedure can be repeated until each \mathbf{w}_i is converted into a word that contains all the letters y_1, \dots, y_n . Now for each i , the letters of \mathbf{w}_i are non-simple in \mathbf{w} and so can be arranged by the identities (15.1b) in any manner. Hence each \mathbf{w}_i can be converted by the identities (15.1b) into a word of the form $\mathbf{x}_i y_1^{e_1} \dots y_n^{e_n}$ with $e_1, \dots, e_n \geq 1$ and $\mathbf{x}_i \in \{x_1, \dots, x_m\}^*$. Since $\text{occ}(y_1, \mathbf{w}), \dots, \text{occ}(y_n, \mathbf{w}) \geq 3$, the identities (15.1a) can be used to replace each exponent e_i by 3. Hence the identities (15.1) can be used to convert \mathbf{w} into the word

$$\mathbf{w}' = \mathbf{p} \cdot \mathbf{x}_1 \mathbf{y} \mathbf{s}_1 \cdot \mathbf{x}_2 \mathbf{y} \mathbf{s}_2 \dots \mathbf{x}_r \mathbf{y} \mathbf{s}_r.$$

If the factor \mathbf{x}_2 of \mathbf{w}' is nonempty, then it can be moved to the left by the identities (15.1) until it immediately follows the factor \mathbf{x}_1 :

$$\begin{aligned} \mathbf{w}' &= \mathbf{p} \cdot \mathbf{x}_1 \mathbf{y} \mathbf{s}_1 \cdot \overbrace{z_1 \dots z_p}^{\mathbf{x}_2} \mathbf{y} \mathbf{s}_2 \dots && \text{for some } z_1, \dots, z_p \in \{x_1, \dots, x_m\} \\ (15.1a) \quad &\approx \mathbf{p} \cdot \mathbf{x}_1 y_1^p \mathbf{y} \mathbf{s}_1 \cdot z_1 \dots z_p \mathbf{y} \mathbf{s}_2 \dots && \text{since } y_1 = h(\mathbf{y}) \text{ and } \text{occ}(y_1, \mathbf{w}') \geq 3 \\ (15.1b) \quad &\approx \mathbf{p} \cdot \mathbf{x}_1 z_1 \dots z_p \mathbf{y} \mathbf{s}_1 \cdot y_1^p \mathbf{y} \mathbf{s}_2 \dots && \text{since } y_1, z_1, \dots, z_p \text{ are non-simple in } \mathbf{w}' \\ (15.1a) \quad &\approx \mathbf{p} \cdot \mathbf{x}_1 \mathbf{x}_2 \mathbf{y} \mathbf{s}_1 \cdot \mathbf{y} \mathbf{s}_2 \dots \end{aligned}$$

By the same argument, if the factor \mathbf{x}_3 is nonempty, then it can be moved to the left by the identities (15.1) until it immediately follows the factor $\mathbf{x}_1 \mathbf{x}_2$. Continuing in this manner, \mathbf{w}' can be converted by the identities (15.1) into the word

$$\mathbf{w}'' = \mathbf{p}(\mathbf{x}_1 \mathbf{x}_2 \dots \mathbf{x}_r) \cdot \mathbf{y} \mathbf{s}_1 \cdot \mathbf{y} \mathbf{s}_2 \dots \mathbf{y} \mathbf{s}_r.$$

It is easily seen that when converting \mathbf{w} into \mathbf{w}'' , the number of occurrences of each x_i remained unchanged throughout, that is, $\text{occ}(x_i, \mathbf{x}_1 \mathbf{x}_2 \dots \mathbf{x}_r) = \text{occ}(x_i, \mathbf{w}) = 2$. Therefore if the letters of the factor $\mathbf{x}_1 \mathbf{x}_2 \dots \mathbf{x}_r$ are alphabetically ordered by the identities (15.1b), then it becomes $\mathbf{x} = x_1^2 \dots x_m^2$. The resulting word is in β -canonical form. \square

15.4. Proof of Proposition 15.1

Let $S \in \{\mathcal{B}_4, \mathcal{B}_5, \mathcal{B}_6, \mathcal{B}_9, \mathcal{B}_{10}, \mathcal{E}_1\}$. It is routinely checked that S satisfies the identities (15.1). Hence it suffices to show that any identity $\mathbf{w} \approx \mathbf{w}'$ satisfied by S is implied by the identities (15.1). If either \mathbf{w} or \mathbf{w}' is a simple word, then it follows from Lemma 15.2 parts (i) and (iv) that the identity $\mathbf{w} \approx \mathbf{w}'$ is trivial and so is vacuously implied by the identities (15.1). Therefore assume that \mathbf{w} and \mathbf{w}' are both non-simple words. By Lemma 15.2(i), it can further be assumed that:

- (a) $\text{con}_2(\mathbf{w}) = \text{con}_2(\mathbf{w}') = \{x_1, \dots, x_m\}$ where x_1, \dots, x_m are in alphabetical order;
- (b) $\text{con}_3^\infty(\mathbf{w}) = \text{con}_3^\infty(\mathbf{w}') = \{y_1, \dots, y_n\}$ where y_1, \dots, y_n are in alphabetical order.

There are two cases.

Case 1: $\text{con}_3^\infty(\mathbf{w}) = \text{con}_3^\infty(\mathbf{w}') = \emptyset$. Then by Lemma 15.3, the words \mathbf{w} and \mathbf{w}' can be chosen to be in α -canonical form, whence by (a),

$$\mathbf{w} = \mathbf{p} \cdot x_1 \mathbf{s}_1 x_1 \mathbf{t}_1 \dots x_r \mathbf{s}_r x_r \mathbf{t}_r \cdot x_{r+1}^2 \dots x_m^2 \cdot \mathbf{q}$$

and

$$\mathbf{w}' = \mathbf{p}' \cdot x_1 \mathbf{s}'_1 x_1 \mathbf{t}'_1 \dots x_{r'} \mathbf{s}'_{r'} x_{r'} \mathbf{t}'_{r'} \cdot x_{r'+1}^2 \dots x_m^2 \cdot \mathbf{q}'.$$

Therefore $\mathbf{p} = \mathbf{p}'$ and $\mathbf{q} = \mathbf{q}'$ by parts (ii)–(iv) of Lemma 15.2, and

(c) $\{\mathbf{s}_1, \mathbf{t}_1, \dots, \mathbf{s}_r, \mathbf{t}_r\} = \{\mathbf{s}'_1, \mathbf{t}'_1, \dots, \mathbf{s}'_{r'}, \mathbf{t}'_{r'}\}$
 by part (iv) of the same lemma, whence $r = r'$. Consequently,

$$\mathbf{w}' = \mathbf{p} \cdot x_1 \mathbf{s}'_1 x_1 \mathbf{t}'_1 \dots x_r \mathbf{s}'_r x_r \mathbf{t}'_r \cdot x_{r+1}^2 \dots x_m^2 \cdot \mathbf{q}.$$

It now follows from (c) that \mathbf{w}' can be converted by the identities (15.1e) into \mathbf{w} .

Case 2: $\text{con}_3^\infty(\mathbf{w}) = \text{con}_3^\infty(\mathbf{w}') \neq \emptyset$. Then by Lemma 15.4, the words \mathbf{w} and \mathbf{w}' can be chosen to be in β -canonical form, whence by (a) and (b),

$$\mathbf{w} = \mathbf{p}\mathbf{x} \cdot \mathbf{y}\mathbf{s}_1 \dots \mathbf{y}\mathbf{s}_{r-1} \cdot \mathbf{y}\mathbf{s}_r \quad \text{and} \quad \mathbf{w}' = \mathbf{p}'\mathbf{x}' \cdot \mathbf{y}'\mathbf{s}'_1 \dots \mathbf{y}'\mathbf{s}'_{r'-1} \cdot \mathbf{y}'\mathbf{s}'_{r'}$$

where $\mathbf{x} = x_1^2 \dots x_m^2$ and $\mathbf{y} = y_1^3 \dots y_n^3$. Therefore $\mathbf{p} = \mathbf{p}'$ and $\mathbf{s}_r = \mathbf{s}'_{r'}$ by parts (ii)–(iv) of Lemma 15.2, and

(d) $\{\mathbf{s}_1, \dots, \mathbf{s}_{r-1}\} = \{\mathbf{s}'_1, \dots, \mathbf{s}'_{r'-1}\}$
 by part (iv) of the same lemma, whence $r = r'$. Consequently,

$$\mathbf{w}' = \mathbf{p}\mathbf{x} \cdot \mathbf{y}'\mathbf{s}'_1 \dots \mathbf{y}'\mathbf{s}'_{r-1} \cdot \mathbf{y}\mathbf{s}_r.$$

It now follows from (d) that \mathbf{w}' can be converted by the identity (15.1f) into \mathbf{w} .

16. $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4, \mathcal{D}_3, \mathcal{D}_5$

This section establishes the finite basis property of the following semigroups.

\mathcal{C}_1	1 2 3 4 5 6	\mathcal{C}_2	1 2 3 4 5 6	\mathcal{C}_3	1 2 3 4 5 6
1	1 1 1 1 1 1	1	1 1 1 1 1 1	1	1 1 1 1 1 1
2	1 1 1 1 1 2	2	1 1 1 1 1 2	2	1 1 1 1 1 2
3	1 1 1 1 1 3	3	1 1 1 1 1 3	3	1 1 1 1 1 3
4	1 1 2 1 1 1	4	1 1 2 1 1 1	4	1 1 2 1 1 2
5	5 5 5 5 5 5	5	5 5 5 5 5 5	5	5 5 5 5 5 5
6	1 2 1 4 5 6	6	1 2 2 4 5 6	6	1 2 1 4 5 6
\mathcal{C}_4	1 2 3 4 5 6	\mathcal{D}_3	1 2 3 4 5 6	\mathcal{D}_5	1 2 3 4 5 6
1	1 1 1 1 1 1	1	1 1 1 1 1 1	1	1 1 1 1 1 1
2	1 1 1 1 1 2	2	1 1 1 1 1 2	2	1 1 1 1 1 2
3	1 1 1 1 1 3	3	1 1 1 1 2 1	3	1 1 1 1 2 2
4	1 1 2 1 1 2	4	4 4 4 4 4 4	4	4 4 4 4 4 4
5	5 5 5 5 5 5	5	4 4 4 4 4 5	5	4 4 4 4 4 5
6	1 2 2 4 5 6	6	1 2 3 4 4 6	6	1 2 3 4 4 6

PROPOSITION 16.1. *The variety generated by any $S \in \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4, \mathcal{D}_3, \mathcal{D}_5\}$ is defined by the identities*

$$x^3 \approx x^2, \quad x^2yx \approx xyx, \quad xyx^2 \approx xyx, \tag{16.1a}$$

$$xyxz^2 \approx x^2yz^2, \tag{16.1b}$$

$$x^2Hy^2 \approx xHy^2x, \tag{16.1c}$$

$$xHyKxy \approx xHyKyx. \tag{16.1d}$$

The proof of Proposition 16.1 is given in § 16.3.

16.1. Identities satisfied by $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4, \mathcal{D}_3$, and \mathcal{D}_5

LEMMA 16.2. Let $S \in \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4, \mathcal{D}_3, \mathcal{D}_5\}$. Suppose that $\mathbf{w} \approx \mathbf{w}'$ is any identity satisfied by S . Then:

- (i) $\text{ini}(\mathbf{w}) = \text{ini}(\mathbf{w}')$;
- (ii) $\text{con}(\mathbf{w}) = \text{con}(\mathbf{w}')$ and $\text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$;
- (iii) $\text{t}(\mathbf{w}) \in \text{sim}(\mathbf{w})$ if and only if $\text{t}(\mathbf{w}') \in \text{sim}(\mathbf{w}')$;
- (iv) $F_{SS}(\mathbf{w}) = F_{SS}(\mathbf{w}')$.

Proof. (i) The subsemigroup $\{1, 5, 6\}$ of $S' \in \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4\}$ and the subsemigroup $\{1, 4, 6\}$ of $S'' \in \{\mathcal{D}_3, \mathcal{D}_5\}$ are isomorphic to L_2^1 . Therefore the result follows from Lemma 2.1(iii).

(ii) This follows from Lemma 2.1(v) since the subsemigroup $\{1, 2, 6\}$ of S is isomorphic to N_2^1 .

(iii) This follows from Lemma 2.2 since the subsemigroup $\{1, 4, 6\}$ of $S' \in \{\mathcal{C}_1, \mathcal{C}_2\}$, the divisor $\{1, 2, 4, 6\}/\{1, 2\}$ of $S'' \in \{\mathcal{C}_3, \mathcal{C}_4\}$, the subsemigroup $\{1, 3, 6\}$ of \mathcal{D}_3 , and the divisor $\{1, 2, 3, 6\}/\{1, 2\}$ of \mathcal{D}_5 are isomorphic to J .

(iv) The semigroup S does not satisfy the identity (2.3) with $n = 2$ because

$$6^2 \cdot 4 \cdot 6^2 \cdot 3 \cdot 6^2 \neq 6^2 \cdot 4 \cdot 3 \cdot 6^2 \quad \text{in } S' \in \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4\}$$

and

$$6^2 \cdot 3 \cdot 6^2 \cdot 5 \cdot 6^2 \neq 6^2 \cdot 3 \cdot 5 \cdot 6^2 \quad \text{in } S'' \in \{\mathcal{D}_3, \mathcal{D}_4\}.$$

Therefore $F_{SS}(\mathbf{w}) = F_{SS}(\mathbf{w}')$ by part (ii) and Lemma 2.10. □

16.2. A canonical form

Let \mathbf{w} be any non-simple word and let x_0 be the first non-simple letter of \mathbf{w} . Then there exist $\mathbf{p} \in \mathcal{X}^*$ and $\mathbf{q} \in \mathcal{X}^+$ such that

$$\mathbf{w} = \mathbf{p}x_0\mathbf{q}, \tag{16.2}$$

where the prefix \mathbf{p} , if nonempty, consists of simple letters of \mathbf{w} . Note that the suffix \mathbf{q} is nonempty since it contains all non-first occurrences of x_0 . In this section, such a non-simple word \mathbf{w} is said to be in *canonical form* if

$$\mathbf{w} = \mathbf{p}x_0^2 \prod_{i=1}^m (x_i^{e_i} x_0^{f_i}) = \mathbf{p}x_0^2 \cdot x_1^{e_1} x_0^{f_1} \dots x_m^{e_m} x_0^{f_m}, \tag{16.3}$$

where all of the following are satisfied:

- (I) the letters x_0, x_1, \dots, x_m are all distinct;
- (II) $e_1, \dots, e_m \in \{1, 2\}$;
- (III) $f_1, \dots, f_m \in \{0, 2\}$;
- (IV) if $f_i = 2$, then $e_i = e_{i+1} = 1$.

Note that $\text{occ}(x_i, \mathbf{w}) = e_i$ for each $i \in \{1, \dots, m\}$. Therefore

- (V) for each $i \in \{1, \dots, m\}$, the letter x_i is simple in \mathbf{w} if and only if $e_i = 1$.

LEMMA 16.3. Let \mathbf{w} be any non-simple word. Then there exists some word $\overline{\mathbf{w}}$ in canonical form such that the identities (16.1) imply the identity $\mathbf{w} \approx \overline{\mathbf{w}}$.

Proof. It suffices to convert \mathbf{w} in (16.2), using the identities (16.1), into a word in canonical form. Suppose $\text{ini}(x_0\mathbf{q}) = x_0x_1 \dots x_m$ for some $m \geq 0$. Then \mathbf{w} can be written in the form

$$\mathbf{w} = \mathbf{p}x_0^r \prod_{i=1}^m (x_i \mathbf{w}_i) = \mathbf{p}x_0^r \cdot x_1 \mathbf{w}_1 \dots x_m \mathbf{w}_m,$$

where $r \geq 1$ and $\mathbf{w}_i \in \{x_0, \dots, x_i\}^*$. Since the letter x_0 is non-simple in \mathbf{w} , the identities (16.1a) can be used to replace the exponent r by 2. For each i , since the letters of \mathbf{w}_i are non-first occurrences, the identities (16.1d) can be used to arrange them, within \mathbf{w}_i , in any order. Specifically, any occurrence of x_i in \mathbf{w}_i can be moved to the left and grouped with the x_i in \mathbf{w} that immediately precedes \mathbf{w}_i . Hence

$$\mathbf{w} \stackrel{(16.1d)}{\approx} \underbrace{\mathbf{p}x_0^2 \cdot x_1^{e_1} \mathbf{w}'_1 \dots x_m^{e_m} \mathbf{w}'_m}_{\mathbf{w}'}$$

for some $e_i \geq 1$ and $\mathbf{w}'_i \in \{x_0, \dots, x_{i-1}\}^*$.

Consider an arbitrary $k \in \{1, \dots, m\}$ such that $\mathbf{w}'_k \neq \emptyset$. Then $\mathbf{w}'_k = y_1 \dots y_n$ for some $y_1, \dots, y_n \in \{x_0, \dots, x_{k-1}\}$. Write

$$\mathbf{w}' = \mathbf{p}x_0^2 \cdot \mathbf{a} \cdot x_k^{e_k} \mathbf{w}'_k \cdot \mathbf{b}$$

where $\mathbf{a} = \prod_{i=1}^{k-1} (x_i^{e_i} \mathbf{w}'_i)$ and $\mathbf{b} = \prod_{i=k+1}^m (x_i^{e_i} \mathbf{w}'_i)$. Since y_n is non-simple in \mathbf{w}' ,

$$\begin{aligned} \mathbf{w}' &\stackrel{(16.1a)}{\approx} \mathbf{p}x_0^2 \cdot \mathbf{a} \cdot x_k^{e_k} y_1 \dots y_{n-1} y_n^2 \cdot \mathbf{b} \\ &\stackrel{(16.1c)}{\approx} \mathbf{p}x_0 \cdot \mathbf{a} \cdot x_k^{e_k} y_1 \dots y_{n-1} y_n^2 x_0 \cdot \mathbf{b} \\ &\stackrel{(16.1a)}{\approx} \underbrace{\mathbf{p}x_0^2 \cdot \mathbf{a} \cdot x_k^{e_k} y_1 \dots y_{n-1} y_n x_0^2 \cdot \mathbf{b}}_{\mathbf{w}''} \end{aligned}$$

If the letter y_n in \mathbf{w}'' is x_0 , then it can be eliminated by the first identity from (16.1a). If the letter y_n in \mathbf{w}'' is not x_0 , then it is one of x_1, \dots, x_{k-1} and hence can be gathered, using the identity (16.1b), with one of the factors $x_1^{e_1}, \dots, x_{k-1}^{e_{k-1}}$ in \mathbf{a} . In any case, the letter y_n in \mathbf{w}'' is eliminated from \mathbf{w}'_k . Repeat the same argument to eliminate the letters $y_{n-1}, y_{n-2}, \dots, y_1$ from \mathbf{w}'_k . Hence

$$\mathbf{w}'' \stackrel{(16.1)}{\approx} \mathbf{p}x_0^2 \cdot \mathbf{a} \cdot x_k^{e_k} x_0^2 \cdot \mathbf{b},$$

that is, the factor \mathbf{w}'_k of \mathbf{w} is converted into x_0^2 . Since $k \in \{1, \dots, m\}$ is arbitrary, the same argument can be repeated on any \mathbf{w}'_i that is nonempty. Therefore \mathbf{w} can be converted by the identities (16.1) into the word (16.3) with (I) and (III) satisfied and $e_1, \dots, e_m \geq 1$. If $e_i \geq 3$, then the identities (16.1a) can be used to reduce e_i to 2. Hence (II) is satisfied.

If $e_i = 2$ and $f_i = 2$, then

$$\begin{aligned} \mathbf{w} &= \mathbf{p}x_0^2 \dots x_i^{e_i-1} x_0^{f_i-1} \cdot x_i^2 x_0^2 \cdot x_{i+1}^{e_{i+1}} x_0^{f_{i+1}} \dots \\ &\stackrel{(16.1c)}{\approx} \mathbf{p}x_0^4 \dots x_i^{e_i-1} x_0^{f_i-1} \cdot x_i^2 \cdot x_{i+1}^{e_{i+1}} x_0^{f_{i+1}} \dots \\ &\stackrel{(16.1a)}{\approx} \mathbf{p}x_0^2 \dots x_i^{e_i-1} x_0^{f_i-1} \cdot x_i^2 \cdot x_{i+1}^{e_{i+1}} x_0^{f_{i+1}} \dots, \end{aligned}$$

so that f_i is reduced to 0. Similarly, if $e_{i+1} = 2$ and $f_i = 2$, then

$$\begin{aligned} \mathbf{w} &= \mathbf{p}x_0^2 \dots x_i^{e_i-1} x_0^{f_i-1} \cdot x_i^{e_i} x_0^2 \cdot x_{i+1}^2 x_0^{f_{i+1}} \dots \\ &\stackrel{(16.1b)}{\approx} \mathbf{p}x_0^4 \dots x_i^{e_i-1} x_0^{f_i-1} \cdot x_i^{e_i} \cdot x_{i+1}^2 x_0^{f_{i+1}} \dots \\ &\stackrel{(16.1a)}{\approx} \mathbf{p}x_0^2 \dots x_i^{e_i-1} x_0^{f_i-1} \cdot x_i^{e_i} \cdot x_{i+1}^2 x_0^{f_{i+1}} \dots, \end{aligned}$$

so that f_i is reduced to 0. Consequently, (IV) is satisfied. □

16.3. Proof of Proposition 16.1

Let $S \in \{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4, \mathcal{D}_3, \mathcal{D}_5\}$. It is routinely checked that S satisfies the identities (16.1). Hence it suffices to show that any identity $\mathbf{w} \approx \mathbf{w}'$ satisfied by S is implied by the identities (16.1). Note that $\text{ini}(\mathbf{w}) = \text{ini}(\mathbf{w}')$ and $\text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$ by Lemma 16.2 parts (i) and (ii). Therefore if either \mathbf{w} or \mathbf{w}' is a simple word, then the identity $\mathbf{w} \approx \mathbf{w}'$ is trivial and so is vacuously implied by the identities (16.1). Hence assume that \mathbf{w} and \mathbf{w}' are both non-simple words and so, by Lemma 16.3, can be chosen to be in canonical form. Since $\text{ini}(\mathbf{w}) = \text{ini}(\mathbf{w}')$ and $\text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$, it follows from (V) that

$$\mathbf{w} = \mathbf{p}x_0^2 \cdot x_1^{e_1}x_0^{f_1} \dots x_m^{e_m}x_0^{f_m} \quad \text{and} \quad \mathbf{w}' = \mathbf{p}x_0^2 \cdot x_1^{e_1}x_0^{f'_1} \dots x_m^{e_m}x_0^{f'_m}.$$

Case 1: $f_m \neq f'_m$, say $(f_m, f'_m) = (2, 0)$. Then $e_m = 1$ by (IV), so that

$$\mathbf{w} = \mathbf{p}x_0^2 \cdot x_1^{e_1}x_0^{f_1} \dots x_{m-1}^{e_{m-1}}x_0^{f_{m-1}} \cdot x_mx_0^2 \quad \text{and} \quad \mathbf{w}' = \mathbf{p}x_0^2 \cdot x_1^{e_1}x_0^{f'_1} \dots x_{m-1}^{e_{m-1}}x_0^{f'_{m-1}} \cdot x_m.$$

But $t(\mathbf{w}) = x_0 \notin \text{sim}(\mathbf{w})$ and $t(\mathbf{w}') = x_m \in \text{sim}(\mathbf{w}')$ violate Lemma 16.2(iii).

Case 2: $f_i \neq f'_i$ for some $i < m$, say $(f_i, f'_i) = (2, 0)$. Then $e_i = e_{i+1} = 1$ by (IV), so that

$$\mathbf{w} = \mathbf{p}x_0^2 \cdot x_1^{e_1}x_0^{f_1} \dots x_ix_0^2 \cdot x_{i+1}x_0^{f_{i+1}} \dots \quad \text{and} \quad \mathbf{w}' = \mathbf{p}x_0^2 \cdot x_1^{e_1}x_0^{f'_1} \dots x_i \cdot x_{i+1}x_0^{f'_{i+1}} \dots$$

But $x_ix_{i+1} \notin F_{SS}(\mathbf{w})$ and $x_ix_{i+1} \in F_{SS}(\mathbf{w}')$ violate Lemma 16.2(iv).

Since both cases are impossible, $f_i = f'_i$ for all i . Therefore the identity $\mathbf{w} \approx \mathbf{w}'$ is trivial and so is implied by the identities (16.1).

17. $\mathcal{C}_5, \mathcal{C}_6, \mathcal{C}_{10}, \mathcal{D}_1$

This section establishes the finite basis property of the following semigroups.

\mathcal{C}_5	1	2	3	4	5	6	\mathcal{C}_6	1	2	3	4	5	6	\mathcal{C}_{10}	1	2	3	4	5	6	\mathcal{D}_1	1	2	3	4	5	6
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
2	1	1	1	1	1	2	2	1	1	1	1	1	2	2	1	1	1	1	1	2	2	1	1	1	1	2	
3	1	1	1	1	1	3	3	1	1	1	1	1	3	3	1	1	1	1	1	3	3	1	1	1	1	3	
4	1	1	2	1	2	1	4	1	1	2	1	2	2	4	1	1	3	4	5	4	4	1	1	3	4	5	5
5	1	1	2	1	2	3	5	1	1	2	1	2	3	5	1	3	3	4	5	4	5	1	3	3	4	5	5
6	1	2	1	4	4	6	6	1	2	2	4	4	6	6	1	1	3	4	5	6	6	1	3	3	4	5	6

17.1. Finite basis property of \mathcal{C}_5 and \mathcal{C}_6

PROPOSITION 17.1. The variety generated by $S \in \{\mathcal{C}_5, \mathcal{C}_6\}$ is defined by the identities

$$x^2yx \approx xyx, \quad xyx^2 \approx xyx, \tag{17.1a}$$

$$x^2Hy^2 \approx y^2Hx^2, \tag{17.1b}$$

$$x^3Hy^2 \approx y^2Hx^3, \tag{17.1c}$$

$$x^3Hy^3 \approx y^3Hx^3, \tag{17.1d}$$

$$x^3yh^2 \approx x^3yx^3h^2, \tag{17.1e}$$

$$h^2yx^3 \approx h^2x^3yx^3, \tag{17.1f}$$

$$h^2x^2yk^2 \approx h^2yx^2k^2, \tag{17.1g}$$

$$hxyh \approx hyhxh, \tag{17.1h}$$

$$h^2xk^2yt^2 \approx h^2yk^2xt^2. \tag{17.1i}$$

A non-simple letter x of a word \mathbf{w} is said to be *restricted* if $\mathbf{w} = \mathbf{a}x^2\mathbf{b}$ for some $\mathbf{a}, \mathbf{b} \in \mathcal{X}^*$ such that $x \notin \text{con}(\mathbf{ab})$; in this case, x^2 is called a *restricted square* of \mathbf{w} . A non-simple letter that is not restricted is said to be *unrestricted*. For any word \mathbf{w} , let $\text{res}(\mathbf{w})$ denote the set of restricted letters of \mathbf{w} and let $\text{unres}(\mathbf{w})$ denote the set of unrestricted letters of \mathbf{w} . Note that the disjoint union $\text{res}(\mathbf{w}) \cup \text{unres}(\mathbf{w})$ coincides with the set of non-simple letters of \mathbf{w} .

LEMMA 17.2. *Let $S \in \{\mathcal{C}_5, \mathcal{C}_6\}$. Suppose that $\mathbf{w} \approx \mathbf{w}'$ is any identity satisfied by S . Then:*

- (i) $\text{con}(\mathbf{w}) = \text{con}(\mathbf{w}')$ and $\text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$;
- (ii) either (a) $\text{t}(\mathbf{w}) \in \text{sim}(\mathbf{w})$, $\text{t}(\mathbf{w}') \in \text{sim}(\mathbf{w}')$, and $\text{t}(\mathbf{w}) = \text{t}(\mathbf{w}')$, or (b) $\text{t}(\mathbf{w}) \notin \text{sim}(\mathbf{w})$ and $\text{t}(\mathbf{w}') \notin \text{sim}(\mathbf{w}')$;
- (iii) either (a) $\text{h}(\mathbf{w}) \in \text{sim}(\mathbf{w})$, $\text{h}(\mathbf{w}') \in \text{sim}(\mathbf{w}')$, and $\text{h}(\mathbf{w}) = \text{h}(\mathbf{w}')$, or (b) $\text{h}(\mathbf{w}) \notin \text{sim}(\mathbf{w})$ and $\text{h}(\mathbf{w}') \notin \text{sim}(\mathbf{w}')$;
- (iv) $\text{res}(\mathbf{w}) = \text{res}(\mathbf{w}')$ and $\text{unres}(\mathbf{w}) = \text{unres}(\mathbf{w}')$;
- (v) $F_{SS}(\mathbf{w}) = F_{SS}(\mathbf{w}')$.

Proof. (i) This follows from Lemma 2.1(v) since the subsemigroup $\{1, 2, 6\}$ of S is isomorphic to N_2^1 .

(ii) This follows from Lemma 2.2 since the subsemigroup $\{1, 4, 6\}$ of \mathcal{C}_5 and the divisor $\{1, 2, 4, 6\}/\{1, 2\}$ of \mathcal{C}_6 are isomorphic to J .

(iii) This follows from the dual result of Lemma 2.2 since the subsemigroup $\{1, 3, 6\}$ of \mathcal{C}_5 and the divisor $\{1, 2, 3, 6\}/\{1, 2\}$ of \mathcal{C}_6 are isomorphic to the dual semigroup of J .

(iv) By part (i), the set of non-simple letters of \mathbf{w} coincides with the set of non-simple letters of \mathbf{w}' . Therefore it suffices to verify that $\text{res}(\mathbf{w}) = \text{res}(\mathbf{w}')$. Suppose $x \in \text{res}(\mathbf{w})$. Then $\mathbf{w} = \mathbf{a}x^2\mathbf{b}$ for some $\mathbf{a}, \mathbf{b} \in \mathcal{X}^*$ such that $x \notin \text{con}(\mathbf{ab})$. Let $\varphi : \mathcal{X} \rightarrow S$ denote the substitution

$$z \mapsto \begin{cases} 5 & \text{if } z = x, \\ 6 & \text{otherwise.} \end{cases}$$

Then $\mathbf{w}\varphi = \dots 5^2 \dots = 2$. By part (i), the letter x is non-simple in \mathbf{w}' , so $\text{occ}(x, \mathbf{w}') = k \geq 2$. If two occurrences of x in \mathbf{w}' sandwich some other letter, say $\mathbf{w}' = \mathbf{c}x\mathbf{d}x\mathbf{e}$ for some $\mathbf{c}, \mathbf{e} \in \mathcal{X}^*$ and $\mathbf{d} \in \mathcal{X}^+$ such that $x \notin \text{con}(\mathbf{d})$, then $\mathbf{w}'\varphi = \dots 5 \cdot 6 \cdot 5 \dots = 1$ is a contradiction. Therefore all k occurrences of x form the factor x^k in \mathbf{w}' , whence $\mathbf{w}' = \mathbf{f}x^k\mathbf{g}$ for some $\mathbf{f}, \mathbf{g} \in \mathcal{X}^*$ such that $x \notin \text{con}(\mathbf{fg})$. If $k \geq 3$, then $\mathbf{w}'\varphi = \dots 5^k \dots = 1$ is a contradiction. It follows that $k = 2$ and $x \in \text{res}(\mathbf{w}')$. Consequently, the inclusion $\text{res}(\mathbf{w}) \subseteq \text{res}(\mathbf{w}')$ holds. The inclusion $\text{res}(\mathbf{w}') \subseteq \text{res}(\mathbf{w})$ holds by symmetry.

(v) The identity (2.3) with $n = 3$ is not satisfied by S since $6^3 \cdot 4 \cdot 6^3 \cdot 3 \cdot 6^3 \neq 6^3 \cdot 4 \cdot 3 \cdot 6^3$. Therefore $F_{SS}(\mathbf{w}) = F_{SS}(\mathbf{w}')$ by Lemma 2.10 and part (i). □

Let \mathbf{w} be any word with $\text{res}(\mathbf{w}) = \{x_1, \dots, x_m\}$ for some $m \geq 1$ and $\text{unres}(\mathbf{w}) = \emptyset$. Suppose that the letters x_1, \dots, x_m are in alphabetical order. In this subsection, such a word \mathbf{w} is said to be in *α -canonical form* if

$$\begin{aligned} \mathbf{w} &= \mathbf{p} \left(\prod_{i=1}^r (x_i^2 \mathbf{s}_i) \right) \left(\prod_{i=r+1}^m x_i^2 \right) \mathbf{q} \\ &= \mathbf{p} \cdot x_1^2 \mathbf{s}_1 \dots x_r^2 \mathbf{s}_r \cdot x_{r+1}^2 \dots x_m^2 \cdot \mathbf{q}, \end{aligned}$$

where all of the following are satisfied:

- (I) $r \in \{0, \dots, m\}$;
- (II) the letters of $\mathbf{p}, \mathbf{q} \in \mathcal{X}^*$ and $\mathbf{s}_1, \dots, \mathbf{s}_r \in \mathcal{X}^+$ are simple in \mathbf{w} ;
- (III) if $r = m$, then $\mathbf{q} = \emptyset$.

Note the extreme cases:

$$\mathbf{w} = \begin{cases} \mathbf{p} \cdot x_1^2 \dots x_m^2 \cdot \mathbf{q} & \text{if } r = 0, \\ \mathbf{p} \cdot x_1^2 \mathbf{s}_1 \dots x_m^2 \mathbf{s}_m & \text{if } r = m. \end{cases}$$

LEMMA 17.3. *Let \mathbf{w} be any non-simple word such that $\text{unres}(\mathbf{w}) = \emptyset$. Then there exists some word $\bar{\mathbf{w}}$ in α -canonical form such that the identities (17.1) imply the identity $\mathbf{w} \approx \bar{\mathbf{w}}$.*

Proof. It suffices to convert \mathbf{w} , using the identities (17.1), into a word in α -canonical form. Generality is not lost by assuming $\text{res}(\mathbf{w}) = \{x_1, \dots, x_m\}$ with x_1, \dots, x_m in alphabetical order. Hence \mathbf{w} is a product, in some order, of the restricted squares x_1^2, \dots, x_m^2 and some simple letters. The identities (17.1b) can be used to arrange these restricted squares so that they appear in alphabetical order, that is,

$$\mathbf{w} \stackrel{(17.1b)}{\approx} \underbrace{\dots x_1^2 \dots x_2^2 \dots x_m^2 \dots}_{\mathbf{w}'} \in \mathcal{X}^* x_1^2 \mathcal{X}^* x_2^2 \mathcal{X}^* \dots \mathcal{X}^* x_m^2 \mathcal{X}^*.$$

If the squares x_i^2 and x_{i+1}^2 are adjacent but the squares x_{i+1}^2 and x_{i+2}^2 are not adjacent, so that $\mathbf{w}' = \mathbf{a}x_i^2x_{i+1}^2\mathbf{b}x_{i+2}^2\mathbf{c}$ for some $\mathbf{a}, \mathbf{c} \in \mathcal{X}^*$ and $\mathbf{b} \in \mathcal{X}^+$, then the squares x_i^2 and x_{i+1}^2 can be separated by the identity (17.1g):

$$\mathbf{w}' = \mathbf{a}x_i^2x_{i+1}^2\mathbf{b}x_{i+2}^2\mathbf{c} \stackrel{(17.1g)}{\approx} \mathbf{a}x_i^2\mathbf{b}x_{i+1}^2x_{i+2}^2\mathbf{c}.$$

It is easily seen how this procedure can be repeated until \mathbf{w}' is converted by the identity (17.1g) into a word in α -canonical form. □

Let \mathbf{w} be any word with $\text{res}(\mathbf{w}) = \{x_1, \dots, x_m\}$ for some $m \geq 0$ and $\text{unres}(\mathbf{w}) = \{y_1, \dots, y_n\}$ for some $n \geq 1$. Suppose that the letters x_1, \dots, x_m are in alphabetical order and the letters y_1, \dots, y_n are in alphabetical order. In this subsection, such a word \mathbf{w} is said to be in β -canonical form if

$$\mathbf{w} = \mathbf{p}\mathbf{x} \prod_{i=1}^r (\mathbf{y}\mathbf{s}_i) = \mathbf{p}\mathbf{x} \cdot \mathbf{y}\mathbf{s}_1 \dots \mathbf{y}\mathbf{s}_r,$$

where all of the following are satisfied:

- (I) $r \geq 1$;
- (II) the letters of $\mathbf{p}, \mathbf{s}_r \in \mathcal{X}^*$ and $\mathbf{s}_1, \dots, \mathbf{s}_{r-1} \in \mathcal{X}^+$ are simple in \mathbf{w} ;
- (III) $\mathbf{x} = x_1^2 \dots x_m^2$;
- (IV) $\mathbf{y} = y_1^3 \dots y_n^3$.

Note that \mathbf{x} is empty if $m = 0$, but \mathbf{y} is nonempty because $\text{unres}(\mathbf{w}) \neq \emptyset$.

LEMMA 17.4. *Let \mathbf{w} be any non-simple word such that $\text{unres}(\mathbf{w}) \neq \emptyset$. Then there exists some word $\bar{\mathbf{w}}$ in β -canonical form such that the identities (17.1) imply the identity $\mathbf{w} \approx \bar{\mathbf{w}}$.*

Proof. It suffices to convert \mathbf{w} , using the identities (17.1), into a word in β -canonical form. Since \mathbf{w} is non-simple, generality is not lost by assuming $\text{res}(\mathbf{w}) = \{x_1, \dots, x_m\}$ with x_1, \dots, x_m in alphabetical order, and $\text{unres}(\mathbf{w}) = \{y_1, \dots, y_n\}$ with y_1, \dots, y_n in alphabetical order. Then \mathbf{w} can be written in the form

$$\mathbf{w} = \mathbf{p}\mathbf{w}_1\mathbf{s}_1 \dots \mathbf{w}_r\mathbf{s}_r,$$

where (I) and (II) are satisfied and the letters of $\mathbf{w}_1, \dots, \mathbf{w}_r \in \mathcal{X}^+$ are all non-simple in \mathbf{w} . The identities (17.1a) can be used to replace each unrestricted letter y_i by y_i^3 . Hence it can further be assumed that $\mathbf{w}_i \in \{x_1^2, \dots, x_m^2, y_1^3, \dots, y_n^3\}^+$ for all i .

Let $k \in \{1, \dots, n\}$. By assumption, y_k^3 is a factor of some \mathbf{w}_i , so that $\mathbf{w}_i = \mathbf{a}y_k^3\mathbf{b}$ for some $\mathbf{a}, \mathbf{b} \in \{x_1^2, \dots, x_m^2, y_1^3, \dots, y_n^3\}^*$. Suppose that y_k^3 is not a factor of \mathbf{w}_{i+1} . Let $h = h(\mathbf{w}_{i+1})$. Then \mathbf{w}_{i+1} begins with h^2 or h^3 depending on whether or not h is restricted. In any case, $\mathbf{w}_{i+1} = h^2\mathbf{w}'_{i+1}$ for some $\mathbf{w}'_{i+1} \in \mathcal{X}^*$. Hence

$$\begin{aligned} \mathbf{w} &= \dots \overbrace{\mathbf{a}y_k^3\mathbf{b}}^{\mathbf{w}_i} \mathbf{s}_i \cdot \overbrace{h^2\mathbf{w}'_{i+1}}^{\mathbf{w}_{i+1}} \mathbf{s}_{i+1} \dots \quad (17.1e) \quad \dots \mathbf{a}y_k^3\mathbf{b}\mathbf{s}_i \cdot y_k^3 h^2 \mathbf{w}'_{i+1} \mathbf{s}_{i+1} \dots \\ &= \dots \mathbf{w}_i \mathbf{s}_i \cdot (y_k^3 \mathbf{w}_{i+1}) \mathbf{s}_{i+1} \dots, \end{aligned}$$

that is, \mathbf{w}_{i+1} is converted by the identity (17.1e) into $y_k^3\mathbf{w}_{i+1}$. If y_k^3 is not a factor of \mathbf{w}_{i-1} , then by a symmetrical argument, \mathbf{w}_{i-1} can be converted by the identity (17.1f) into $\mathbf{w}_{i-1}y_k^3$. Since k is arbitrary in $\{1, \dots, n\}$, it is easily seen how this procedure can be repeated until

(a) every \mathbf{w}_i is converted into a word that contains all the factors y_1^3, \dots, y_n^3 .
 Now the letters x_1, \dots, x_m are restricted, so that each of the restricted squares x_1^2, \dots, x_m^2 occurs precisely once in \mathbf{w} as a factor of some \mathbf{w}_i . The identities {(17.1b), (17.1c), (17.1g)} can thus be used to move the squares x_1^2, \dots, x_m^2 to the left until they are grouped with \mathbf{w}_1 . Therefore in view of (a):

(b) \mathbf{w}_1 is converted into a product involving $x_1^2, \dots, x_m^2, y_1^3, \dots, y_n^3$;
 (c) each of $\mathbf{w}_2, \dots, \mathbf{w}_r$ is converted into a product involving only y_1^3, \dots, y_n^3 .
 It then easily follows from (b) and (c) that the identities (17.1a)–(17.1d) can be used to convert \mathbf{w}_1 into \mathbf{xy} , and every one of $\mathbf{w}_2, \dots, \mathbf{w}_r$ to \mathbf{y} . Hence (III) and (IV) are satisfied, and the resulting word is in β -canonical form. \square

Proof of Proposition 17.1. Let $S \in \{\mathcal{C}_5, \mathcal{C}_6\}$. It is routinely checked that S satisfies the identities (17.1). Hence it suffices to show that any identity $\mathbf{w} \approx \mathbf{w}'$ satisfied by S is implied by the identities (17.1). If either \mathbf{w} or \mathbf{w}' is a simple word, then it follows from Lemma 17.2 parts (i) and (v) that the identity $\mathbf{w} \approx \mathbf{w}'$ is trivial and so is vacuously implied by (17.1). Therefore assume that \mathbf{w} and \mathbf{w}' are both non-simple words. By Lemma 17.2 parts (i) and (iv), it can further be assumed that:

(a) $\text{res}(\mathbf{w}) = \text{res}(\mathbf{w}') = \{x_1, \dots, x_m\}$ where x_1, \dots, x_m are in alphabetical order;
 (b) $\text{unres}(\mathbf{w}) = \text{unres}(\mathbf{w}') = \{y_1, \dots, y_n\}$ where y_1, \dots, y_n are in alphabetical order.
 There are two cases.

Case 1: $\text{unres}(\mathbf{w}) = \text{unres}(\mathbf{w}') = \emptyset$. Then by Lemma 17.3, the words \mathbf{w} and \mathbf{w}' can be chosen to be in α -canonical form, whence by (a),

$$\mathbf{w} = \mathbf{p} \cdot x_1^2 \mathbf{s}_1 \dots x_r^2 \mathbf{s}_r \cdot x_{r+1}^2 \dots x_m^2 \cdot \mathbf{q} \quad \text{and} \quad \mathbf{w}' = \mathbf{p}' \cdot x_1^2 \mathbf{s}'_1 \dots x_{r'}^2 \mathbf{s}'_{r'} \cdot x_{r'+1}^2 \dots x_m^2 \cdot \mathbf{q}'.$$

Therefore $\mathbf{p} = \mathbf{p}'$ and $\mathbf{q} = \mathbf{q}'$ by parts (ii), (iii), and (v) of Lemma 17.2, and
 (c) $\{\mathbf{s}_1, \dots, \mathbf{s}_r\} = \{\mathbf{s}'_1, \dots, \mathbf{s}'_{r'}\}$
 by part (v) of the same lemma, whence $r = r'$. Thus $\mathbf{w}' = \mathbf{p} \cdot x_1^2 \mathbf{s}'_1 \dots x_r^2 \mathbf{s}'_r \cdot x_{r+1}^2 \dots x_m^2 \cdot \mathbf{q}$. It now follows from (c) that \mathbf{w}' can be converted by the identity (17.1i) into \mathbf{w} .

Case 2: $\text{unres}(\mathbf{w}) = \text{unres}(\mathbf{w}') \neq \emptyset$. Then by Lemma 17.4, the words \mathbf{w} and \mathbf{w}' can be chosen to be in β -canonical form, whence by (a) and (b),

$$\mathbf{w} = \mathbf{px} \cdot \mathbf{ys}_1 \dots \mathbf{ys}_{r-1} \cdot \mathbf{ys}_r \quad \text{and} \quad \mathbf{w}' = \mathbf{p}'\mathbf{x} \cdot \mathbf{ys}'_1 \dots \mathbf{ys}'_{r'-1} \cdot \mathbf{ys}'_{r'}$$

where $\mathbf{x} = x_1^2 \dots x_m^2$ and $\mathbf{y} = y_1^3 \dots y_n^3$. Therefore $\mathbf{p} = \mathbf{p}'$ and $\mathbf{s}_r = \mathbf{s}'_{r'}$ by parts (ii), (iii), and (v) of Lemma 17.2, and

$$(d) \{s_1, \dots, s_{r-1}\} = \{s'_1, \dots, s'_{r'-1}\}$$

by part (v) of the same lemma, whence $r = r'$. Thus $\mathbf{w}' = \mathbf{p}\mathbf{x} \cdot \mathbf{y}\mathbf{s}'_1 \dots \mathbf{y}\mathbf{s}'_{r-1} \cdot \mathbf{y}\mathbf{s}_r$. It now follows from (d) that \mathbf{w}' can be converted by the identity (17.1h) into \mathbf{w} . \square

17.2. Finite basis property of \mathcal{C}_{10} and \mathcal{D}_1

PROPOSITION 17.5. The variety generated by $S \in \{\mathcal{C}_{10}, \mathcal{D}_1\}$ is defined by the identities

$$x^3 \approx x^2, \quad x^2yx \approx xyx, \quad xyx^2 \approx xyx, \tag{17.2a}$$

$$x^2y^2z^2 \approx y^2x^2z^2, \tag{17.2b}$$

$$x^2Hy^2 \approx yx^2Hy, \tag{17.2c}$$

$$xyHxKy \approx yxHxKy. \tag{17.2d}$$

For any word \mathbf{w} , let $F_{LS}(\mathbf{w})$ denote the set of factors of \mathbf{w} of length two that begin with the last occurrence of some non-simple letter and end with a simple letter:

$$F_{LS}(\mathbf{w}) = \{xy \in \mathcal{X}^2 \mid \mathbf{w} \in \mathcal{X}^*xy(\mathcal{X} \setminus \{x\})^*, x \notin \text{sim}(\mathbf{w}), y \in \text{sim}(\mathbf{w})\}.$$

LEMMA 17.6. Let $S \in \{\mathcal{C}_{10}, \mathcal{D}_1\}$. Suppose that $\mathbf{w} \approx \mathbf{w}'$ is any identity satisfied by S . Then:

- (i) $t(\mathbf{w}) = t(\mathbf{w}')$;
- (ii) $\text{con}(\mathbf{w}) = \text{con}(\mathbf{w}')$ and $\text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$;
- (iii) for any $x \in \text{con}(\mathbf{w}) = \text{con}(\mathbf{w}')$ and $y \in \text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$, the conditions $x \prec_{\mathbf{w}} y$ and $x \prec_{\mathbf{w}'} y$ are equivalent;
- (iv) $\mathbf{w}_{\text{sim}} = \mathbf{w}'_{\text{sim}}$;
- (v) $F_{LS}(\mathbf{w}) = F_{LS}(\mathbf{w}')$;
- (vi) $F_{SS}(\mathbf{w}) = F_{SS}(\mathbf{w}')$.

Proof. The subsemigroup $\{4, 5\}$ of S is isomorphic to R_2 , so part (i) holds by Lemma 2.1(ii). Parts (ii)–(iv) hold by Lemma 2.3 because the subsemigroup $\{1, 3, 4, 6\}$ of \mathcal{C}_{10} and the divisor $\{1, 3, 4, 5, 6\}/\{4, 5\}$ of \mathcal{D}_1 are isomorphic to J^1 .

(v) Suppose $F_{LS}(\mathbf{w}) \neq F_{LS}(\mathbf{w}')$, say $xy \in F_{LS}(\mathbf{w}) \setminus F_{LS}(\mathbf{w}')$. Then

(a) $\mathbf{w} = \mathbf{a}xy\mathbf{b}$ for some $\mathbf{a}, \mathbf{b} \in \mathcal{X}^*$ such that $x \in \text{con}(\mathbf{a}) \setminus \text{con}(\mathbf{b})$ and $y \notin \text{con}(\mathbf{ab})$.

By part (ii), the letter x is non-simple in both \mathbf{w} and \mathbf{w}' , while the letter y is simple in both \mathbf{w} and \mathbf{w}' . Further, since $x \prec_{\mathbf{w}} y$, it follows from part (iii) that $x \prec_{\mathbf{w}'} y$. Hence

(b) $\mathbf{w}' = \mathbf{a}'t\mathbf{y}\mathbf{b}'$ for some $t \in \mathcal{X}$ and some $\mathbf{a}', \mathbf{b}' \in \mathcal{X}^*$ such that $x \in \text{con}(\mathbf{a}') \setminus \text{con}(t\mathbf{b}')$ and $y \notin \text{con}(\mathbf{a}'t\mathbf{b}')$.

Let φ_1 and φ_2 denote the following substitutions into \mathcal{C}_{10} and \mathcal{D}_1 , respectively:

$$z \mapsto \begin{cases} 5 & \text{if } z = x, \\ 2 & \text{if } z = y, \\ 6 & \text{otherwise;} \end{cases} \quad z \mapsto \begin{cases} 4 & \text{if } z = x, \\ 2 & \text{if } z = y, \\ 6 & \text{otherwise.} \end{cases}$$

Then $\mathbf{a}\varphi_1 \in \{5, 6\}^+ \subseteq \{4, 5, 6\}$ and $\mathbf{b}\varphi_1 \in \{6\}^*$ by (a), so that

$$\mathbf{w}\varphi_1 = \mathbf{a}\varphi_1 \cdot 5 \cdot 2 \cdot \mathbf{b}\varphi_1 = 3 \quad \text{and} \quad \mathbf{w}'\varphi_1 = \dots 6 \cdot 2 \dots = 1,$$

and $\mathbf{a}'\varphi_2 \in \{4, 6\}^+ \subseteq \{4, 5, 6\}$ and $\mathbf{b}'\varphi_2 \in \{6\}^*$ by (b), so that

$$\mathbf{w}'\varphi_2 = \mathbf{a}'\varphi_2 \cdot 6 \cdot 2 \cdot \mathbf{b}'\varphi_2 = 3 \quad \text{and} \quad \mathbf{w}\varphi_2 = \dots 4 \cdot 2 \dots = 1.$$

Both of these are impossible. Hence $F_{LS}(\mathbf{w}) = F_{LS}(\mathbf{w}')$.

(vi) Suppose $F_{SS}(\mathbf{w}) \neq F_{SS}(\mathbf{w}')$, say $xy \in F_{SS}(\mathbf{w}) \setminus F_{SS}(\mathbf{w}')$. Then it is easily seen from part (ii) that $xy \in F_{LS}(x\mathbf{w}) \setminus F_{LS}(x\mathbf{w}')$. It then follows from part (v) that S does not satisfy the identity $x\mathbf{w} \approx x\mathbf{w}'$, which is impossible. \square

Let \mathbf{w} be any non-simple word with distinct non-simple letters x_1, \dots, x_r . In this subsection, such a word is said to be in *canonical form* if

$$\mathbf{w} = \prod_{i=1}^m (\mathbf{s}_i \mathbf{w}_i), \tag{17.3}$$

where all of the following are satisfied:

- (I) the letters of $\mathbf{s}_1 \in \mathcal{X}^*$ and $\mathbf{s}_2, \dots, \mathbf{s}_m \in \mathcal{X}^+$ are simple in \mathbf{w} ;
- (II) $\mathbf{w}_1, \dots, \mathbf{w}_{m-1} \in \mathcal{X}^+$ and $\mathbf{w}_m \in \mathcal{X}^*$ are such that

$$\mathbf{w}_1, \dots, \mathbf{w}_m \in \{y_1^2 \dots y_\ell^2 \mid y_1, \dots, y_\ell \text{ are distinct in } \{x_1, \dots, x_r\} \text{ with } \ell \geq 0\};$$

- (III) $\text{con}(\mathbf{w}_1) \supseteq \dots \supseteq \text{con}(\mathbf{w}_m)$.

LEMMA 17.7. *Let \mathbf{w} be any non-simple word. Then there exists some word $\bar{\mathbf{w}}$ in canonical form such that the identities (17.2) imply the identity $\mathbf{w} \approx \bar{\mathbf{w}}$.*

Proof. It suffices to convert \mathbf{w} , using the identities (17.2), into a word in canonical form. Clearly \mathbf{w} can be written in the form (17.3) with (I) satisfied, $\mathbf{w}_1, \dots, \mathbf{w}_{m-1} \in \{x_1, \dots, x_r\}^+$, and $\mathbf{w}_m \in \{x_1, \dots, x_r\}^*$. If $y \in \text{con}(\mathbf{w}_{i+1}) \setminus \text{con}(\mathbf{w}_i)$, say $\mathbf{w}_{i+1} = \mathbf{a}y\mathbf{b}$ for some $\mathbf{a}, \mathbf{b} \in \mathcal{X}^*$, then the identities (17.2) can be used to convert \mathbf{w}_i into $y\mathbf{w}_i$:

$$\begin{aligned} \mathbf{w} &= \dots \mathbf{s}_i \mathbf{w}_i \cdot \mathbf{s}_{i+1} \mathbf{w}_{i+1} \dots \stackrel{(17.2a)}{\approx} \dots \mathbf{s}_i (\mathbf{h}(\mathbf{w}_i))^2 \mathbf{w}_i \cdot \mathbf{s}_{i+1} \mathbf{a}y^2 \mathbf{b} \dots \\ &\stackrel{(17.2c)}{\approx} \dots \mathbf{s}_i y (\mathbf{h}(\mathbf{w}_i))^2 \mathbf{w}_i \cdot \mathbf{s}_{i+1} \mathbf{a}y\mathbf{b} \dots \stackrel{(17.2a)}{\approx} \dots \mathbf{s}_i (y\mathbf{w}_i) \cdot \mathbf{s}_{i+1} \mathbf{w}_{i+1} \dots \end{aligned}$$

Hence (III) is satisfied by applying the identities (17.2).

Choose any $k \in \{1, \dots, m\}$. It is convenient to let $\mathbf{p} = \prod_{i=1}^{k-1} (\mathbf{s}_i \mathbf{w}_i)$ and $\mathbf{q} = \prod_{i=k+1}^m (\mathbf{s}_i \mathbf{w}_i)$, so that $\mathbf{w} = \mathbf{p} \mathbf{s}_k \mathbf{w}_k \mathbf{q}$. Suppose $t = \mathbf{t}(\mathbf{w}_k)$ and $\text{con}(\mathbf{w}_k) = \{y_1, \dots, y_\ell, t\} \subseteq \{x_1, \dots, x_r\}$. Let φ denote the substitution $x \mapsto x^2$ for all $x \in \mathcal{X}$. Since the letters of \mathbf{w}_k are non-simple in \mathbf{w} ,

$$\begin{aligned} \mathbf{w} &\stackrel{(17.2a)}{\approx} \mathbf{p} \mathbf{s}_k (\mathbf{w}_k \varphi) \mathbf{q} \stackrel{(17.2b)}{\approx} \mathbf{p} \mathbf{s}_k y_1^{2e_1} \dots y_\ell^{2e_\ell} t^{2e} \mathbf{q} \quad \text{for some } e_1, \dots, e_\ell, e \geq 1 \\ &\stackrel{(17.2a)}{\approx} \mathbf{p} \mathbf{s}_k y_1^2 \dots y_\ell^2 t^2 \mathbf{q}. \end{aligned}$$

Hence \mathbf{w}_k satisfies (II). \square

LEMMA 17.8. *Let $S \in \{\mathcal{C}_{10}, \mathcal{D}_1\}$. Suppose that S satisfies an identity $\mathbf{w} \approx \mathbf{w}'$, where*

$$\mathbf{w} = \prod_{i=1}^m (\mathbf{s}_i \mathbf{w}_i) \quad \text{and} \quad \mathbf{w}' = \prod_{i=1}^m (\mathbf{s}_i \mathbf{w}'_i)$$

are in canonical form. Then $\text{con}(\mathbf{w}_i) = \text{con}(\mathbf{w}'_i)$ for all i .

Proof. First note that $\bigcup_{i=1}^m \text{con}(\mathbf{w}_i) = \bigcup_{i=1}^m \text{con}(\mathbf{w}'_i)$ holds by Lemma 17.6(ii). Suppose $\text{con}(\mathbf{w}_1) \neq \text{con}(\mathbf{w}'_1)$, say $x \in \text{con}(\mathbf{w}_1) \setminus \text{con}(\mathbf{w}'_1)$. Since the letter x is non-simple in \mathbf{w} by (II), it is also non-simple in \mathbf{w}' by Lemma 17.6(ii). Hence by (II), there exists some $i > 1$ such that $x \in \text{con}(\mathbf{w}'_i)$. But then (III) implies $x \in \text{con}(\mathbf{w}'_1)$, contradicting the assumption. Therefore $\text{con}(\mathbf{w}_1) = \text{con}(\mathbf{w}'_1)$.

Now suppose $\text{con}(\mathbf{w}_k) \neq \text{con}(\mathbf{w}'_k)$ for some $k > 1$, say $x \in \text{con}(\mathbf{w}_k) \setminus \text{con}(\mathbf{w}'_k)$. Then (III) implies $x \in \text{con}(\mathbf{w}_1 \dots \mathbf{w}_k)$ and $x \notin \text{con}(\mathbf{w}'_k \dots \mathbf{w}'_m)$, whence $x \not\prec_{\mathbf{w}} h(\mathbf{s}_k)$ and $x \prec_{\mathbf{w}'} h(\mathbf{s}_k)$. But this violates Lemma 17.6(iii). \square

LEMMA 17.9. Let $S \in \{\mathcal{C}_{10}, \mathcal{D}_1\}$. Suppose that S satisfies an identity $\mathbf{w} \approx \mathbf{w}'$, where

$$\mathbf{w} = \prod_{i=1}^m (\mathbf{s}_i \mathbf{w}_i) \quad \text{and} \quad \mathbf{w}' = \prod_{i=1}^m (\mathbf{s}_i \mathbf{w}'_i)$$

are in canonical form. Then the identities (17.2) imply the identity $\mathbf{w} \approx \mathbf{w}'$.

Proof. It suffices to show that for any $k \in \{1, \dots, m\}$, the identities (17.2) can be used to convert the factor \mathbf{w}_k of \mathbf{w} into \mathbf{w}'_k . For convenience, write

$$\mathbf{w} = \mathbf{p} \mathbf{s}_k \mathbf{w}_k \mathbf{q} \quad \text{and} \quad \mathbf{w}' = \mathbf{p}' \mathbf{s}_k \mathbf{w}'_k \mathbf{q}'$$

where $\mathbf{p} = \prod_{i=1}^{k-1} (\mathbf{s}_i \mathbf{w}_i)$, $\mathbf{q} = \prod_{i=k+1}^m (\mathbf{s}_i \mathbf{w}_i)$, $\mathbf{p}' = \prod_{i=1}^{k-1} (\mathbf{s}_i \mathbf{w}'_i)$, and $\mathbf{q}' = \prod_{i=k+1}^m (\mathbf{s}_i \mathbf{w}'_i)$. Then Lemma 17.8 and the assumption of this lemma imply

- (a) $\text{con}(\mathbf{w}_k) = \text{con}(\mathbf{w}'_k) \subseteq \{x_1, \dots, x_r\}$,
- (b) $\text{con}(\mathbf{q}) = \text{con}(\mathbf{q}')$,

and $\text{con}(\mathbf{p}) = \text{con}(\mathbf{p}')$. There are two cases.

Case 1: $t(\mathbf{w}_k) = t(\mathbf{w}'_k) = t$. By (a) and (II), there exist distinct $y_1, \dots, y_\ell \in \{x_1, \dots, x_r\}$ with $\ell \geq 0$ such that $\mathbf{w}_k = y_1^2 \dots y_\ell^2 t^2$ and $\mathbf{w}'_k = y_{1\pi}^2 \dots y_{\ell\pi}^2 t^2$, where π is some permutation on $\{1, \dots, \ell\}$. Then

$$\mathbf{w} = \mathbf{p} \mathbf{s}_k y_1^2 \dots y_\ell^2 t^2 \mathbf{q} \stackrel{(17.2b)}{\approx} \mathbf{p} \mathbf{s}_k y_{1\pi}^2 \dots y_{\ell\pi}^2 t^2 \mathbf{q} = \mathbf{p} \mathbf{s}_k \mathbf{w}'_k \mathbf{q}.$$

Case 2: $t(\mathbf{w}_k) = t \neq t' = t(\mathbf{w}'_k)$. Since $t(\mathbf{w}_m) = t(\mathbf{w}) = t(\mathbf{w}') = t(\mathbf{w}'_m)$ by Lemma 17.6(i), it follows that $k < m$ and $\mathbf{q} \neq \emptyset$. Recall that $F_{LS}(\mathbf{w}) = F_{LS}(\mathbf{w}')$ by Lemma 17.6(v). Hence if

- (c) the last occurrence of t in \mathbf{w}_k is the last occurrence of t in \mathbf{w} ,

so that $t h(\mathbf{s}_{k+1}) \in F_{LS}(\mathbf{w})$, then $t h(\mathbf{s}_{k+1}) \in F_{LS}(\mathbf{w}')$ is a contradiction because the letter in \mathbf{w}' that immediately precedes the simple letter $h(\mathbf{s}_{k+1})$ is $t' = t(\mathbf{w}'_k)$. By symmetry, the same contradiction is obtained if

- (d) the last occurrence of t' in \mathbf{w}'_k is the last occurrence of t' in \mathbf{w}' .

Therefore neither (c) nor (d) holds, whence $t, t' \in \text{con}(\mathbf{q})$ by (b). Now by (a) and (II), there exist distinct letters $y_1, \dots, y_\ell, z_1, \dots, z_s \in \{x_1, \dots, x_r\}$, where $\ell, s \geq 0$, such that

$$\mathbf{w}_k = y_1^2 \dots y_\ell^2 (t')^2 z_1^2 \dots z_s^2 t^2$$

and \mathbf{w}'_k is a product of $y_1^2, \dots, y_\ell^2, z_1^2, \dots, z_s^2, t^2, (t')^2$ in some order ending with $(t')^2$. Then

$$\begin{aligned} \mathbf{w} &= \mathbf{p} \mathbf{s}_k y_1^2 \dots y_\ell^2 (t')^2 z_1^2 \dots z_s^2 t^2 \mathbf{q} \stackrel{(17.2b)}{\approx} \mathbf{p} \mathbf{s}_k y_1^2 \dots y_\ell^2 z_1^2 \dots z_s^2 (t')^2 t^2 \mathbf{q} \\ &\stackrel{(17.2d)}{\approx} \mathbf{p} \mathbf{s}_k \underbrace{y_1^2 \dots y_\ell^2 z_1^2 \dots z_s^2 t^2 (t')^2}_{\mathbf{u}} \mathbf{q} \quad \text{because } t, t' \in \text{con}(\mathbf{q}). \end{aligned}$$

Since $\text{con}(\mathbf{u}) = \text{con}(\mathbf{w}'_k)$ and $t(\mathbf{u}) = t(\mathbf{w}'_k)$, the procedure in Case 1 can be repeated to give $\mathbf{p} \mathbf{s}_k \mathbf{u} \mathbf{q} \stackrel{(17.2)}{\approx} \mathbf{p} \mathbf{s}_k \mathbf{w}'_k \mathbf{q}$. \square

Proof of Proposition 17.5. Let $S \in \{\mathcal{C}_{10}, \mathcal{D}_1\}$. It is routinely checked that S satisfies the identities (17.2). Hence it suffices to show that any identity $\mathbf{w} \approx \mathbf{w}'$ satisfied by S is implied by the identities (17.2). If either \mathbf{w} or \mathbf{w}' is a simple word, then it follows from Lemma 17.6 parts (ii) and (iv) that the identity $\mathbf{w} \approx \mathbf{w}'$ is trivial and so is vacuously implied by (17.2).

Therefore assume that \mathbf{w} and \mathbf{w}' are both non-simple and so by Lemma 17.7, can be chosen to be in canonical form. Hence

$$\mathbf{w} = \prod_{i=1}^m (\mathbf{s}_i \mathbf{w}_i) \quad \text{and} \quad \mathbf{w}' = \prod_{i=1}^{m'} (\mathbf{s}'_i \mathbf{w}'_i).$$

Since $\mathbf{s}_1 \dots \mathbf{s}_m = \mathbf{w}_{\text{sim}} = \mathbf{w}'_{\text{sim}} = \mathbf{s}'_1 \dots \mathbf{s}'_{m'}$ by Lemma 17.6(iv), it follows from Lemma 17.6(vi) that $m = m'$ and $\mathbf{s}_i = \mathbf{s}'_i$ for all i , that is, $\mathbf{w}' = \prod_{i=1}^m (\mathbf{s}_i \mathbf{w}'_i)$. By Lemma 17.9, the identities (17.2) imply the identity $\mathbf{w} \approx \mathbf{w}'$. \square

18. \mathcal{C}_7

This section establishes the finite basis property of the following semigroup.

\mathcal{C}_7	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	1	1	1	1	2
3	1	1	1	1	1	3
4	1	1	2	1	4	1
5	1	1	3	1	5	1
6	1	2	1	4	1	6

PROPOSITION 18.1. *The variety generated by \mathcal{C}_7 is defined by the identities*

$$x^3 \approx x^2, \quad x^2yx \approx xyx, \quad xyx^2 \approx xyx, \tag{18.1a}$$

$$x^2y^2 \approx y^2x^2, \tag{18.1b}$$

$$hx^2yh \approx hyx^2h, \tag{18.1c}$$

$$hxyh \approx hyh, \tag{18.1d}$$

$$xHyKxTy \approx xHyKxTy, \tag{18.1e}$$

$$xHyKxTy \approx yxHyKxTy. \tag{18.1f}$$

The proof of Proposition 18.1 is given in § 18.3.

18.1. *A canonical form*

LEMMA 18.2. *Let $\mathbf{w} = \mathbf{s}_1 \mathbf{w}_1 \mathbf{s}_2$ with $\mathbf{s}_1, \mathbf{s}_2 \in \mathcal{X}^*$ and $\mathbf{w}_1 \in \mathcal{X}^+$. Suppose that the letters of \mathbf{w}_1 are non-simple in \mathbf{w} with $\text{con}(\mathbf{w}_1) = \{x_1, \dots, x_r\}$. Then for any permutation π on $\{1, \dots, r\}$, the identities (18.1) imply the identity*

$$\mathbf{s}_1 \mathbf{w}_1 \mathbf{s}_2 \approx \mathbf{s}_1 x_{1\pi}^2 \dots x_{r\pi}^2 \mathbf{s}_2. \tag{18.2}$$

Proof. Suppose $\mathbf{w}_1 = y_1 \dots y_m$ where $y_1, \dots, y_m \in \{x_1, \dots, x_r\}$. Then since y_1, \dots, y_m are non-simple letters in \mathbf{w} ,

$$\begin{aligned} \mathbf{s}_1 \mathbf{w}_1 \mathbf{s}_2 &\stackrel{(18.1a)}{\approx} \mathbf{s}_1 y_1^2 \dots y_m^2 \mathbf{s}_2 \stackrel{(18.1b)}{\approx} \mathbf{s}_1 x_{1\pi}^{2e_1} \dots x_{r\pi}^{2e_r} \mathbf{s}_2 \quad \text{for some } e_1, \dots, e_r \geq 1 \\ &\stackrel{(18.1a)}{\approx} \mathbf{s}_1 x_{1\pi}^2 \dots x_{r\pi}^2 \mathbf{s}_2. \end{aligned}$$

Therefore the identities (18.1) imply the identity (18.2). \square

COROLLARY 18.3. *Suppose that \mathbf{w} and \mathbf{w}' are any words such that $\text{con}(\mathbf{w}) = \text{con}(\mathbf{w}')$ and $\text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}') = \emptyset$. Then the identities (18.1) imply the identity $\mathbf{w} \approx \mathbf{w}'$.*

LEMMA 18.4. Let $\mathbf{w} = \mathbf{s}_1\mathbf{w}_1\mathbf{s}_2\mathbf{w}_2\mathbf{s}_3$ where the letters of $\mathbf{s}_1, \mathbf{s}_3 \in \mathcal{X}^*$ and $\mathbf{s}_2 \in \mathcal{X}^+$ are simple in \mathbf{w} and the letters of $\mathbf{w}_1, \mathbf{w}_2 \in \mathcal{X}^+$ are non-simple in \mathbf{w} . Suppose $\text{con}(\mathbf{w}_1) \cap \text{con}(\mathbf{w}_2) \neq \emptyset$. Then the identities (18.1) imply the identity

$$\mathbf{s}_1\mathbf{w}_1\mathbf{s}_2\mathbf{w}_2\mathbf{s}_3 \approx \mathbf{s}_1\mathbf{w}_1\mathbf{w}_2\mathbf{s}_2\mathbf{w}_1\mathbf{w}_2\mathbf{s}_3. \tag{18.3}$$

Proof. Generality is not lost by assuming

$$\text{con}(\mathbf{w}_1) = \{x_1, \dots, x_k, y_1, \dots, y_m\} \quad \text{and} \quad \text{con}(\mathbf{w}_2) = \{x_1, \dots, x_k, z_1, \dots, z_n\},$$

where $k \geq 1$ and $m, n \geq 0$ with $y_i \neq z_j$ for all i and j . For brevity, write $\mathbf{y} = y_1^2 \dots y_m^2$ and $\mathbf{z} = z_1^2 \dots z_n^2$. Then

$$\begin{aligned} \mathbf{s}_1\mathbf{w}_1\mathbf{s}_2\mathbf{w}_2\mathbf{s}_3 &\stackrel{(18.2)}{\approx} \mathbf{s}_1 \cdot x_1^2 \dots x_k^2 \mathbf{y} \cdot \mathbf{s}_2 \cdot x_1^2 \dots x_{k-1}^2 x_k^2 \mathbf{z} \cdot \mathbf{s}_3 \\ &\stackrel{(18.2)}{\approx} \mathbf{s}_1 \cdot \mathbf{y} x_1^2 \dots x_k^2 \mathbf{y}^2 \cdot \mathbf{s}_2 \cdot \mathbf{z} x_1^2 \dots x_{k-1}^2 x_k^2 \mathbf{z}^2 \cdot \mathbf{s}_3 \\ &\stackrel{(18.1c)}{\approx} \mathbf{s}_1 \cdot \mathbf{y} x_1^2 \dots x_k^2 \cdot \mathbf{s}_2 \cdot \mathbf{z} x_1^2 \dots x_{k-1}^2 \mathbf{y}^2 x_k^2 \mathbf{z}^2 \cdot \mathbf{s}_3 \\ &\stackrel{(18.1b)}{\approx} \mathbf{s}_1 \cdot \mathbf{y} x_1^2 \dots x_k^2 \cdot \mathbf{s}_2 \cdot \mathbf{z} x_1^2 \dots x_{k-1}^2 \mathbf{y}^2 \mathbf{z}^2 x_k^2 \cdot \mathbf{s}_3 \\ &\stackrel{(18.1c)}{\approx} \mathbf{s}_1 \cdot \mathbf{y} x_1^2 \dots x_k^2 \mathbf{z}^2 \cdot \mathbf{s}_2 \cdot \mathbf{z} x_1^2 \dots x_{k-1}^2 \mathbf{y}^2 x_k^2 \cdot \mathbf{s}_3 \\ &\stackrel{(18.2)}{\approx} \mathbf{s}_1 \cdot x_1^2 \dots x_k^2 \mathbf{y} \mathbf{z} \cdot \mathbf{s}_2 \cdot x_1^2 \dots x_k^2 \mathbf{y} \mathbf{z} \cdot \mathbf{s}_3 \\ &\stackrel{(18.2)}{\approx} \mathbf{s}_1\mathbf{w}_1\mathbf{w}_2\mathbf{s}_2\mathbf{w}_1\mathbf{w}_2\mathbf{s}_3. \end{aligned}$$

Therefore the identities (18.1) imply the identity (18.3). □

Let \mathbf{w} be any non-simple word such that $\text{sim}(\mathbf{w}) \neq \emptyset$ and $\text{h}(\mathbf{w}), \text{t}(\mathbf{w}) \notin \text{sim}(\mathbf{w})$, and let x_1, \dots, x_r be the distinct non-simple letters of \mathbf{w} listed in alphabetical order. In this section, such a word \mathbf{w} is said to be in *canonical form* if

$$\mathbf{w} = \mathbf{w}_0 \prod_{i=1}^m (\mathbf{s}_i \mathbf{w}_i)$$

for some $m \geq 1$, where the following are satisfied:

- (I) the letters of $\mathbf{s}_1, \dots, \mathbf{s}_m \in \mathcal{X}^+$ are simple in \mathbf{w} ;
- (II) $\mathbf{w}_0, \dots, \mathbf{w}_m \in \{x_{n_1}^2 \dots x_{n_k}^2 \mid 1 \leq n_1 < \dots < n_k \leq r \text{ and } k \geq 1\}$;
- (III) if $i \neq j$, then either $\mathbf{w}_i = \mathbf{w}_j$ or $\text{con}(\mathbf{w}_i) \cap \text{con}(\mathbf{w}_j) = \emptyset$;
- (IV) if $\mathbf{w}_i = \mathbf{w}_k$ and $\mathbf{w}_j = \mathbf{w}_\ell$ where $i < j < k < \ell$, then $\mathbf{w}_i = \mathbf{w}_j = \mathbf{w}_k = \mathbf{w}_\ell$.

LEMMA 18.5. Let \mathbf{w} be any non-simple word such that $\text{sim}(\mathbf{w}) \neq \emptyset$. Suppose that $\text{h}(\mathbf{w})$ and $\text{t}(\mathbf{w})$ are non-simple letters of \mathbf{w} . Then there exists some word $\bar{\mathbf{w}}$ in canonical form such that the identities (18.1) imply the identity $\mathbf{w} \approx \bar{\mathbf{w}}$.

Proof. Suppose that x_1, \dots, x_r are the distinct non-simple letters of \mathbf{w} listed in alphabetical order. By Lemmas 18.2 and 18.4, it suffices to convert \mathbf{w} , using the identities (18.1)–(18.3), into a word in canonical form. It is easily seen that \mathbf{w} can be written as a word of the form $\mathbf{w} = \mathbf{w}_0 \prod_{i=1}^m (\mathbf{s}_i \mathbf{w}_i)$ with (I) satisfied and $\mathbf{w}_i \in \{x_1, \dots, x_r\}^+$. The letters of \mathbf{w}_i are non-simple in \mathbf{w} , so that (II) is satisfied by applying the identity (18.2).

Suppose that the factors \mathbf{w}_i and \mathbf{w}_j , where $i < j$, do not satisfy (III). Then $\mathbf{w}_i \neq \mathbf{w}_j$ and $\text{con}(\mathbf{w}_i) \cap \text{con}(\mathbf{w}_j) \neq \emptyset$. Generality is not lost by assuming $\text{con}(\mathbf{w}_i \mathbf{w}_j) = \{x_{n_1}, \dots, x_{n_k}\}$ where

$1 \leq n_1 < \dots < n_k \leq r$. Since

$$\begin{aligned} \mathbf{w} &= \dots \mathbf{s}_i \mathbf{w}_i \dots \mathbf{s}_j \mathbf{w}_j \dots \quad (\text{the prefix } \dots \mathbf{s}_i \text{ is empty if } i = 0) \\ &\stackrel{(18.3)}{\approx} \dots \mathbf{s}_i \mathbf{w}_i \mathbf{w}_j \dots \mathbf{s}_j \mathbf{w}_i \mathbf{w}_j \dots \\ &\stackrel{(18.2)}{\approx} \dots \mathbf{s}_i (x_{n_1}^2 \dots x_{n_k}^2) \dots \mathbf{s}_j (x_{n_1}^2 \dots x_{n_k}^2) \dots, \end{aligned}$$

the factors \mathbf{w}_i and \mathbf{w}_j are converted by the identities $\{(18.2), (18.3)\}$ into the same word $x_{n_1}^2 \dots x_{n_k}^2$. Hence (III) is satisfied.

Suppose that $\mathbf{w}_i, \mathbf{w}_j, \mathbf{w}_k$, and \mathbf{w}_ℓ , where $i < j < k < \ell$, violate (IV). Then $\mathbf{w}_i = \mathbf{w}_k = \mathbf{a}$ and $\mathbf{w}_j = \mathbf{w}_\ell = \mathbf{b}$ with $\mathbf{a} \neq \mathbf{b}$. Generality is not lost by assuming $\text{con}(\mathbf{ab}) = \{x_{n_1}, \dots, x_{n_k}\}$ where $1 \leq n_1 < \dots < n_k \leq r$. Then

$$\begin{aligned} \mathbf{w} &= \dots \mathbf{s}_i \mathbf{w}_i \dots \mathbf{s}_j \mathbf{w}_j \dots \mathbf{s}_k \mathbf{w}_k \dots \mathbf{s}_\ell \mathbf{w}_\ell \dots \quad (\text{the prefix } \dots \mathbf{s}_i \text{ is empty if } i = 0) \\ &= \dots \mathbf{s}_i \mathbf{a} \dots \mathbf{s}_j \mathbf{b} \dots \mathbf{s}_k \mathbf{a} \dots \mathbf{s}_\ell \mathbf{b} \dots \\ &\stackrel{(18.1e)}{\approx} \dots \mathbf{s}_i \mathbf{a} \dots \mathbf{s}_j \mathbf{b} \dots \mathbf{s}_k \mathbf{a} \dots \mathbf{s}_\ell \mathbf{b} \mathbf{a} \dots \\ &\stackrel{(18.1f)}{\approx} \dots \mathbf{s}_i \mathbf{b} \mathbf{a} \dots \mathbf{s}_j \mathbf{b} \dots \mathbf{s}_k \mathbf{a} \dots \mathbf{s}_\ell \mathbf{b} \mathbf{a} \dots \\ &\stackrel{(18.3)}{\approx} \dots \mathbf{s}_i \mathbf{b} \mathbf{a} \mathbf{b} \dots \mathbf{s}_j \mathbf{b} \mathbf{a} \mathbf{b} \dots \mathbf{s}_k \mathbf{a} \dots \mathbf{s}_\ell \mathbf{b} \mathbf{a} \dots \quad \text{since } \text{con}(\mathbf{b} \mathbf{a}) \cap \text{con}(\mathbf{b}) \neq \emptyset \\ &\stackrel{(18.3)}{\approx} \dots \mathbf{s}_i \mathbf{b} \mathbf{a} \mathbf{b} \dots \mathbf{s}_j \mathbf{b} \mathbf{a} \mathbf{b} \dots \mathbf{s}_k \mathbf{a} \mathbf{b} \mathbf{a} \dots \mathbf{s}_\ell \mathbf{b} \mathbf{a} \mathbf{b} \dots \quad \text{since } \text{con}(\mathbf{a}) \cap \text{con}(\mathbf{b} \mathbf{a}) \neq \emptyset \\ &\stackrel{(18.2)}{\approx} \dots \mathbf{s}_i \mathbf{c} \dots \mathbf{s}_j \mathbf{c} \dots \mathbf{s}_k \mathbf{c} \dots \mathbf{s}_\ell \mathbf{c} \dots \quad \text{where } \mathbf{c} = x_{n_1}^2 \dots x_{n_k}^2. \end{aligned}$$

Hence the factors $\mathbf{w}_i, \mathbf{w}_j, \mathbf{w}_k$ and \mathbf{w}_ℓ are converted by the identities (18.1)–(18.3) into the same word \mathbf{c} , whence (IV) is no longer violated. \square

LEMMA 18.6. *Let \mathbf{w} be any product of pairwise disjoint connected words with $\text{sim}(\mathbf{w}) \neq \emptyset$. Then there exists some connected word $\bar{\mathbf{w}} = \mathbf{w}_0 \prod_{i=1}^m (\mathbf{s}_i \mathbf{w}_i)$ in canonical form, with $\mathbf{w}_0 = \mathbf{w}_m$, such that the identities (18.1) imply the identity $\mathbf{w} \approx \bar{\mathbf{w}}$.*

Proof. By Lemmas 18.2 and 18.4, it suffices to convert \mathbf{w} , using the identities (18.1)–(18.3), into a connected word in canonical form. First suppose that \mathbf{w} is a single connected word with $\text{sim}(\mathbf{w}) \neq \emptyset$. Then there exists a sequence $\mathbf{h}(\mathbf{w}) = x_1, x_2, \dots, x_m = \mathbf{t}(\mathbf{w})$ of non-simple letters of \mathbf{w} occurring in an overlapping pattern such that the first x_{i+1} occurs between two occurrences of x_i and the last x_{i+1} occurs to the right of all occurrences of x_i . This sequence can be chosen to have minimal length among all such sequences, so that

$$\mathbf{w} = x_1 \mathbf{p}_1 x_2 \mathbf{p}_2 x_1 \mathbf{q}_1 x_3 \mathbf{p}_3 x_2 x_1 \mathbf{q}_2 x_4 \mathbf{p}_4 x_3 \mathbf{q}_3 \dots \mathbf{q}_{m-1} x_m$$

for some $\mathbf{p}_1, \dots, \mathbf{p}_m, \mathbf{q}_1, \dots, \mathbf{q}_{m-1} \in \mathcal{X}^*$. (Note that the factor \mathbf{p}_i follows the first x_i , the factor \mathbf{q}_i follows the last x_i , and at least one simple letter of \mathbf{w} belongs to some \mathbf{p}_i or \mathbf{q}_i .) Then

$$\begin{aligned} \mathbf{w} &\stackrel{(18.1e)}{\approx} x_1 \mathbf{p}_1 x_2 \mathbf{p}_2 x_1 \mathbf{q}_1 x_3 \mathbf{p}_3 x_2 x_1 \mathbf{q}_2 x_4 \mathbf{p}_4 x_3 \mathbf{q}_3 \dots \mathbf{q}_{m-1} x_m \\ &\stackrel{(18.1e)}{\approx} x_1 \mathbf{p}_1 x_2 \mathbf{p}_2 x_1 \mathbf{q}_1 x_3 \mathbf{p}_3 x_2 x_1 \mathbf{q}_2 x_4 \mathbf{p}_4 x_3 x_1 \mathbf{q}_3 \dots \mathbf{q}_{m-1} x_m \\ &\quad \vdots \\ &\stackrel{(18.1e)}{\approx} \underbrace{x_1 \mathbf{p}_1 x_2 \mathbf{p}_2 x_1 \mathbf{q}_1 x_3 \mathbf{p}_3 x_2 x_1 \mathbf{q}_2 x_4 \mathbf{p}_4 x_3 x_1 \mathbf{q}_3 \dots \mathbf{q}_{m-1} x_m x_1}_{\mathbf{w}'} \end{aligned}$$

where \mathbf{w}' is a word that satisfies $\text{sim}(\mathbf{w}') \neq \emptyset$ and $\mathbf{h}(\mathbf{w}') = \mathbf{t}(\mathbf{w}')$. Repeat the proof of Lemma 18.5 to convert \mathbf{w}' , using the identities (18.1)–(18.3), into a word $\bar{\mathbf{w}} = \mathbf{w}_0 \prod_{i=1}^m (\mathbf{s}_i \mathbf{w}_i)$ in canonical form. The letter x_1 is easily seen to belong to both \mathbf{w}_0 and \mathbf{w}_m , so that $\mathbf{w}_0 = \mathbf{w}_m$ by (III). It follows that $\bar{\mathbf{w}}$ is connected.

Now suppose that \mathbf{w} is a product of pairwise disjoint connected words with $\text{sim}(\mathbf{w}) \neq \emptyset$. Then $\mathbf{w} = \mathbf{c}_1 \dots \mathbf{c}_n$ where $\mathbf{c}_1, \dots, \mathbf{c}_n$ are pairwise disjoint connected words. For each i , since \mathbf{c}_i is connected, the letters $h_i = \mathbf{h}(\mathbf{c}_i)$ and $t_i = \mathbf{t}(\mathbf{c}_i)$ are non-simple in \mathbf{c}_i . Therefore

$$\begin{aligned} \mathbf{w} &\stackrel{(18.1a)}{\approx} h_1^2 \mathbf{c}_1 t_1^2 \cdot h_2^2 \mathbf{c}_2 t_2^2 \cdot h_3^2 \mathbf{c}_3 t_3^2 \dots h_{n-1}^2 \mathbf{c}_{n-1} t_{n-1}^2 \cdot h_n^2 \mathbf{c}_n t_n^2 \\ &\stackrel{(18.1b)}{\approx} \underbrace{h_1^2 \mathbf{c}_1 h_2^2 \cdot t_1^2 \mathbf{c}_2 h_3^2 \cdot t_2^2 \mathbf{c}_3 h_4^2 \dots t_{n-2}^2 \mathbf{c}_{n-1} h_n^2 \cdot t_{n-1}^2 \mathbf{c}_n t_n^2}_{\mathbf{w}''}, \end{aligned}$$

that is, the word \mathbf{w} is converted by the identities (18.1) into the connected word \mathbf{w}'' with $\text{sim}(\mathbf{w}'') \neq \emptyset$. The procedure in the previous paragraph can then be repeated to convert \mathbf{w}'' into a word with the required properties. \square

18.2. Identities satisfied by \mathcal{C}_7

LEMMA 18.7. Suppose that $\mathbf{w} \approx \mathbf{w}'$ is any identity satisfied by \mathcal{C}_7 . Then:

- (i) $\text{con}(\mathbf{w}) = \text{con}(\mathbf{w}')$ and $\text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$;
- (ii) $F_{SS}(\mathbf{w}) = F_{SS}(\mathbf{w}')$.

Proof. (i) This follows from Lemma 2.1(v) because the subsemigroup $\{1, 2, 6\}$ of \mathcal{C}_7 is isomorphic to N_2^1 .

(ii) Since $6^2 \cdot 4 \cdot 6^2 \cdot 3 \cdot 6^2 \neq 6^2 \cdot 4 \cdot 3 \cdot 6^2$ in \mathcal{C}_7 , the identity (2.3) with $n = 2$ is not satisfied by \mathcal{C}_7 . Therefore $F_{SS}(\mathbf{w}) = F_{SS}(\mathbf{w}')$ by Lemma 2.10 and part (i). \square

For any word

$$\mathbf{w} = \mathbf{w}_0 \prod_{i=1}^m (\mathbf{s}_i \mathbf{w}_i) \tag{18.4}$$

in canonical form, it is convenient to call $\mathbf{s}_1, \dots, \mathbf{s}_m$ the *maximal simple factors* of \mathbf{w} , and call $\mathbf{w}_0, \dots, \mathbf{w}_m$ the *maximal non-simple factors* of \mathbf{w} . Recall from (III) that any two maximal non-simple factors of \mathbf{w} are either identical or disjoint.

For any distinct maximal non-simple factors \mathbf{a} and \mathbf{b} of the word \mathbf{w} in (18.4), write $\mathbf{a} \in_{\mathbf{w}} \mathbf{b}$ to indicate that within \mathbf{w} , all occurrences of \mathbf{a} are sandwiched between two occurrences of \mathbf{b} . Note that if $\mathbf{a} \in_{\mathbf{w}} \mathbf{b}$, then (IV) implies that no occurrence of \mathbf{b} is sandwiched between any two occurrences of \mathbf{a} .

LEMMA 18.8. Suppose that $\mathbf{w} \approx \mathbf{w}'$ is any identity satisfied by \mathcal{C}_7 , where

$$\mathbf{w} = \mathbf{w}_0 \prod_{i=1}^m (\mathbf{s}_i \mathbf{w}_i) \quad \text{and} \quad \mathbf{w}' = \mathbf{w}'_0 \prod_{i=1}^{m'} (\mathbf{s}'_i \mathbf{w}'_i)$$

are in canonical form. Then:

- (i) $m = m'$;
- (ii) $(\mathbf{s}_1, \dots, \mathbf{s}_m) = (\mathbf{s}'_{1\pi}, \dots, \mathbf{s}'_{m\pi})$ for some permutation π on $\{1, \dots, m\}$;
- (iii) $(\mathbf{w}_0, \dots, \mathbf{w}_m) = (\mathbf{w}'_{0\tau}, \dots, \mathbf{w}'_{m\tau})$ for some permutation τ on $\{0, \dots, m\}$;
- (iv) $\mathbf{s}_i \mathbf{w}_j$ is a factor of \mathbf{w} if and only if $\mathbf{s}_i \mathbf{w}_j$ is a factor of \mathbf{w}' ;
- (v) $\mathbf{w}_i \mathbf{s}_j$ is a factor of \mathbf{w} if and only if $\mathbf{w}_i \mathbf{s}_j$ is a factor of \mathbf{w}' .

Further, if the words \mathbf{w} and \mathbf{w}' are connected, then

$$(vi) \quad \mathbf{w}_0 = \mathbf{w}_m = \mathbf{w}'_0 = \mathbf{w}'_m.$$

Proof. Parts (i) and (ii) follow from Lemma 18.7.

Suppose that x^2 and y^2 are factors of different maximal non-simple factors of \mathbf{w} . Generality is not lost by assuming that x^2 is a factor of \mathbf{a} and y^2 is a factor of \mathbf{b} with $\mathbf{b} \notin_{\mathbf{w}} \mathbf{a}$. Let p be the least integer such that $\mathbf{w}_p = \mathbf{a}$ and let q be the greatest integer such that $\mathbf{w}_q = \mathbf{a}$. If $\mathbf{w}_i = \mathbf{b}$ for some i with $p < i < q$, then it follows from (IV) that all occurrences of \mathbf{b} are sandwiched between $\mathbf{w}_p = \mathbf{a}$ and $\mathbf{w}_q = \mathbf{a}$, whence the contradiction $\mathbf{b} \in_{\mathbf{w}} \mathbf{a}$ is deduced. Therefore $\mathbf{b} \notin \{\mathbf{w}_p, \dots, \mathbf{w}_q\}$, so that $\mathbf{b} \in \{\mathbf{w}_0, \dots, \mathbf{w}_{p-1}\} \cup \{\mathbf{w}_{q+1}, \dots, \mathbf{w}_m\}$, whence $(p, q) \neq (0, m)$. Let $\varphi_1 : \mathcal{X} \rightarrow \mathcal{C}_7$ denote the substitution

$$z \mapsto \begin{cases} 4 & \text{if } z = h(\mathbf{s}_p), \\ 3 & \text{if } z = t(\mathbf{s}_{q+1}), \\ 5 & \text{if } z \in \text{con}(\mathbf{s}_p \mathbf{w}_p \mathbf{s}_{p+1} \mathbf{w}_{p+1} \dots \mathbf{s}_q \mathbf{w}_q \mathbf{s}_{q+1}) \text{ and } z \neq h(\mathbf{s}_p), t(\mathbf{s}_{q+1}), \\ 6 & \text{otherwise.} \end{cases}$$

(Note that $\mathbf{s}_p = \emptyset$ if $p = 0$, and $\mathbf{s}_{q+1} = \emptyset$ if $q = m$.) If $p = 0$ and $q < m$, then

$$\begin{aligned} \mathbf{w}\varphi_1 &= \left(\mathbf{w}_0 \prod_{i=1}^q (\mathbf{s}_i \mathbf{w}_i) \right) \varphi_1 \cdot \mathbf{s}_{q+1} \varphi_1 \cdot \left(\mathbf{w}_{q+1} \prod_{i=q+2}^m (\mathbf{s}_i \mathbf{w}_i) \right) \varphi_1 \\ &= (5 \dots 5) \cdot (5 \dots 5 \cdot 3) \cdot (6 \dots 6) = 3. \end{aligned}$$

If $0 < p$ and $q = m$, then

$$\begin{aligned} \mathbf{w}\varphi_1 &= \left(\mathbf{w}_0 \prod_{i=1}^{p-1} (\mathbf{s}_i \mathbf{w}_i) \right) \varphi_1 \cdot \mathbf{s}_p \varphi_1 \cdot \left(\mathbf{w}_p \prod_{i=p+1}^m (\mathbf{s}_i \mathbf{w}_i) \right) \varphi_1 \\ &= (6 \dots 6) \cdot (4 \cdot 5 \dots 5) \cdot (5 \dots 5) = 4. \end{aligned}$$

If $0 < p$ and $q < m$, then

$$\begin{aligned} \mathbf{w}\varphi_1 &= \left(\mathbf{w}_0 \prod_{i=1}^{p-1} (\mathbf{s}_i \mathbf{w}_i) \right) \varphi_1 \cdot \mathbf{s}_p \varphi_1 \cdot \left(\mathbf{w}_p \prod_{i=p+1}^q (\mathbf{s}_i \mathbf{w}_i) \right) \varphi_1 \cdot \mathbf{s}_{q+1} \varphi_1 \cdot \left(\mathbf{w}_{q+1} \prod_{i=q+2}^m (\mathbf{s}_i \mathbf{w}_i) \right) \varphi_1 \\ &= (6 \dots 6) \cdot (4 \cdot 5 \dots 5) \cdot (5 \dots 5) \cdot (5 \dots 5 \cdot 3) \cdot (6 \dots 6) = 2. \end{aligned}$$

Therefore $\mathbf{w}\varphi_1 \neq 1$. Now it follows from Lemma 18.7(i), (I), and (II) that x^2 and y^2 are factors of \mathbf{w}' . If x^2 and y^2 are factors of the same maximal non-simple factor of \mathbf{w}' , say $\mathbf{w}'_k = \dots x^2 \dots y^2 \dots$, then

$$\mathbf{w}'\varphi_1 \in \dots \underbrace{\{5, 6\}^* \cdot 5^2 \cdot \{5, 6\}^* \cdot 6^2 \cdot \{5, 6\}^*}_{\mathbf{w}'_k \varphi_1} \dots = \{1\}$$

is a contradiction. Therefore x^2 and y^2 are factors of different maximal non-simple factors of \mathbf{w}' . By symmetry, if x^2 and y^2 are factors of different maximal non-simple factors of \mathbf{w}' , then they are factors of different maximal non-simple factors of \mathbf{w} . Consequently,

(a) x^2 and y^2 are factors of the same \mathbf{w}_i if and only if x^2 and y^2 are factors of the same \mathbf{w}'_j .

Let $x^2 y^2$ be a factor of some \mathbf{w}_i . Then $\mathbf{w}_i = \dots x^2 y^2 \dots$ and x alphabetically precedes y by (II). By (a), the words x^2 and y^2 are factors of some \mathbf{w}'_j . Suppose that $x^2 y^2$ is not a factor of \mathbf{w}'_j . Then $\mathbf{w}'_j = \dots x^2 \cdot z_1^2 z_2^2 \dots z_k^2 \cdot y^2 \dots$ for some letters z_1, z_2, \dots, z_k such that

$x, z_1, z_2, \dots, z_k, y$ are in alphabetical order. By (II), the letter z_1 cannot belong to \mathbf{w}_i , so that x^2 and z_1^2 are factors of different maximal non-simple factors of \mathbf{w} . But this contradicts (a) since x^2 and z_1^2 are factors of \mathbf{w}'_j . Therefore x^2y^2 is a factor of \mathbf{w}'_j . By symmetry, if x^2y^2 is a factor of some \mathbf{w}'_j , then it is a factor of some \mathbf{w}_i . Consequently, x^2y^2 is a factor of some \mathbf{w}_i if and only if x^2y^2 is a factor of some \mathbf{w}'_j . It then follows from Lemma 18.7(i), (I), and (II) that \mathbf{w} and \mathbf{w}' share the same set of maximal non-simple factors, that is,

$$(b) \bigcup_{i=0}^m \{\mathbf{w}_i\} = \bigcup_{i=0}^m \{\mathbf{w}'_i\}.$$

Let $\mathbf{s}_i\mathbf{a}$ be a factor of \mathbf{w} and $\mathbf{s}_i\mathbf{b}$ be a factor of \mathbf{w}' , where $\mathbf{a}, \mathbf{b} \in \bigcup_{i=0}^m \{\mathbf{w}_i\}$. Suppose $\mathbf{a} \neq \mathbf{b}$. Choose any letter $z \notin \text{con}(\mathbf{w}) = \text{con}(\mathbf{w}')$ that alphabetically precedes all of x_1, \dots, x_r . Let φ_2 denote the substitution $t(\mathbf{s}_i) \mapsto t(\mathbf{s}_i)z^2$. Then $\mathbf{w}\varphi_2 = \dots z^2\mathbf{a}\dots$ and $\mathbf{w}'\varphi_2 = \dots z^2\mathbf{b}\dots$. Following the proof of Lemma 18.6, the words $\mathbf{w}\varphi_2$ and $\mathbf{w}'\varphi_2$ can be converted by the identities (18.1) into words $\overline{\mathbf{w}\varphi_2}$ and $\overline{\mathbf{w}'\varphi_2}$ in canonical form. Since the identities (18.1) are satisfied by \mathcal{C}_7 , the identity $\overline{\mathbf{w}\varphi_2} \approx \overline{\mathbf{w}'\varphi_2}$ is also satisfied by \mathcal{C}_7 . It is easily seen that $\overline{\mathbf{w}\varphi_2}$ is obtained from \mathbf{w} by replacing all occurrences of \mathbf{a} with $z^2\mathbf{a}$, while $\overline{\mathbf{w}'\varphi_2}$ is obtained from \mathbf{w}' by replacing all occurrences of \mathbf{b} with $z^2\mathbf{b}$. It follows that $\overline{\mathbf{w}\varphi_2}$ and $\overline{\mathbf{w}'\varphi_2}$ do not share the same maximal non-simple factors, and this contradicts the result obtained in the previous paragraph. Therefore $\mathbf{a} = \mathbf{b}$. Consequently, for any $\mathbf{a} \in \bigcup_{i=0}^m \{\mathbf{w}_i\} = \bigcup_{i=0}^m \{\mathbf{w}'_i\}$,

(c) $\mathbf{s}_i\mathbf{a}$ is a factor of \mathbf{w} if and only if $\mathbf{s}_i\mathbf{a}$ is a factor of \mathbf{w}' .

By a symmetrical argument,

(d) $\mathbf{a}\mathbf{s}_i$ is a factor of \mathbf{w} if and only if $\mathbf{a}\mathbf{s}_i$ is a factor of \mathbf{w}' .

Parts (iii)–(v) now follow from (b)–(d).

Finally, suppose that \mathbf{w} and \mathbf{w}' are connected. Then $\mathbf{w}_0 = \mathbf{w}_m$ and $\mathbf{w}'_0 = \mathbf{w}'_m$ by Lemma 18.6. Suppose $\mathbf{a} = \mathbf{w}_0 = \mathbf{w}_m$ and $\mathbf{b} = \mathbf{w}'_0 = \mathbf{w}'_m$ with $\mathbf{a} \neq \mathbf{b}$. Then by picking any distinct letters $x, y, z \notin \text{con}(\mathbf{w}) = \text{con}(\mathbf{w}')$, the words $\tilde{\mathbf{w}} = xy\mathbf{w}zx$ and $\tilde{\mathbf{w}}' = xy\mathbf{w}'zx$ are in canonical form such that the identity $\tilde{\mathbf{w}} \approx \tilde{\mathbf{w}}'$ is satisfied by \mathcal{C}_7 . Now $y \in \text{sim}(\tilde{\mathbf{w}}) = \text{sim}(\tilde{\mathbf{w}}')$ and $y\mathbf{a}$ is a factor of $\tilde{\mathbf{w}}$ that is not a factor of $\tilde{\mathbf{w}}'$. But this is impossible by part (iv). Therefore $\mathbf{a} = \mathbf{b}$, so that part (vi) holds. \square

LEMMA 18.9. *The semigroup \mathcal{C}_7 has a basis of the form $\{(18.1)\} \cup \overline{\Sigma}$, where $\overline{\Sigma}$ is some set of identities formed by connected words in canonical form.*

Proof. It is easily verified that \mathcal{C}_7 satisfies the identities (18.1) and is idempotent-separable. Let \mathbf{V} denote the variety generated by \mathcal{C}_7 . Then $B_0 \in \mathbf{V}$ because the subsemigroup $\{1, 3, 5, 6\}$ of \mathcal{C}_7 is isomorphic to B_0 . The semigroup A_0 does not satisfy the identity (18.1b) because $a^2b^2 \neq b^2a^2$ in A_0 . Therefore $A_0 \notin \mathbf{V}$, and it follows from Lemma 2.6(ii) that \mathcal{C}_7 has a basis Σ that consists of identities formed by words that are products of pairwise disjoint connected words. Hence $\{(18.1)\} \cup \Sigma$ is also a basis for \mathcal{C}_7 .

Let $\mathbf{w} \approx \mathbf{w}'$ be any identity in Σ . Then $\text{con}(\mathbf{w}) = \text{con}(\mathbf{w}')$ and $\text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$ by Lemma 18.7(i). If $\text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}') = \emptyset$, then by Corollary 18.3, the identity $\mathbf{w} \approx \mathbf{w}'$ is implied by the identities (18.1) and hence is redundant in the basis of \mathcal{C}_7 . Therefore it can further be assumed that $\text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}') \neq \emptyset$. By Lemma 18.6, there exist connected words $\overline{\mathbf{w}}$ and $\overline{\mathbf{w}'}$ in canonical form such that the identities (18.1) imply the identities $\mathbf{w} \approx \overline{\mathbf{w}}$ and $\mathbf{w}' \approx \overline{\mathbf{w}'}$. Hence the sets $\{(18.1), \mathbf{w} \approx \mathbf{w}'\}$ and $\{(18.1), \overline{\mathbf{w}} \approx \overline{\mathbf{w}'}\}$ define the same variety.

Now since the identity $\mathbf{w} \approx \mathbf{w}'$ is arbitrary in Σ , the construction of $\overline{\mathbf{w}} \approx \overline{\mathbf{w}'}$ from $\mathbf{w} \approx \mathbf{w}'$ in the preceding paragraph can be repeated on every other identity in Σ to obtain the set $\overline{\Sigma} = \{\overline{\mathbf{w}} \approx \overline{\mathbf{w}'} \mid \mathbf{w} \approx \mathbf{w}' \in \Sigma\}$ with the property that $\{(18.1)\} \cup \overline{\Sigma}$ is a basis for \mathcal{C}_7 . \square

18.3. Proof of Proposition 18.1

By Lemma 18.9, the semigroup \mathcal{C}_7 has a basis of the form $\{(18.1)\} \cup \overline{\Sigma}$ where $\overline{\Sigma}$ is some set of identities formed by connected words in canonical form. Let $\mathbf{w} \approx \mathbf{w}'$ be any identity from $\overline{\Sigma}$.

Then by Lemma 18.8 parts (i)–(iii),

$$\mathbf{w} = \mathbf{w}_0 \prod_{i=1}^m (\mathbf{s}_i \mathbf{w}_i) \quad \text{and} \quad \mathbf{w}' = \mathbf{w}'_0 \prod_{i=1}^m (\mathbf{s}'_i \mathbf{w}'_i)$$

where:

- $(\mathbf{s}_1, \dots, \mathbf{s}_m) = (\mathbf{s}'_{1\pi}, \dots, \mathbf{s}'_{m\pi})$ for some permutation π on $\{1, \dots, m\}$;
- $(\mathbf{w}_0, \dots, \mathbf{w}_m) = (\mathbf{w}'_{0\tau}, \dots, \mathbf{w}'_{m\tau})$ for some permutation τ on $\{0, \dots, m\}$.

In the remainder of this proof, it is shown that the identities (18.1) can be used to convert \mathbf{w}' into \mathbf{w} . It follows that the identity $\mathbf{w} \approx \mathbf{w}'$ is implied by the identities (18.1). Consequently, the identities (18.1) imply every identity in $\bar{\Sigma}$ and so constitute a basis for \mathcal{C}_7 .

Since the words \mathbf{w} and \mathbf{w}' are connected, $\mathbf{w}_0 = \mathbf{w}'_0$ by Lemma 18.8(vi). Suppose that $\ell \geq 1$ is such that $(\mathbf{s}_1 \mathbf{w}_1, \dots, \mathbf{s}_{\ell-1} \mathbf{w}_{\ell-1}) = (\mathbf{s}'_1 \mathbf{w}'_1, \dots, \mathbf{s}'_{\ell-1} \mathbf{w}'_{\ell-1})$ and $\mathbf{s}_\ell \mathbf{w}_\ell \neq \mathbf{s}'_\ell \mathbf{w}'_\ell$. Then

$$\mathbf{w} = \mathbf{has}_\ell \mathbf{w}_\ell \dots \mathbf{s}_m \mathbf{w}_m \quad \text{and} \quad \mathbf{w}' = \mathbf{has}'_\ell \mathbf{w}'_\ell \dots \mathbf{s}'_m \mathbf{w}'_m \tag{18.5}$$

where $\mathbf{h} = \mathbf{w}_0 (\prod_{i=1}^{\ell-2} (\mathbf{s}_i \mathbf{w}_i)) \mathbf{s}_{\ell-1} = \mathbf{w}'_0 (\prod_{i=1}^{\ell-2} (\mathbf{s}'_i \mathbf{w}'_i)) \mathbf{s}'_{\ell-1}$ and $\mathbf{a} = \mathbf{w}_{\ell-1} = \mathbf{w}'_{\ell-1}$. (Note that if $\ell = 1$, then $\mathbf{h} = \emptyset$ and $\mathbf{a} = \mathbf{w}_0 = \mathbf{w}'_0$.) If $\mathbf{s}_\ell \neq \mathbf{s}'_\ell$, then it is shown in Lemma 18.10 below that the identities (18.1) can be used to convert \mathbf{w}' into a word $\mathbf{w}' = \mathbf{has}_\ell \dots$ in canonical form. Hence it can be assumed that $\mathbf{s}_\ell = \mathbf{s}'_\ell$. It then follows from Lemma 18.8(iv) that $\mathbf{w}_\ell = \mathbf{w}'_\ell$, whence $\mathbf{s}_\ell \mathbf{w}_\ell = \mathbf{s}'_\ell \mathbf{w}'_\ell$. The argument in this paragraph can be repeated until \mathbf{w}' is converted by the identities (18.1) into \mathbf{w} .

LEMMA 18.10. *Suppose $\mathbf{s}_\ell \neq \mathbf{s}'_\ell$ in (18.5). Then the identities (18.1) can be used to convert \mathbf{w}' in (18.5) into $\mathbf{has}_\ell \dots$ in canonical form.*

Proof. By assumption, $\mathbf{as}_\ell \mathbf{w}_\ell \neq \mathbf{as}'_\ell \mathbf{w}'_\ell$ where $\mathbf{as}_\ell \mathbf{w}_\ell$ is a factor of \mathbf{w} and $\mathbf{as}'_\ell \mathbf{w}'_\ell$ is a factor of \mathbf{w}' . Then it follows from Lemma 18.8 parts (iv) and (v) that:

- $\mathbf{as}'_\ell \mathbf{w}'_\ell$ is a factor of \mathbf{w} , say $\mathbf{as}'_\ell \mathbf{w}'_\ell = \mathbf{w}_p \mathbf{s}_{p+1} \mathbf{w}_{p+1}$ for some $p \geq \ell$;
- $\mathbf{as}_\ell \mathbf{w}_\ell$ is a factor of \mathbf{w}' , say $\mathbf{as}_\ell \mathbf{w}_\ell = \mathbf{w}'_q \mathbf{s}'_{q+1} \mathbf{w}'_{q+1}$ for some $q \geq \ell$.

Thus

$$\mathbf{w} = \mathbf{ha} \cdot \mathbf{s}_\ell \mathbf{w}_\ell \cdot \mathbf{s}_{\ell+1} \mathbf{w}_{\ell+1} \dots \mathbf{s}_p \overbrace{\mathbf{a} \cdot \mathbf{s}'_\ell \mathbf{w}'_\ell}^{\mathbf{w}_p \cdot \mathbf{s}_{p+1} \mathbf{w}_{p+1}} \cdot \mathbf{s}_{p+2} \mathbf{w}_{p+2} \dots \mathbf{s}_m \mathbf{w}_m$$

and

$$\mathbf{w}' = \mathbf{ha} \cdot \mathbf{s}'_\ell \mathbf{w}'_\ell \cdot \mathbf{s}'_{\ell+1} \mathbf{w}'_{\ell+1} \dots \mathbf{s}'_q \underbrace{\mathbf{a} \cdot \mathbf{s}_\ell \mathbf{w}_\ell}_{\mathbf{w}'_q \cdot \mathbf{s}'_{q+1} \mathbf{w}'_{q+1}} \cdot \mathbf{s}'_{q+2} \mathbf{w}'_{q+2} \dots \mathbf{s}'_m \mathbf{w}'_m.$$

Note that by (I),

(a) $\text{con}(\mathbf{h}) \cap \text{con}(\mathbf{s}_\ell \mathbf{s}_{\ell+1} \dots \mathbf{s}_p) = \emptyset$.

Further, the maximal non-simple factors $\mathbf{w}'_\ell, \mathbf{w}'_{\ell+1}, \dots, \mathbf{w}'_{q-1}$ of \mathbf{w}' are sandwiched between $\mathbf{w}'_{\ell-1} = \mathbf{a}$ and $\mathbf{w}'_q = \mathbf{a}$. Therefore by (IV),

(b) $\{\mathbf{w}'_\ell, \mathbf{w}'_{\ell+1}, \dots, \mathbf{w}'_{q-1}\} \cap \{\mathbf{w}'_{q+1}, \mathbf{w}'_{q+2}, \dots, \mathbf{w}'_m\}$ is either $\{\mathbf{a}\}$ or empty.

Suppose that

(†) \mathbf{a} is not equal to any of the factors $\mathbf{w}'_{q+1}, \mathbf{w}'_{q+2}, \dots, \mathbf{w}'_m$ of \mathbf{w}' .

By Lemma 18.8(iv), the factor $\mathbf{s}_p \mathbf{a}$ of \mathbf{w} is a factor of \mathbf{w}' . Hence by (a) and (†),

(c) \mathbf{s}_p is a factor of $\mathbf{s}'_\ell \mathbf{w}'_\ell \cdot \mathbf{s}'_{\ell+1} \mathbf{w}'_{\ell+1} \dots \mathbf{s}'_q \mathbf{w}'_q$.

By Lemma 18.8(v), the factor $\mathbf{w}_{p-1} \mathbf{s}_p$ of \mathbf{w} is a factor of \mathbf{w}' . Hence (c) implies that

(d) \mathbf{w}_{p-1} is a factor of $\mathbf{a} \cdot \mathbf{s}'_\ell \mathbf{w}'_\ell \cdot \mathbf{s}'_{\ell+1} \mathbf{w}'_{\ell+1} \dots \mathbf{s}'_q \mathbf{w}'_q$.

If $\mathbf{w}_{p-1} = \mathbf{a}$, then (†) implies that

(e) \mathbf{w}_{p-1} is not equal to any of $\mathbf{w}'_{q+1}, \mathbf{w}'_{q+2}, \dots, \mathbf{w}'_m$.

If $\mathbf{w}_{p-1} \neq \mathbf{a}$, then $\mathbf{a} = \mathbf{w}'_q$ and (d) imply $\mathbf{w}_{p-1} \in \{\mathbf{w}'_\ell, \mathbf{w}'_{\ell+1}, \dots, \mathbf{w}'_{q-1}\}$, which together with (b), imply (e). Therefore (e) holds in any case. By Lemma 18.8(iv), the factor $\mathbf{s}_{p-1}\mathbf{w}_{p-1}$ of \mathbf{w} is a factor of \mathbf{w}' . Hence (d) and (e) imply that the factor \mathbf{s}_{p-1} of \mathbf{w} is a factor of $\mathbf{s}'_\ell \mathbf{w}'_\ell \cdot \mathbf{s}'_{\ell+1} \mathbf{w}'_{\ell+1} \cdots \mathbf{s}'_q \mathbf{w}'_q$.

The argument in the previous paragraph can be repeated to deduce that the factors $\mathbf{s}_{p-2}, \mathbf{s}_{p-3}, \dots, \mathbf{s}_{\ell+1}$ of \mathbf{w} are factors of $\mathbf{s}'_\ell \mathbf{w}'_\ell \cdot \mathbf{s}'_{\ell+1} \mathbf{w}'_{\ell+1} \cdots \mathbf{s}'_q \mathbf{w}'_q$. It follows that $\mathbf{s}_\ell \mathbf{w}_\ell \mathbf{s}_{\ell+1}$ is not a factor of \mathbf{w}' . However, $\mathbf{s}_\ell \mathbf{w}_\ell \mathbf{s}_{\ell+1}$ is a factor of \mathbf{w} , whence Lemma 18.8 parts (iv) and (v) are violated. Therefore the assumption (†) cannot hold, whence $\mathbf{a} = \mathbf{w}'_j$ for some $j \in \{q + 1, q + 2, \dots, m\}$. Consequently,

$$\begin{aligned} \mathbf{w}' &= \mathbf{h}(\mathbf{a} \cdot \mathbf{s}'_\ell \mathbf{w}'_\ell \cdots \mathbf{s}'_q \cdot \mathbf{a} \cdot \mathbf{s}_\ell \mathbf{w}_\ell \cdots \mathbf{s}'_j \cdot \mathbf{a}) \mathbf{s}'_{j+1} \mathbf{w}'_{j+1} \cdots \mathbf{s}'_m \mathbf{w}'_m \\ &\stackrel{(18.1d)}{\approx} \mathbf{h}(\mathbf{a} \cdot \mathbf{s}_\ell \mathbf{w}_\ell \cdots \mathbf{s}'_j \cdot \mathbf{a} \cdot \mathbf{s}'_\ell \mathbf{w}'_\ell \cdots \mathbf{s}'_q \cdot \mathbf{a}) \mathbf{s}'_{j+1} \mathbf{w}'_{j+1} \cdots \mathbf{s}'_m \mathbf{w}'_m, \end{aligned}$$

where the latter word begins with \mathbf{has}_ℓ and is in canonical form. □

19. \mathcal{C}_8

This section establishes the finite basis property of the following semigroup.

\mathcal{C}_8	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	1	1	1	1	2
3	1	1	1	1	1	3
4	1	1	2	1	4	1
5	1	1	3	1	5	1
6	1	2	2	4	4	6

PROPOSITION 19.1. *The variety generated by \mathcal{C}_8 is defined by the identities*

$$x^3 \approx x^2, \quad x^2yx \approx xyx, \quad xyx^2 \approx xyx, \tag{19.1a}$$

$$hxhyh \approx hyhxx, \tag{19.1b}$$

$$xHyKxTy \approx xHxKxTy^2x, \tag{19.1c}$$

$$xHy^2x \approx y^2xHy^2x, \quad xHy^2xKx \approx xHy^2xKy^2x, \tag{19.1d}$$

$$xHyKxy \approx xHyKyx, \quad xHyKxy \approx yHxKxy, \tag{19.1e}$$

$$xHy^2xKy \approx xHy^2xKx, \quad yHxKy^2x \approx xHxKy^2x, \quad xHyKy^2x \approx xHxKy^2x. \tag{19.1f}$$

The proof of Proposition 19.1 is given in § 19.4.

19.1. \mathcal{Z} -perfect words

For any nonempty set $\mathcal{Z} = \{z_1, \dots, z_r\}$ of letters where z_1, \dots, z_r are in alphabetical order, define the *perfect \mathcal{Z} -square* to be the word

$$\mathcal{Z}^{\boxtimes} = (z_1 \dots z_r)^2.$$

More generally, a *perfect square* is a perfect \mathcal{Z} -square for some nonempty set \mathcal{Z} of letters.

Let \mathbf{w} be any non-simple word and let \mathcal{Z} be some nonempty set of non-simple letters of \mathbf{w} . Then \mathbf{w} can be written in the form

$$\mathbf{w} = \mathbf{a}_0 \prod_{i=1}^n (\mathbf{z}_i \mathbf{a}_i) = \mathbf{a}_0 \mathbf{z}_1 \mathbf{a}_1 \mathbf{z}_2 \mathbf{a}_2 \dots \mathbf{z}_n \mathbf{a}_n$$

for some $n \geq 1$, where $\mathbf{a}_0, \mathbf{a}_n \in \mathcal{X}^*$, $\mathbf{a}_1, \dots, \mathbf{a}_{n-1} \in \mathcal{X}^+$, and $\mathbf{z}_1, \dots, \mathbf{z}_n \in \mathcal{Z}^+$ are such that $\text{con}(\mathbf{a}_0 \dots \mathbf{a}_n) \cap \mathcal{Z} = \emptyset$. In this case, the words $\mathbf{z}_1, \dots, \mathbf{z}_n$ are called *maximal \mathcal{Z} -factors* of \mathbf{w} . If $\mathbf{z}_1 = \dots = \mathbf{z}_n = \mathcal{Z}^\boxtimes$, then \mathbf{w} is said to be *\mathcal{Z} -perfect*.

For any distinct non-simple letters x and y of a word \mathbf{w} , write $x \sim_{\mathbf{w}} y$ to indicate that any one of the following conditions holds:

- (C1) $\mathbf{w} \in \mathcal{X}^* x \mathcal{X}^* y x \mathcal{X}^*$ or $\mathbf{w} \in \mathcal{X}^* y \mathcal{X}^* x y \mathcal{X}^*$;
- (C2) $\mathbf{w} \in \mathcal{X}^* x \mathcal{X}^* y \mathcal{X}^* x \mathcal{X}^* y \mathcal{X}^*$ or $\mathbf{w} \in \mathcal{X}^* y \mathcal{X}^* x \mathcal{X}^* y \mathcal{X}^* x \mathcal{X}^*$.

Note that $x \sim_{\mathbf{w}} y$ if and only if $y \sim_{\mathbf{w}} x$.

LEMMA 19.2. *Let \mathbf{w} be any non-simple word. Suppose that z_1 and z_2 are any distinct non-simple letters of \mathbf{w} such that $z_1 \sim_{\mathbf{w}} z_2$. Then there exists some $\{z_1, z_2\}$ -perfect word \mathbf{w}' such that the identities (19.1) imply the identity $\mathbf{w} \approx \mathbf{w}'$.*

Proof. By assumption,

$$\mathbf{w} = \mathbf{a}_0 \prod_{i=1}^n (\mathbf{z}_i \mathbf{a}_i) = \mathbf{a}_0 \mathbf{z}_1 \mathbf{a}_1 \mathbf{z}_2 \mathbf{a}_2 \dots \mathbf{z}_n \mathbf{a}_n$$

for some $n \geq 1$, where $\mathbf{a}_0, \mathbf{a}_n \in \mathcal{X}^*$, $\mathbf{a}_1, \dots, \mathbf{a}_{n-1} \in \mathcal{X}^+$, and $\mathbf{z}_1, \dots, \mathbf{z}_n \in \{z_1, z_2\}^+$ are such that $z_1, z_2 \notin \text{con}(\mathbf{a}_0 \dots \mathbf{a}_n)$. Generality is not lost by assuming that z_1 alphabetically precedes z_2 , so that $\{z_1, z_2\}^\boxtimes = (z_1 z_2)^2$. Then it suffices to convert \mathbf{w} , using the identities (19.1), into the $\{z_1, z_2\}$ -perfect word

$$\mathbf{w}' = \mathbf{a}_0 \prod_{i=1}^n ((z_1 z_2)^2 \mathbf{a}_i).$$

There are two cases depending on whether (C1) or (C2) holds.

Case 1: (C1) holds. Then $\mathbf{w} \in \mathcal{X}^* z_j \mathcal{X}^* z_k z_j \mathcal{X}^*$ where $\{z_j, z_k\} = \{z_1, z_2\}$.

1.1. $n = 1$. Then

$$\mathbf{w} = \mathbf{a}_0 \underbrace{\mathbf{b} z_j \mathbf{c} z_k z_j \mathbf{d}}_{\mathbf{z}_1} \mathbf{a}_1$$

for some $\mathbf{b}, \mathbf{c}, \mathbf{d} \in \{z_1, z_2\}^*$. Hence

$$\begin{aligned} \mathbf{w} &\stackrel{(19.1a)}{\approx} \mathbf{a}_0 \mathbf{b} z_j \mathbf{c} z_k^2 z_j \mathbf{d} \mathbf{a}_1 \stackrel{(19.1f)}{\approx} \mathbf{a}_0 z_j^{|\mathbf{b}|} z_j z_j^{|\mathbf{c}|} z_k^2 z_j z_j^{|\mathbf{d}|} \mathbf{a}_1 \stackrel{(19.1a)}{\approx} \mathbf{a}_0 z_j z_k^2 z_j \mathbf{a}_1 \\ &\stackrel{(19.1e)}{\approx} \mathbf{a}_0 (z_1 z_2)^2 \mathbf{a}_1 = \mathbf{w}'. \end{aligned}$$

1.2. $n \geq 2$. Then

$$\mathbf{w} = \mathbf{a}_0 \left(\prod_{i=1}^{\ell-1} (\mathbf{z}_i \mathbf{a}_i) \right) \underbrace{\mathbf{b} z_k z_j \mathbf{c} \mathbf{a}_\ell}_{\mathbf{z}_\ell} \left(\prod_{i=\ell+1}^n (\mathbf{z}_i \mathbf{a}_i) \right)$$

for some $\mathbf{b}, \mathbf{c} \in \{z_1, z_2\}^*$ with $\mathbf{a}_0 \left(\prod_{i=1}^{\ell-1} (\mathbf{z}_i \mathbf{a}_i) \right) \mathbf{b} \in \mathcal{X}^* z_j \mathcal{X}^*$. Hence

$$\begin{aligned} \mathbf{w} &\stackrel{(19.1a)}{\approx} \mathbf{a}_0 \left(\prod_{i=1}^{\ell-1} (\mathbf{z}_i \mathbf{a}_i) \right) \mathbf{b} z_k^2 z_j \mathbf{c} \mathbf{a}_\ell \left(\prod_{i=\ell+1}^n (\mathbf{z}_i \mathbf{a}_i) \right) \\ &\stackrel{(19.1f)}{\approx} \mathbf{a}_0 \left(\prod_{i=1}^{\ell-1} (z_j^{|\mathbf{z}_i|} \mathbf{a}_i) \right) z_j^{|\mathbf{b}|} z_k^2 z_j z_j^{|\mathbf{c}|} \mathbf{a}_\ell \left(\prod_{i=\ell+1}^n (z_j^{|\mathbf{z}_i|} \mathbf{a}_i) \right) \end{aligned}$$

$$\begin{aligned}
 & \stackrel{(19.1d)}{\approx} \mathbf{a}_0 \left(\prod_{i=1}^{\ell-1} (z_k^2 z_j^{|\mathbf{z}_i|} \mathbf{a}_i) \right) z_j^{|\mathbf{b}|} z_k^2 z_j z_j^{|\mathbf{c}|} \mathbf{a}_\ell \left(\prod_{i=\ell+1}^n (z_k^2 z_j^{|\mathbf{z}_i|} \mathbf{a}_i) \right) \\
 & \stackrel{(19.1e)}{\approx} \mathbf{a}_0 \left(\prod_{i=1}^{\ell-1} (z_k^2 z_j^{|\mathbf{z}_i|} \mathbf{a}_i) \right) z_k^2 z_j^{|\mathbf{b}|+1+|\mathbf{c}|} \mathbf{a}_\ell \left(\prod_{i=\ell+1}^n (z_k^2 z_j^{|\mathbf{z}_i|} \mathbf{a}_i) \right) \\
 & \stackrel{(19.1a)}{\approx} \mathbf{a}_0 \left(\prod_{i=1}^{\ell-1} (z_k^2 z_j^2 \mathbf{a}_i) \right) z_k^2 z_j^2 \mathbf{a}_\ell \left(\prod_{i=\ell+1}^n (z_k^2 z_j^2 \mathbf{a}_i) \right) \\
 & \stackrel{(19.1e)}{\approx} \mathbf{a}_0 \left(\prod_{i=1}^{\ell-1} ((z_1 z_2)^2 \mathbf{a}_i) \right) (z_1 z_2)^2 \mathbf{a}_\ell \left(\prod_{i=\ell+1}^n ((z_1 z_2)^2 \mathbf{a}_i) \right) = \mathbf{w}'.
 \end{aligned}$$

Case 2: (C2) holds. Then $\mathbf{w} \in \mathcal{X}^* z_j \mathcal{X}^* z_k \mathcal{X}^* z_j \mathcal{X}^* z_k \mathcal{X}^*$ where $\{z_j, z_k\} = \{z_1, z_2\}$. Hence

$$\mathbf{w} = \dots z_j \dots z_k \dots z_j \dots z_k \dots \stackrel{(19.1c)}{\approx} \underbrace{\dots z_j \dots z_j \dots z_j \dots z_k^2 z_j \dots}_{\mathbf{u}},$$

where \mathbf{u} belongs to $\mathcal{X}^* z_j \mathcal{X}^* z_k z_j \mathcal{X}^*$. By Case 1, the word \mathbf{u} can be converted by the identities (19.1) into \mathbf{w}' . □

Let \mathbf{w} be any non-simple word and let \mathcal{Z} and \mathcal{Z}' be some disjoint nonempty sets of non-simple letters of \mathbf{w} . Then $\sim_{\mathbf{w}}$ can be generalized: for any perfect \mathcal{Z} -square \mathbf{x} and any perfect \mathcal{Z}' -square \mathbf{y} , write $\mathbf{x} \sim_{\mathbf{w}} \mathbf{y}$ whenever one of the following generalized conditions holds:

- (G1) $\mathbf{w} \in \mathcal{X}^* \mathbf{x} \mathcal{X}^* \mathbf{y} \mathcal{X}^*$ or $\mathbf{w} \in \mathcal{X}^* \mathbf{y} \mathcal{X}^* \mathbf{x} \mathcal{X}^*$;
- (G2) $\mathbf{w} \in \mathcal{X}^* \mathbf{x} \mathcal{X}^* \mathbf{y} \mathcal{X}^* \mathbf{x} \mathcal{X}^* \mathbf{y} \mathcal{X}^*$ or $\mathbf{w} \in \mathcal{X}^* \mathbf{y} \mathcal{X}^* \mathbf{x} \mathcal{X}^* \mathbf{y} \mathcal{X}^* \mathbf{x} \mathcal{X}^*$.

19.2. A canonical form

Let \mathbf{w} be any connected word and let x_1, \dots, x_t be the distinct non-simple letters of \mathbf{w} listed in alphabetical order. In this section, such a word \mathbf{w} is said to be in *canonical form* if some partition $\mathcal{Z}_1, \dots, \mathcal{Z}_r$ of $\{x_1, \dots, x_t\}$ exists and

$$\mathbf{w} = \mathbf{z}_0 \prod_{i=1}^n (\mathbf{s}_i \mathbf{z}_i) \tag{19.2}$$

for some $n \geq 0$ such that the following are satisfied:

- (I) the letters of $\mathbf{s}_1, \dots, \mathbf{s}_n \in \mathcal{X}^*$ are simple in \mathbf{w} ;
- (II) the word \mathbf{w} is \mathcal{Z}_i -perfect for each $i \in \{1, \dots, r\}$ and $\mathbf{z}_0, \dots, \mathbf{z}_n \in \{\mathcal{Z}_1^{\boxtimes}, \dots, \mathcal{Z}_r^{\boxtimes}\}$ (so that $\text{con}(\mathbf{z}_0 \dots \mathbf{z}_n) = \{x_1, \dots, x_t\}$);
- (III) if $\mathbf{z}_i \neq \mathbf{z}_j$, then $\mathbf{z}_i \not\sim_{\mathbf{w}} \mathbf{z}_j$;
- (IV) if $\mathbf{z}_i = \mathbf{z}_j$ with $i < j$, then $\mathbf{s}_j \neq \emptyset$;
- (V) $\mathbf{z}_0 = \mathbf{z}_n$.

Note that since $\mathcal{Z}_1, \dots, \mathcal{Z}_r$ form a partition of $\{x_1, \dots, x_t\}$, it follows from (II) that

- (VI) for any i and j , either $\mathbf{z}_i = \mathbf{z}_j$ or $\text{con}(\mathbf{z}_i) \cap \text{con}(\mathbf{z}_j) = \emptyset$.

In view of (I) and (II), it is convenient to call the words $\mathbf{s}_1, \dots, \mathbf{s}_n$ *maximal simple factors* of \mathbf{w} .

LEMMA 19.3. *Let \mathbf{w} be any connected word. Then there exists some word $\bar{\mathbf{w}}$ in canonical form such that the identities (19.1) imply the identity $\mathbf{w} \approx \bar{\mathbf{w}}$.*

Proof. It suffices to convert \mathbf{w} , using the identities (19.1), into a word in canonical form. Let x_1, \dots, x_t be the distinct non-simple letters of \mathbf{w} listed in alphabetical order. The proof is trivial if $t = 1$. Therefore assume $t \geq 2$.

Since \mathbf{w} is connected, there exist distinct non-simple letters $\mathbf{h}(\mathbf{w}) = x_{r_1}, x_{r_2}, \dots, x_{r_\ell} = \mathbf{t}(\mathbf{w})$ from $\{x_1, \dots, x_t\}$ occurring in \mathbf{w} in an overlapping pattern such that the first $x_{r_{i+1}}$ occurs between two occurrences of x_{r_i} and the last $x_{r_{i+1}}$ occurs to the right of all occurrences of x_{r_i} . This sequence can be chosen to have minimal length among all such sequences, so that

$$\mathbf{w} = x_{r_1} \mathbf{p}_1 x_{r_2} \mathbf{p}_2 x_{r_1} \mathbf{q}_1 x_{r_3} \mathbf{p}_3 x_{r_2} \mathbf{q}_2 x_{r_4} \mathbf{p}_4 x_{r_3} \mathbf{q}_3 \dots \mathbf{q}_{\ell-1} x_{r_\ell}$$

for some $\mathbf{p}_1, \dots, \mathbf{p}_\ell, \mathbf{q}_1, \dots, \mathbf{q}_{\ell-1} \in \mathcal{X}^*$. (Note that the factor \mathbf{p}_i follows the first x_{r_i} and the factor \mathbf{q}_i follows the last x_{r_i} .) Then $x_{r_1} \sim_{\mathbf{w}} x_{r_2} \sim_{\mathbf{w}} \dots \sim_{\mathbf{w}} x_{r_\ell}$ by (G2). By Lemma 19.2, the identities (19.1) can be used to convert \mathbf{w} into some $\{x_{r_1}, x_{r_2}\}$ -perfect word \mathbf{w}' . Note that \mathbf{w}' is obtained by replacing all maximal $\{x_{r_1}, x_{r_2}\}$ -factors of \mathbf{w} by the perfect square $\{x_{r_1}, x_{r_2}\}^{\boxtimes} = (x_{r_1}x_{r_2})^2$. It is clear that $\{x_{r_1}, x_{r_2}\}^{\boxtimes} \sim_{\mathbf{w}'} x_{r_3}$. Therefore the proof of Lemma 19.2 can be repeated to convert \mathbf{w}' , using the identities (19.1), into some $\{x_{r_1}, x_{r_2}, x_{r_3}\}$ -perfect word. Continuing in this manner, \mathbf{w} is converted by the identities (19.1) into an $\{x_{r_1}, \dots, x_{r_\ell}\}$ -perfect word. This can be further repeated until \mathbf{w} is converted by the identities (19.1) into a \mathcal{Z}_1 -perfect word, where \mathcal{Z}_1 is some subset of $\{x_1, \dots, x_t\}$ that contains $x_{r_1}, \dots, x_{r_\ell}$ and

(a) $\mathcal{Z}_1^{\boxtimes} \sim_{\mathbf{w}} x_i$ if and only if $x_i \in \mathcal{Z}_1$.

Hence it can be assumed that

$$\mathbf{w} = \mathcal{Z}_1^{\boxtimes} \mathbf{w}_1 \mathcal{Z}_1^{\boxtimes} \mathbf{w}_2 \dots \mathcal{Z}_1^{\boxtimes} \mathbf{w}_m \mathcal{Z}_1^{\boxtimes}, \quad (19.3)$$

where

(b) $\mathbf{w}_1, \dots, \mathbf{w}_m \in \mathcal{X}^+$ are such that $\text{con}(\mathbf{w}_1 \dots \mathbf{w}_m) \cap \mathcal{Z}_1 = \emptyset$.

If $x \in \text{con}(\mathbf{w}_i) \cap \text{con}(\mathbf{w}_j)$ for some distinct i and j , then $\mathcal{Z}_1^{\boxtimes} \sim_{\mathbf{w}} x$ by (G2), and this contradicts (a) and (b). If $\mathbf{t}(\mathbf{w}_i) \notin \text{sim}(\mathbf{w})$ for some i , then $\mathcal{Z}_1^{\boxtimes} \sim_{\mathbf{w}} \mathbf{t}(\mathbf{w}_i)$ by (G1), and this again contradicts (a) and (b). Therefore

(c) $\mathbf{w}_1, \dots, \mathbf{w}_m$ are pairwise disjoint and $\mathbf{t}(\mathbf{w}_1), \dots, \mathbf{t}(\mathbf{w}_m) \in \text{sim}(\mathbf{w})$.

If $x_i \sim_{\mathbf{w}} x_j$ for some $x_i, x_j \in \text{con}(\mathbf{w}_k)$ (so that $x_i, x_j \notin \mathcal{Z}_1$ by (c)), then by Lemma 19.2, the identities (19.1) can be used to convert \mathbf{w} into some $\{x_i, x_j\}$ -perfect word. This can be repeated until the word \mathbf{w} in (19.3) is converted into a \mathcal{Z}_2 -perfect word, where \mathcal{Z}_2 is some subset of $\{x_1, \dots, x_t\} \setminus \mathcal{Z}_1$ such that $\mathcal{Z}_2^{\boxtimes} \sim_{\mathbf{w}} x_i$ if and only if $x_i \in \mathcal{Z}_2$. (Note that by (c), all perfect \mathcal{Z}_2 -squares $\mathcal{Z}_2^{\boxtimes}$ are factors of some \mathbf{w}_k .) It is easily seen how this argument can be repeated until a partition $\mathcal{Z}_1, \dots, \mathcal{Z}_r$ of $\{x_1, \dots, x_t\}$ is obtained, and the word \mathbf{w} in (19.3) is converted by the identities (19.1) into a word $\bar{\mathbf{w}}$ that is simultaneously \mathcal{Z}_i -perfect for all i with $\mathcal{Z}_i^{\boxtimes} \not\sim_{\bar{\mathbf{w}}} \mathcal{Z}_j^{\boxtimes}$ whenever $i \neq j$.

Consequently, $\bar{\mathbf{w}}$ is of the form (19.2) with (I), (II), (III), and (V) satisfied. Suppose $\mathbf{z}_i = \mathbf{z}_j$ with $i < j$ and $\mathbf{s}_j = \emptyset$, so that $\bar{\mathbf{w}} = \dots \mathbf{s}_i \mathbf{z}_i \dots \mathbf{s}_{j-1} \mathbf{z}_{j-1} \cdot \mathbf{z}_i \dots$. If $i < j - 1$, then $\mathbf{z}_i \sim_{\bar{\mathbf{w}}} \mathbf{z}_{j-1}$ by (G1), which contradicts (III). Therefore $i = j - 1$, whence

$$\bar{\mathbf{w}} = \dots \mathbf{s}_i \mathbf{z}_i \cdot \mathbf{z}_i \dots \stackrel{(19.1a)}{\approx} \dots \mathbf{s}_i \mathbf{z}_i \dots,$$

that is, the factor \mathbf{z}_j is eliminated from $\bar{\mathbf{w}}$. Hence (IV) is satisfied. \square

LEMMA 19.4. *The semigroup \mathcal{C}_8 has a basis of the form $\{(19.1)\} \cup \Sigma$, where Σ is some set of identities formed by connected words in canonical form.*

Proof. It is routinely verified that \mathcal{C}_8 satisfies the identities (19.1) and is idempotent-separable. Since the subsemigroup $\{1, 4, 5, 6\}$ of \mathcal{C}_8 is isomorphic to A_0 , the variety generated by \mathcal{C}_8 contains A_0 . It follows from Lemma 2.6(i) that \mathcal{C}_8 has a basis Σ that consists of

identities formed by connected words. Therefore the set $\{(19.1)\} \cup \Sigma$ is also a basis for \mathcal{C}_8 . By Lemma 19.3, the words that form the identities in Σ can be chosen to be connected and in canonical form. \square

19.3. *Identities satisfied by \mathcal{C}_8*

Let \mathbf{x} and \mathbf{y} be any distinct perfect squares of a word \mathbf{w} in canonical form. Then write $\mathbf{x} \Subset_{\mathbf{w}} \mathbf{y}$ to indicate that within \mathbf{w} , all occurrences of \mathbf{x} are sandwiched between two occurrences of \mathbf{y} . Note that if $\mathbf{x} \Subset_{\mathbf{w}} \mathbf{y}$, then (III) implies that no occurrence of \mathbf{y} is sandwiched between any two occurrences of \mathbf{x} .

A generalized version of the order $\prec_{\mathbf{w}}$ introduced in §2 is also required: write $\mathbf{x} \prec_{\mathbf{w}} \mathbf{y}$ to indicate that within \mathbf{w} , each occurrence of \mathbf{x} precedes the first occurrence of \mathbf{y} .

REMARK 19.5. Suppose that \mathbf{x} and \mathbf{y} are distinct perfect squares of a word \mathbf{w} in canonical form. Then $\mathbf{x} \not\prec_{\mathbf{w}} \mathbf{y}$ by (III), whence by (G1) and (G2), precisely one of the following holds:

- (1 $\cancel{\neq}$) $\mathbf{x} \prec_{\mathbf{w}} \mathbf{y}$;
- (2 $\cancel{\neq}$) $\mathbf{y} \prec_{\mathbf{w}} \mathbf{x}$;
- (3 $\cancel{\neq}$) $\mathbf{x} \Subset_{\mathbf{w}} \mathbf{y}$ and \mathbf{xy} is not a factor of \mathbf{w} ;
- (4 $\cancel{\neq}$) $\mathbf{y} \Subset_{\mathbf{w}} \mathbf{x}$ and \mathbf{yx} is not a factor of \mathbf{w} .

For the remainder of this subsection, suppose that $\mathbf{w} \approx \mathbf{w}'$ is any identity satisfied by the semigroup \mathcal{C}_8 , where the words

$$\mathbf{w} = \mathbf{z}_0 \prod_{i=1}^n (\mathbf{s}_i \mathbf{z}_i) \quad \text{and} \quad \mathbf{w}' = \mathbf{z}'_0 \prod_{i=1}^{n'} (\mathbf{s}'_i \mathbf{z}'_i)$$

are in canonical form. Let

$$\mathbf{Z} = \bigcup_{i=0}^n \{\mathbf{z}_i\}, \quad \mathbf{S} = \bigcup_{i=1}^n \{\mathbf{s}_i\}, \quad \mathbf{Z}' = \bigcup_{i=0}^{n'} \{\mathbf{z}'_i\} \quad \text{and} \quad \mathbf{S}' = \bigcup_{i=1}^{n'} \{\mathbf{s}'_i\}.$$

Note that there are repetitions in $\mathbf{z}_0, \dots, \mathbf{z}_n$ and in $\mathbf{z}'_0, \dots, \mathbf{z}'_{n'}$, since $\mathbf{z}_0 = \mathbf{z}_n$ and $\mathbf{z}'_0 = \mathbf{z}'_{n'}$ by (V). But by (I), the only repetition that can possibly occur in \mathbf{S} and in \mathbf{S}' is when two or more words equal the empty word.

LEMMA 19.6. *The equality $\mathbf{S} = \mathbf{S}'$ holds.*

Proof. The subsemigroup $\{1, 2, 6\}$ of \mathcal{C}_8 is isomorphic to N_2^1 . Therefore $\text{con}(\mathbf{w}) = \text{con}(\mathbf{w}')$ and $\text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$ by Lemma 2.1(v). Further, \mathcal{C}_8 does not satisfy the identity (2.3) with $n = 2$ because $6^2 \cdot 4 \cdot 6^2 \cdot 3 \cdot 6^2 \neq 6^2 \cdot 4 \cdot 3 \cdot 6^2$, so that $F_{\mathbf{SS}}(\mathbf{w}) = F_{\mathbf{SS}}(\mathbf{w}')$ by Lemma 2.10. The present lemma thus follows. \square

LEMMA 19.7. *The equality $\mathbf{Z} = \mathbf{Z}'$ holds.*

Proof. Since $\text{con}(\mathbf{w}) = \text{con}(\mathbf{w}')$ and $\text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$ by the proof of Lemma 19.6,

(a) $\text{con}(\mathbf{z}_0 \dots \mathbf{z}_n) = \text{con}(\mathbf{z}'_0 \dots \mathbf{z}'_{n'})$ and $\text{con}(\mathbf{s}_1 \dots \mathbf{s}_n) = \text{con}(\mathbf{s}'_1 \dots \mathbf{s}'_{n'})$.

Let x and y be any distinct letters of some perfect square \mathbf{z}' from \mathbf{Z}' , say

$$\mathbf{z}' = (\dots x \dots y \dots)^2.$$

It is shown in the remainder of this proof that the letters x and y belong to the same perfect square in \mathbf{Z} . Hence

(b) if $x, y \in \text{con}(\mathbf{z}')$ for some $\mathbf{z}' \in \mathbf{Z}'$, then $x, y \in \text{con}(\mathbf{z})$ for some $\mathbf{z} \in \mathbf{Z}$.

The converse of (b) holds by symmetry. It then follows from (a), (II), and (VI) that $\mathbf{Z} = \mathbf{Z}'$.

Suppose that x and y are letters from distinct perfect squares $\mathbf{x}, \mathbf{y} \in Z$, that is, $\mathbf{x} \neq \mathbf{y}$ with $\mathbf{x} = (\dots x \dots)^2$ and $\mathbf{y} = (\dots y \dots)^2$. Then $\mathbf{x} \not\sim_{\mathbf{w}} \mathbf{y}$ by (III) and $\text{con}(\mathbf{x}) \cap \text{con}(\mathbf{y}) = \emptyset$ by (VI). As observed in Remark 19.5, both (3_{\neq}) and (4_{\neq}) cannot simultaneously hold. By symmetry, assume that (4_{\neq}) does not hold. Hence it follows from either (1_{\neq}) , (2_{\neq}) , or (3_{\neq}) that

(c) no occurrence of \mathbf{y} in \mathbf{w} is sandwiched between any two occurrences of \mathbf{x} .

Let p be the least integer such that $\mathbf{z}_p = \mathbf{x}$ and let q be the greatest integer such that $\mathbf{z}_q = \mathbf{x}$. Then

$$\mathbf{w} = \mathbf{z}_0 \underbrace{\left(\prod_{i=1}^{p-1} (\mathbf{s}_i \mathbf{z}_i) \right)}_{\mathbf{a}} \mathbf{s}_p \mathbf{x} \underbrace{\left(\prod_{i=p+1}^{q-1} (\mathbf{s}_i \mathbf{z}_i) \right)}_{\mathbf{b}} \mathbf{s}_q \mathbf{x} \underbrace{\left(\prod_{i=q+1}^n (\mathbf{s}_i \mathbf{z}_i) \right)}_{\mathbf{c}}.$$

By (V), either $p = 0$ or $q = n$ implies $\mathbf{z}_0 = \mathbf{z}_n = \mathbf{x}$, whence (c) is violated. It thus follows that $0 < p \leq q < n$ and $\mathbf{a} \neq \emptyset \neq \mathbf{c}$. Now $\text{con}(\mathbf{x}) \cap \text{con}(\mathbf{ac}) = \emptyset$ by the choice of p and q . Further, since (III) implies $\mathbf{x} \not\sim_{\mathbf{w}} \mathbf{h}$ for all $\mathbf{h} \in Z \setminus \{\mathbf{x}\}$, it follows from (I) and (G2) that

(d) $\text{con}(\mathbf{b}) \cap \text{con}(\mathbf{ac}) = \emptyset$.

Since $\mathbf{z}_0 = \mathbf{z}_n$ by (V), the sets $\{\mathbf{z}_0, \dots, \mathbf{z}_{p-1}\}$ and $\{\mathbf{z}_{q+1}, \dots, \mathbf{z}_n\}$ share some common element. Let $m \in \{0, \dots, p-1\}$ be the greatest integer such that

$$\mathbf{z}_m \in \{\mathbf{z}_0, \dots, \mathbf{z}_{p-1}\} \cap \{\mathbf{z}_{q+1}, \dots, \mathbf{z}_n\},$$

and let $r \in \{q+1, \dots, n\}$ be the least integer such that $\mathbf{z}_m = \mathbf{z}_r$. Then

$$\mathbf{w} = \mathbf{z}_0 \underbrace{\left(\prod_{i=1}^{m-1} (\mathbf{s}_i \mathbf{z}_i) \right)}_{\mathbf{a}} \mathbf{s}_m \mathbf{u} \underbrace{\left(\prod_{i=m+1}^{p-1} (\mathbf{s}_i \mathbf{z}_i) \right)}_{\mathbf{b}} \mathbf{s}_p \mathbf{b} \underbrace{\left(\prod_{i=q+1}^{r-1} (\mathbf{s}_i \mathbf{z}_i) \right)}_{\mathbf{c}'} \mathbf{s}_r \mathbf{u} \underbrace{\left(\prod_{i=r+1}^n (\mathbf{s}_i \mathbf{z}_i) \right)}_{\mathbf{c}''}$$

where $\mathbf{u} = \mathbf{z}_m = \mathbf{z}_r$ and $\mathbf{c} = \mathbf{c}' \mathbf{s}_r \mathbf{c}''$. Note that by (IV),

(e) $\mathbf{s}_r \neq \emptyset$.

Now the choice of m and r implies that:

(f) $\mathbf{u} \notin \{\mathbf{z}_{m+1}, \dots, \mathbf{z}_{p-1}\} \cup \{\mathbf{z}_{q+1}, \dots, \mathbf{z}_{r-1}\}$;

(g) $\{\mathbf{z}_{m+1}, \dots, \mathbf{z}_{p-1}\} \cap \{\mathbf{z}_{q+1}, \dots, \mathbf{z}_{r-1}\} = \emptyset$.

If there exists some $\mathbf{k} \in \{\mathbf{z}_0, \dots, \mathbf{z}_{m-1}\} \cap \{\mathbf{z}_{q+1}, \dots, \mathbf{z}_{r-1}\}$, then $\mathbf{k} \neq \mathbf{u}$ by (f) so that $\mathbf{k} \sim_{\mathbf{w}} \mathbf{u}$ by (G2). If there exists some $\mathbf{k}' \in \{\mathbf{z}_{q+1}, \dots, \mathbf{z}_{r-1}\} \cap \{\mathbf{z}_{r+1}, \dots, \mathbf{z}_n\}$, then $\mathbf{k}' \neq \mathbf{u}$ by (f) so that $\mathbf{k}' \sim_{\mathbf{w}} \mathbf{u}$ by (G2). But either of these contradicts (III). Thus \mathbf{k} and \mathbf{k}' do not exist, whence

$$\{\mathbf{z}_0, \dots, \mathbf{z}_{m-1}\} \cap \{\mathbf{z}_{q+1}, \dots, \mathbf{z}_{r-1}\} = \{\mathbf{z}_{q+1}, \dots, \mathbf{z}_{r-1}\} \cap \{\mathbf{z}_{r+1}, \dots, \mathbf{z}_n\} = \emptyset.$$

It then follows from (I), (d), (f), and (g) that

(h) $\text{con}(\mathbf{c}') \cap \text{con}(\mathbf{abs}_r \mathbf{c}'') = \emptyset$.

Let $\varphi : \mathcal{X} \rightarrow \mathcal{C}_8$ denote the substitution

$$z \mapsto \begin{cases} 5 & \text{if } z \in \text{con}(\mathbf{c}'), \\ 3 & \text{if } z = \mathbf{h}(\mathbf{s}_r), \\ 6 & \text{otherwise.} \end{cases}$$

Then by (d), (e), and (h),

$$\begin{aligned} \mathbf{w}\varphi &= \begin{cases} (\mathbf{ab})\varphi \cdot \mathbf{c}'\varphi \cdot \mathbf{s}_r\varphi \cdot \mathbf{c}''\varphi & \text{if } \mathbf{c}' \neq \emptyset \\ (\mathbf{ab})\varphi \cdot \mathbf{s}_r\varphi \cdot \mathbf{c}''\varphi & \text{if } \mathbf{c}' = \emptyset \end{cases} \\ &= \begin{cases} (6 \dots 6) \cdot (5 \dots 5) \cdot (3 \cdot 6 \dots 6) \cdot (6 \dots 6) & \text{if } \mathbf{c}' \neq \emptyset \\ (6 \dots 6) \cdot (3 \cdot 6 \dots 6) \cdot (6 \dots 6) & \text{if } \mathbf{c}' = \emptyset \end{cases} \\ &= 2. \end{aligned}$$

But $\mathbf{z}'\varphi \in (\{5, 6\}^* \cdot 6 \cdot \{5, 6\}^* \cdot 5 \cdot \{5, 6\}^*)^2 = \{1\}$ implies the contradiction $\mathbf{w}'\varphi = 1$. Therefore the assumption that the letters x and y belong to the distinct perfect squares $\mathbf{x}, \mathbf{y} \in Z$ does not hold. Consequently, the letters x and y belong to the same perfect square in Z . \square

COROLLARY 19.8. *Let φ be any endomorphism of \mathcal{X}^+ . Suppose that the words $\mathbf{w}\varphi$ and $\mathbf{w}'\varphi$ are converted by the identities (19.1) into the words $\overline{\mathbf{w}\varphi}$ and $\overline{\mathbf{w}'\varphi}$ in canonical form. Then the set of perfect squares of $\overline{\mathbf{w}\varphi}$ is equal to the set of perfect squares of $\overline{\mathbf{w}'\varphi}$.*

Proof. As observed in the proof of Lemma 19.4, the semigroup \mathcal{C}_8 satisfies the identities (19.1). Then by assumption, \mathcal{C}_8 satisfies the identity $\mathbf{w}\varphi \approx \mathbf{w}'\varphi$. Hence the result follows from Lemma 19.7. \square

LEMMA 19.9. *Suppose that $\mathbf{z} \in Z = Z'$ and $\mathbf{s} \in S = S'$ with $\mathbf{s} \neq \emptyset$. Then \mathbf{sz} is a factor of \mathbf{w} if and only if \mathbf{sz} is a factor of \mathbf{w}' .*

Proof. Let \mathbf{sz} be a factor of \mathbf{w} . Suppose that \mathbf{sz} is not a factor of \mathbf{w}' . Then there exists some $\mathbf{z}' \in Z = Z'$ with $\mathbf{z}' \neq \mathbf{z}$ such that \mathbf{sz}' is a factor of \mathbf{w}' . Hence $\mathbf{z} \not\sim_{\mathbf{w}} \mathbf{z}'$ and $\mathbf{z} \not\sim_{\mathbf{w}'} \mathbf{z}'$ by (III), and $\text{con}(\mathbf{z}) \cap \text{con}(\mathbf{z}') = \emptyset$ by (VI). Since $\mathbf{sz} = \mathbf{s}_k \mathbf{z}_k$ for some $k \in \{1, \dots, n\}$ and $\mathbf{sz}' = \mathbf{s}'_{k'} \mathbf{z}'_{k'}$ for some $k' \in \{1, \dots, n'\}$,

$$\mathbf{w} = \mathbf{z}_0 \left(\prod_{i=1}^{k-1} (\mathbf{s}_i \mathbf{z}_i) \right) \mathbf{sz} \left(\prod_{i=k+1}^n (\mathbf{s}_i \mathbf{z}_i) \right) \quad \text{and} \quad \mathbf{w}' = \mathbf{z}'_0 \left(\prod_{i=1}^{k'-1} (\mathbf{s}'_i \mathbf{z}'_i) \right) \mathbf{sz}' \left(\prod_{i=k'+1}^{n'} (\mathbf{s}'_i \mathbf{z}'_i) \right).$$

Choose any letter $x \notin \text{con}(\mathbf{w}\mathbf{w}')$ and let φ_1 denote the substitution $\mathbf{h}(\mathbf{s}) \mapsto x\mathbf{z}\mathbf{h}(\mathbf{s})$. Then \mathcal{C}_8 satisfies the identity $\mathbf{w}\varphi_1 \approx \mathbf{w}'\varphi_1$. It is easily seen that the set of perfect squares of

$$\mathbf{w}\varphi_1 = \mathbf{z}_0 \left(\prod_{i=1}^{k-1} (\mathbf{s}_i \mathbf{z}_i) \right) x\mathbf{z}\mathbf{s}\mathbf{z} \left(\prod_{i=k+1}^n (\mathbf{s}_i \mathbf{z}_i) \right)$$

is equal to the set Z of perfect squares of \mathbf{w} . Since (III) implies $\mathbf{z} \not\sim_{\mathbf{w}} \mathbf{x}$ for any $\mathbf{x} \in Z \setminus \{\mathbf{z}\}$, it is easily shown that $\mathbf{z} \not\sim_{\mathbf{w}\varphi_1} \mathbf{x}$ for any $\mathbf{x} \in Z \setminus \{\mathbf{z}\}$. Therefore the word $\mathbf{w}\varphi_1 = \overline{\mathbf{w}\varphi_1}$ is already in canonical form and so its set of perfect squares is equal to Z . Similarly, the set of perfect squares of

$$\mathbf{w}'\varphi_1 = \mathbf{z}'_0 \left(\prod_{i=1}^{k'-1} (\mathbf{s}'_i \mathbf{z}'_i) \right) x\mathbf{z}\mathbf{s}\mathbf{z}' \left(\prod_{i=k'+1}^{n'} (\mathbf{s}'_i \mathbf{z}'_i) \right)$$

is equal to the set Z of perfect squares of \mathbf{w}' . Suppose $\mathbf{z} \sim_{\mathbf{w}'\varphi_1} \mathbf{y}$ for some $\mathbf{y} \in Z \setminus \{\mathbf{z}\}$, so that (III) is violated by $\mathbf{w}'\varphi_1$. Then the word $\mathbf{w}'\varphi_1$ is not in canonical form and so by Lemma 19.3, can be converted by the identities (19.1) into some word $\overline{\mathbf{w}'\varphi_1}$ in canonical form. Since $\mathbf{z} \sim_{\mathbf{w}'\varphi_1} \mathbf{y}$, the conversion of $\mathbf{w}'\varphi_1$ into $\overline{\mathbf{w}'\varphi_1}$ will result in the letters of \mathbf{z} and \mathbf{y} being grouped into the same perfect square. It follows that the set Z of perfect squares of $\overline{\mathbf{w}'\varphi_1}$ is not equal to the set of perfect squares of $\overline{\mathbf{w}'\varphi_1}$, whence Corollary 19.8 is contradicted. Therefore \mathbf{y} does not exist, so that $\mathbf{z} \not\sim_{\mathbf{w}'\varphi_1} \mathbf{x}$ for all $\mathbf{x} \in Z \setminus \{\mathbf{z}\}$. Consequently, $\mathbf{w}'\varphi_1$ is in canonical form.

Now let φ_2 denote the substitution $\mathbf{t}(\mathbf{s}) \mapsto \mathbf{t}(\mathbf{s})\mathbf{z}'$. Then \mathcal{C}_8 satisfies $\mathbf{w}\varphi_1\varphi_2 \approx \mathbf{w}'\varphi_1\varphi_2$. Since

$$\mathbf{w}'\varphi_1\varphi_2 = \mathbf{z}'_0 \left(\prod_{i=1}^{k'-1} (\mathbf{s}'_i \mathbf{z}'_i) \right) x\mathbf{z}\mathbf{s}\mathbf{z}'\mathbf{z}' \left(\prod_{i=k'+1}^{n'} (\mathbf{s}'_i \mathbf{z}'_i) \right) \stackrel{(19.1a)}{\approx} \mathbf{w}'\varphi_1$$

where the word $\mathbf{w}'\varphi_1$ is in canonical form, the set of perfect squares of $\overline{\mathbf{w}'\varphi_1\varphi_2} = \mathbf{w}'\varphi_1$ is

equal to Z . On the other hand, the word

$$\mathbf{w}\varphi_1\varphi_2 = \mathbf{z}_0 \left(\prod_{i=1}^{k-1} (\mathbf{s}_i \mathbf{z}_i) \right) \mathbf{x} \mathbf{z} \mathbf{s} \mathbf{z}' \mathbf{z} \left(\prod_{i=k+1}^n (\mathbf{s}_i \mathbf{z}_i) \right)$$

is not in canonical form because $\mathbf{z} \sim_{\mathbf{w}\varphi_1\varphi_2} \mathbf{z}'$ by (G1). Therefore when the word $\mathbf{w}\varphi_1\varphi_2$ is converted by the identities (19.1) into the word $\overline{\mathbf{w}\varphi_1\varphi_2}$ in canonical form, the letters of \mathbf{z} and \mathbf{z}' are grouped into the same perfect square. It follows that the set of perfect squares of $\overline{\mathbf{w}\varphi_1\varphi_2}$ is not equal to the set Z of perfect squares of $\overline{\mathbf{w}'\varphi_1\varphi_2}$, whence Corollary 19.8 is contradicted. Consequently, the assumption that $\mathbf{z}\mathbf{s}$ is not a factor of \mathbf{w}' does not hold. \square

LEMMA 19.10. *Suppose that $\mathbf{z} \in Z = Z'$ and $\mathbf{s} \in S = S'$ with $\mathbf{s} \neq \emptyset$. Then $\mathbf{z}\mathbf{s}$ is a factor of \mathbf{w} if and only if $\mathbf{z}\mathbf{s}$ is a factor of \mathbf{w}' .*

Proof. Let $\mathbf{z}\mathbf{s}$ be a factor of \mathbf{w} . Suppose that $\mathbf{z}\mathbf{s}$ is not a factor of \mathbf{w}' . Then there exists some $\mathbf{z}' \in Z = Z'$ with $\mathbf{z}' \neq \mathbf{z}$ such that $\mathbf{z}'\mathbf{s}$ is a factor of \mathbf{w}' . Hence $\mathbf{z} \not\sim_{\mathbf{w}} \mathbf{z}'$ and $\mathbf{z} \not\sim_{\mathbf{w}'} \mathbf{z}'$ by (III), and $\text{con}(\mathbf{z}) \cap \text{con}(\mathbf{z}') = \emptyset$ by (VI). Now $\mathbf{z}\mathbf{s} = \mathbf{z}_k \mathbf{s}_{k+1}$ for some $k \in \{0, \dots, n-1\}$ and $\mathbf{z}'\mathbf{s} = \mathbf{z}'_{k'} \mathbf{s}'_{k'+1}$ for some $k' \in \{0, \dots, n'-1\}$. Further, $\mathbf{z}_{k+1} = \mathbf{z}'_{k'+1}$ by Lemma 19.9. Thus

$$\mathbf{w} = \mathbf{z}_0 \left(\prod_{i=1}^{k-1} (\mathbf{s}_i \mathbf{z}_i) \right) \mathbf{s}_k \cdot \mathbf{z} \mathbf{s} \cdot \mathbf{x} \left(\prod_{i=k+2}^n (\mathbf{s}_i \mathbf{z}_i) \right)$$

and

$$\mathbf{w}' = \mathbf{z}'_0 \left(\prod_{i=1}^{k'-1} (\mathbf{s}'_i \mathbf{z}'_i) \right) \mathbf{s}'_{k'} \cdot \mathbf{z}' \mathbf{s} \cdot \mathbf{x} \left(\prod_{i=k'+2}^{n'} (\mathbf{s}'_i \mathbf{z}'_i) \right),$$

where $\mathbf{x} = \mathbf{z}_{k+1} = \mathbf{z}'_{k'+1}$.

Suppose $\mathbf{z} \in \{\mathbf{z}'_0, \dots, \mathbf{z}'_{k'-1}\}$. Let φ_1 denote the substitution $\mathbf{h}(\mathbf{s}) \mapsto \mathbf{z} \mathbf{h}(\mathbf{s})$. Then \mathcal{C}_8 satisfies the identity $\mathbf{w}\varphi_1 \approx \mathbf{w}'\varphi_1$. Since

$$\mathbf{w}\varphi_1 = \mathbf{z}_0 \left(\prod_{i=1}^{k-1} (\mathbf{s}_i \mathbf{z}_i) \right) \mathbf{s}_k \cdot \mathbf{z} \mathbf{s} \cdot \mathbf{x} \left(\prod_{i=k+2}^n (\mathbf{s}_i \mathbf{z}_i) \right) \stackrel{(19.1a)}{\approx} \mathbf{w},$$

where the word \mathbf{w} is in canonical form, the set of perfect squares of $\overline{\mathbf{w}\varphi_1} = \mathbf{w}$ is equal to Z . On the other hand,

$$\mathbf{w}'\varphi_1 = \mathbf{z}'_0 \left(\prod_{i=1}^{k'-1} (\mathbf{s}'_i \mathbf{z}'_i) \right) \mathbf{s}'_{k'} \cdot \mathbf{z}' \mathbf{s} \cdot \mathbf{x} \left(\prod_{i=k'+2}^{n'} (\mathbf{s}'_i \mathbf{z}'_i) \right)$$

and $\mathbf{z} \in \{\mathbf{z}'_0, \dots, \mathbf{z}'_{k'-1}\}$ imply $\mathbf{z} \sim_{\mathbf{w}'\varphi_1} \mathbf{z}'$ by (G1). Hence the word $\overline{\mathbf{w}'\varphi_1}$ is not in canonical form, and when it is converted by the identities (19.1) into the word $\mathbf{w}'\varphi_1$ in canonical form, the letters of \mathbf{z} and \mathbf{z}' are grouped into the same perfect square. It follows that the set of perfect squares of $\overline{\mathbf{w}'\varphi_1}$ is not equal to the set Z of perfect squares of $\overline{\mathbf{w}\varphi_1}$, whence Corollary 19.8 is contradicted. Thus $\mathbf{z} \notin \{\mathbf{z}'_0, \dots, \mathbf{z}'_{k'-1}\}$. A symmetrical argument shows $\mathbf{z}' \notin \{\mathbf{z}_0, \dots, \mathbf{z}_{k-1}\}$.

Suppose $\mathbf{x} = \mathbf{z}'$. Let φ_2 denote the substitution $\mathbf{t}(\mathbf{s}) \mapsto \mathbf{t}(\mathbf{s})\mathbf{z}$. Then

$$\mathbf{w}\varphi_2 = \mathbf{z}_0 \left(\prod_{i=1}^{k-1} (\mathbf{s}_i \mathbf{z}_i) \right) \mathbf{s}_k \cdot \mathbf{z} \mathbf{s} \mathbf{z} \cdot \mathbf{z}' \left(\prod_{i=k+2}^n (\mathbf{s}_i \mathbf{z}_i) \right).$$

It is clear that the set of perfect squares of $\mathbf{w}\varphi_2$ is equal to Z . Further, $\mathbf{z} \not\sim_{\mathbf{w}\varphi_2} \mathbf{z}'$ because $\mathbf{z}' \notin \{\mathbf{z}_0, \dots, \mathbf{z}_{k-1}\}$ and $\mathbf{z} \not\sim_{\mathbf{w}} \mathbf{z}'$. It follows that the word $\mathbf{w}\varphi_2 = \overline{\mathbf{w}\varphi_2}$ is in canonical form and its set of perfect squares is equal to Z . On the other hand, the word

$$\mathbf{w}'\varphi_2 = \mathbf{z}'_0 \left(\prod_{i=1}^{k'-1} (\mathbf{s}'_i \mathbf{z}'_i) \right) \mathbf{s}'_{k'} \cdot \mathbf{z}' \mathbf{s} \mathbf{z} \cdot \mathbf{z}' \left(\prod_{i=k'+2}^{n'} (\mathbf{s}'_i \mathbf{z}'_i) \right)$$

is not in canonical form because $\mathbf{z} \sim_{\mathbf{w}'\varphi_2} \mathbf{z}'$ by (G1). Therefore when the word $\mathbf{w}'\varphi_2$ is converted by the identities (19.1) into the word $\overline{\mathbf{w}'\varphi_2}$ in canonical form, the letters of \mathbf{z} and \mathbf{z}' are grouped into the same perfect square. It follows that the set of perfect squares of $\overline{\mathbf{w}'\varphi_2}$ is not equal to the set Z of perfect squares of $\overline{\mathbf{w}\varphi_2}$, whence Corollary 19.8 is contradicted. Hence $\mathbf{x} \neq \mathbf{z}'$. A symmetrical argument shows $\mathbf{x} \neq \mathbf{z}$.

Now the word \mathbf{z}' is a perfect square of \mathbf{w} because $\mathbf{z}' \in Z' = Z$. But since it was shown that $\mathbf{z}' \notin \{\mathbf{z}_0, \dots, \mathbf{z}_{k-1}\}$ and $\mathbf{z}' \neq \mathbf{x}$, each factor \mathbf{z}' of \mathbf{w} is one of $\mathbf{z}_{k+2}, \dots, \mathbf{z}_n$. Let $p \in \{k+2, \dots, n\}$ be the least such that $\mathbf{z}_p = \mathbf{z}'$ and let $q \in \{k+2, \dots, n\}$ be the greatest such that $\mathbf{z}_q = \mathbf{z}'$. Then

$$\mathbf{w} = \underbrace{\mathbf{z}_0 \left(\prod_{i=1}^{k-1} (\mathbf{s}_i \mathbf{z}_i) \right) \mathbf{s}_k \mathbf{z} \mathbf{s}}_{\mathbf{a}} \underbrace{\mathbf{x} \left(\prod_{i=k+2}^{p-1} (\mathbf{s}_i \mathbf{z}_i) \right) \mathbf{s}_p}_{\mathbf{b}} \underbrace{\mathbf{z}' \left(\prod_{i=p+1}^{q-1} (\mathbf{s}_i \mathbf{z}_i) \right) \mathbf{s}_q \mathbf{z}'}_{\mathbf{c}} \underbrace{\left(\prod_{i=q+1}^n (\mathbf{s}_i \mathbf{z}_i) \right)}_{\mathbf{d}}$$

where

(a) any perfect square \mathbf{z}' of \mathbf{w} is a factor of \mathbf{c} .
 Further, it follows from (G2), (I), and (III) that

(b) $\text{con}(\mathbf{c}) \cap \text{con}(\mathbf{abd}) = \emptyset$.

Suppose that the factors \mathbf{b} and \mathbf{d} share some common perfect square. Then there exists the least $r \in \{q+1, \dots, n\}$ such that $\mathbf{z}_r \in \{\mathbf{x}, \mathbf{z}_{k+2}, \dots, \mathbf{z}_{p-1}\}$, whence $\mathbf{s}_r \neq \emptyset$ by (IV). Since $\{\mathbf{z}_{q+1}, \dots, \mathbf{z}_{r-1}\} \cap \{\mathbf{x}, \mathbf{z}_{k+2}, \dots, \mathbf{z}_{p-1}\} = \emptyset$ by the minimality of r , it follows from (I) that

(c) $\text{con}(\mathbf{b}) \cap \text{con}(\prod_{i=q+1}^{r-1} (\mathbf{s}_i \mathbf{z}_i)) = \emptyset$.

Further, since $\mathbf{z}_r \in \{\mathbf{x}, \mathbf{z}_{k+2}, \dots, \mathbf{z}_{p-1}\}$, it follows from (G2), (I), and (III) that

(d) $\text{con}(\mathbf{a}) \cap \text{con}(\prod_{i=q+1}^{r-1} (\mathbf{s}_i \mathbf{z}_i)) = \text{con}(\prod_{i=q+1}^{r-1} (\mathbf{s}_i \mathbf{z}_i)) \cap \text{con}(\prod_{i=r+1}^n (\mathbf{s}_i \mathbf{z}_i)) = \emptyset$.

Let $\varphi_3 : \mathcal{X} \rightarrow \mathcal{C}_8$ denote the substitution

$$z \mapsto \begin{cases} 5 & \text{if } z \in \text{con} \left(\mathbf{c} \prod_{i=q+1}^{r-1} (\mathbf{s}_i \mathbf{z}_i) \right), \\ 3 & \text{if } z = \mathbf{h}(\mathbf{s}_r), \\ 6 & \text{otherwise.} \end{cases}$$

Then by (a)–(d),

$$\begin{aligned} \mathbf{w}\varphi_3 &= (\mathbf{ab})\varphi_3 \cdot \left(\mathbf{c} \prod_{i=q+1}^{r-1} (\mathbf{s}_i \mathbf{z}_i) \right) \varphi_3 \cdot \mathbf{s}_r \varphi_3 \cdot \left(\mathbf{z}_r \prod_{i=r+1}^n (\mathbf{s}_i \mathbf{z}_i) \right) \varphi_3 \\ &= (6 \dots 6) \cdot (5 \dots 5) \cdot (3 \cdot 6 \dots 6) \cdot (6 \dots 6) = 2. \end{aligned}$$

But $(\mathbf{z}'\mathbf{s})\varphi_3 = (5 \dots 5) \cdot (6 \dots 6) = 1$ so that $\mathbf{w}'\varphi_3 = 1$, which is impossible. Hence \mathbf{b} and \mathbf{d} do not share any common perfect squares. It then follows from (I) that

(e) $\text{con}(\mathbf{b}) \cap \text{con}(\mathbf{d}) = \emptyset$.

Now since $\mathbf{z}_0 = \mathbf{z}_n$ by (V), the factors \mathbf{a} and \mathbf{d} share at least one common perfect square. Let $t \in \{q+1, \dots, n\}$ be the least such that $\mathbf{z}_t \in \{\mathbf{z}_0, \dots, \mathbf{z}_{k-1}, \mathbf{z}\}$. Then $\mathbf{s}_t \neq \emptyset$ by (IV). Since

$\{z_{q+1}, \dots, z_{t-1}\} \cap \{z_0, \dots, z_{k-1}, z\} = \emptyset$ by the minimality of t , it follows from (I) that

(f) $\text{con}(\mathbf{a}) \cap \text{con}(\prod_{i=q+1}^{t-1} (s_i z_i)) = \emptyset$.

Let $\varphi_4 : \mathcal{X} \rightarrow \mathcal{C}_8$ denote the substitution

$$z \mapsto \begin{cases} 5 & \text{if } z \in \text{con}\left(\mathbf{c} \prod_{i=q+1}^{t-1} (s_i z_i)\right), \\ 3 & \text{if } z = h(s_t), \\ 6 & \text{otherwise.} \end{cases}$$

Then by (a), (b), (e), and (f),

$$\begin{aligned} \mathbf{w}\varphi_4 &= (\mathbf{ab})\varphi_4 \cdot \left(\mathbf{c} \prod_{i=q+1}^{t-1} (s_i z_i)\right)\varphi_4 \cdot s_t\varphi_4 \cdot \left(z_t \prod_{i=t+1}^n (s_i z_i)\right)\varphi_4 \\ &= (6 \dots 6) \cdot (5 \dots 5) \cdot (3 \cdot 6 \dots 6) \cdot (6 \dots 6) = 2. \end{aligned}$$

But $(z's)\varphi_4 = (5 \dots 5) \cdot (6 \dots 6) = 1$ so that $\mathbf{w}'\varphi_4 = 1$, which is impossible. Consequently, the assumption that \mathbf{zs} is not a factor of \mathbf{w}' does not hold. \square

LEMMA 19.11. *Let $\mathbf{x}, \mathbf{y} \in Z = Z'$ with $\mathbf{x} \neq \mathbf{y}$. Then $\mathbf{x} \in_w \mathbf{y}$ if and only if $\mathbf{x} \in_{w'} \mathbf{y}$.*

Proof. By symmetry, it suffices to show $\mathbf{x} \in_w \mathbf{y}$ implies $\mathbf{x} \in_{w'} \mathbf{y}$. Suppose $\mathbf{x} \in_w \mathbf{y}$. Then all occurrences of \mathbf{x} in \mathbf{w} are sandwiched between two occurrences of \mathbf{y} . Hence there exist p and q with $p \leq q$ such that $\mathbf{z}_p = \mathbf{z}_q = \mathbf{x}$, and any occurrence of \mathbf{x} in \mathbf{w} is one of $\mathbf{z}_p, \dots, \mathbf{z}_q$. Let $k \in \{0, \dots, p-1\}$ be the greatest such that $\mathbf{z}_k = \mathbf{y}$ and let $r \in \{q+1, \dots, m\}$ be the least such that $\mathbf{z}_r = \mathbf{y}$. Then

$$\begin{aligned} \mathbf{w} &= \overbrace{\left(\mathbf{z}_0 \prod_{i=1}^{k-1} (s_i z_i)\right) s_k}^{\mathbf{a}} \mathbf{y} \overbrace{\left(\prod_{i=k+1}^{p-1} (s_i z_i)\right) s_p}^{\mathbf{b}} \mathbf{x} \overbrace{\left(\prod_{i=p+1}^{q-1} (s_i z_i)\right) s_q}^{\mathbf{c}} \mathbf{x} \\ &\quad \cdot \underbrace{\left(\prod_{i=q+1}^{r-1} (s_i z_i)\right) s_r}_{\mathbf{d}} \mathbf{y} \underbrace{\prod_{i=r+1}^n (s_i z_i)}_{\mathbf{e}} \end{aligned}$$

with

(a) $\text{con}(\mathbf{x}) \cap \text{con}(\mathbf{abde}) = \text{con}(\mathbf{y}) \cap \text{con}(\mathbf{bcd}) = \emptyset$.

Further, $s_r \neq \emptyset$ by (IV), and

(b) $\text{con}(\mathbf{b}) \cap \text{con}(\mathbf{ace}) = \text{con}(\mathbf{c}) \cap \text{con}(\mathbf{abde}) = \text{con}(\mathbf{d}) \cap \text{con}(\mathbf{ace}) = \emptyset$

by (G2), (I), and (III). Suppose $\mathbf{x} \notin_{w'} \mathbf{y}$. Then by Remark 19.5, either $(1_{\mathcal{I}})$, $(2_{\mathcal{I}})$, or $(4_{\mathcal{I}})$ holds. But it is shown in the following that none of these is possible. Hence $\mathbf{x} \in_{w'} \mathbf{y}$.

Case 1: $\mathbf{y} \in_{w'} \mathbf{x}$. By Lemma 19.9, the factor $s_r \mathbf{y}$ of \mathbf{w} is also a factor of \mathbf{w}' . By the assumption of this case, all occurrences of \mathbf{y} in \mathbf{w}' are sandwiched between two occurrences of \mathbf{x} . Therefore the factor $s_r \mathbf{y}$ is also sandwiched between those two occurrences of \mathbf{x} , whence $\mathbf{w}' = \dots \mathbf{x} \dots s_r \mathbf{y} \dots \mathbf{x} \dots$. Let $\varphi_1 : \mathcal{X} \rightarrow \mathcal{C}_8$ denote the substitution

$$z \mapsto \begin{cases} 5 & \text{if } z \in \text{con}\left(\mathbf{cx} \prod_{i=q+1}^{r-1} (s_i z_i)\right), \\ 3 & \text{if } z = h(s_r), \\ 6 & \text{otherwise.} \end{cases}$$

Then by (a) and (b),

$$\begin{aligned} \mathbf{w}\varphi_1 &= (\mathbf{ayb})\varphi_1 \cdot \left(\mathbf{xcx} \prod_{i=q+1}^{r-1} (\mathbf{s}_i \mathbf{z}_i) \right) \varphi_1 \cdot \mathbf{s}_r \varphi_1 \cdot (\mathbf{ye})\varphi_1 \\ &= (6 \dots 6) \cdot (5 \dots 5) \cdot (3 \cdot 6 \dots 6) \cdot (6 \dots 6) = 2. \end{aligned}$$

It is routinely shown that $\{5, 6\}^* \cdot 5 \cdot \{5, 6\}^* = \{1, 4, 5\}$ in \mathcal{C}_8 . Therefore

$$\begin{aligned} \mathbf{w}'\varphi_1 &= (\dots \mathbf{x} \dots \mathbf{s}_r \mathbf{y} \dots \mathbf{x} \dots) \varphi_1 \\ &\in \{5, 6\}^* \cdot (5 \dots 5) \cdot \{5, 6\}^* \cdot (3 \cdot 6 \dots 6) \cdot (6 \dots 6) \cdot \{5, 6\}^* \cdot (5 \dots 5) \cdot \{5, 6\}^* \\ &= \{1, 4, 5\} \cdot 3 \cdot \{1, 4, 5\} = \{1\}, \end{aligned}$$

whence the contradiction $\mathbf{w}\varphi_1 \neq \mathbf{w}'\varphi_1$ is obtained.

Case 2: $\mathbf{y} \prec_{\mathbf{w}'} \mathbf{x}$. Then each occurrence of \mathbf{y} in \mathbf{w}' precedes every occurrence of \mathbf{x} . As observed in Case 1, the factor $\mathbf{s}_r \mathbf{y}$ of \mathbf{w} is also a factor of \mathbf{w}' . Hence

$$\begin{aligned} \mathbf{w}'\varphi_1 &= (\dots \mathbf{s}_r \mathbf{y} \dots \mathbf{x} \dots) \varphi_1 \in \{5, 6\}^* \cdot (3 \cdot 6 \dots 6) \cdot (6 \dots 6) \cdot \{5, 6\}^* \cdot (5 \dots 5) \cdot \{5, 6\}^* \\ &= \{1, 4, 5, 6\} \cdot 3 \cdot \{1, 4, 5\} = \{1\}, \end{aligned}$$

which implies the same contradiction as in Case 1.

Case 3: $\mathbf{x} \prec_{\mathbf{w}'} \mathbf{y}$. Then each occurrence of \mathbf{x} in \mathbf{w}' precedes every occurrence of \mathbf{y} . Hence there exist ℓ', m', p' , and q' with $0 \leq \ell' \leq m' < p' \leq q' \leq n'$ such that:

- $\mathbf{z}'_{\ell'} = \mathbf{z}'_{m'} = \mathbf{x}$ and any occurrence of \mathbf{x} in \mathbf{w}' is one of $\mathbf{z}'_{\ell'}, \dots, \mathbf{z}'_{m'}$;
- $\mathbf{z}'_{p'} = \mathbf{z}'_{q'} = \mathbf{y}$ and any occurrence of \mathbf{y} in \mathbf{w}' is one of $\mathbf{z}'_{p'}, \dots, \mathbf{z}'_{q'}$.

Note that $0 < \ell'$ and $q' < n'$ because $\mathbf{z}'_0 = \mathbf{z}'_n$ by (V). It follows that

$$\begin{aligned} \mathbf{w}' &= \overbrace{\left(\mathbf{z}'_0 \prod_{i=1}^{\ell'-1} (\mathbf{s}'_i \mathbf{z}'_i) \right) \mathbf{s}'_{\ell'}}^{\mathbf{a}'} \mathbf{x} \overbrace{\left(\prod_{i=\ell'+1}^{m'-1} (\mathbf{s}'_i \mathbf{z}'_i) \right) \mathbf{s}'_{m'}}^{\mathbf{b}'} \mathbf{x} \\ &\quad \cdot \underbrace{\left(\prod_{i=m'+1}^{p'-1} (\mathbf{s}'_i \mathbf{z}'_i) \right) \mathbf{s}'_{p'}}_{\mathbf{c}'} \mathbf{y} \underbrace{\left(\prod_{i=p'+1}^{q'-1} (\mathbf{s}'_i \mathbf{z}'_i) \right) \mathbf{s}'_{q'}}_{\mathbf{d}'} \mathbf{y} \underbrace{\prod_{i=q'+1}^{n'} (\mathbf{s}'_i \mathbf{z}'_i)}_{\mathbf{e}'} \end{aligned}$$

with

$$(c) \text{ con}(\mathbf{x}) \cap \text{con}(\mathbf{a}'\mathbf{c}'\mathbf{d}'\mathbf{e}') = \text{con}(\mathbf{y}) \cap \text{con}(\mathbf{a}'\mathbf{b}'\mathbf{c}'\mathbf{e}') = \emptyset.$$

It follows from (G2), (I), and (III) that

$$(d) \text{ con}(\mathbf{b}') \cap \text{con}(\mathbf{a}'\mathbf{c}'\mathbf{d}'\mathbf{e}') = \text{con}(\mathbf{d}') \cap \text{con}(\mathbf{a}'\mathbf{b}'\mathbf{c}'\mathbf{e}') = \emptyset.$$

There are two subcases.

3.1. $\text{con}(\mathbf{c}') \cap \text{con}(\mathbf{e}') = \emptyset$. Since $\mathbf{z}'_0 = \mathbf{z}'_{n'}$ by (V), it follows that the sets $\{\mathbf{z}'_0, \dots, \mathbf{z}'_{\ell'-1}\}$ and $\{\mathbf{z}'_{q'+1}, \dots, \mathbf{z}'_n\}$ are not disjoint. Let $t' \in \{q'+1, \dots, n'\}$ be the least such that

$$\mathbf{z}'_{t'} \in \{\mathbf{z}'_0, \dots, \mathbf{z}'_{\ell'-1}\} \cap \{\mathbf{z}'_{q'+1}, \dots, \mathbf{z}'_n\},$$

and let $k' \in \{0, \dots, \ell' - 1\}$ be the greatest such that $\mathbf{z}'_{k'} = \mathbf{z}'_{t'}$. Then

$$\mathbf{w}' = \mathbf{a}' \mathbf{x} \mathbf{b}' \mathbf{x} \mathbf{c}' \mathbf{y} \mathbf{d}' \mathbf{y} \underbrace{\left(\prod_{i=q'+1}^{t'-1} (\mathbf{s}'_i \mathbf{z}'_i) \right) \mathbf{s}'_{t'} \mathbf{z}'_{t'}}_{\mathbf{e}'_1} \underbrace{\left(\prod_{i=t'+1}^{n'} (\mathbf{s}'_i \mathbf{z}'_i) \right)}_{\mathbf{e}'_2}$$

where $\mathbf{z}'_{k'}$ is a factor of \mathbf{a}' . Note that

(e) $\mathbf{s}'_{t'} \neq \emptyset$

by (IV), and that $\{\mathbf{z}'_{q'+1}, \dots, \mathbf{z}'_{t'-1}\} \cap \{\mathbf{z}'_0, \dots, \mathbf{z}'_{\ell'-1}\} = \emptyset$ by (I) and the minimality of t' . Hence

(f) $\text{con}(\mathbf{e}'_1) \cap \text{con}(\mathbf{a}'\mathbf{s}'_{t'}\mathbf{z}'_{t'}) = \emptyset$.

If $\text{con}(\mathbf{e}'_1) \cap \text{con}(\mathbf{e}'_2) \neq \emptyset$ so that $\mathbf{z}'_j \in \{\mathbf{z}'_{q'+1}, \dots, \mathbf{z}'_{t'-1}\} \cap \{\mathbf{z}'_{t'+1}, \dots, \mathbf{z}'_{n'}\}$ for some j , then $\mathbf{z}'_j \sim_{\mathbf{w}'} \mathbf{z}'_{t'}$ by (G2) due to $\mathbf{z}'_{t'} \in \{\mathbf{z}'_0, \dots, \mathbf{z}'_{\ell'-1}\}$, whence (III) is violated. Hence

(g) $\text{con}(\mathbf{e}'_1) \cap \text{con}(\mathbf{e}'_2) = \emptyset$.

Let $\varphi_2 : \mathcal{X} \rightarrow \mathcal{C}_8$ denote the substitution

$$z \mapsto \begin{cases} 5 & \text{if } z \in \text{con}(\mathbf{d}'\mathbf{y}\mathbf{e}'_1), \\ 3 & \text{if } z = \mathbf{h}(\mathbf{s}'_{t'}), \\ 6 & \text{otherwise.} \end{cases}$$

Then $\mathbf{w}\varphi_2 = \mathbf{w}'\varphi_2$ in \mathcal{C}_8 . By (c)–(g) and the assumption of this subcase,

$$\begin{aligned} \mathbf{w}'\varphi_2 &= (\mathbf{a}'\mathbf{x}\mathbf{b}'\mathbf{x}\mathbf{c}')\varphi_2 \cdot (\mathbf{y}\mathbf{d}'\mathbf{y}\mathbf{e}'_1)\varphi_2 \cdot \mathbf{s}'_{t'}\varphi_2 \cdot (\mathbf{z}'_{t'}\mathbf{e}'_2)\varphi_2 \\ &= (6 \dots 6) \cdot (5 \dots 5) \cdot (3 \cdot 6 \dots 6) \cdot (6 \dots 6) = 2. \end{aligned}$$

Therefore $\mathbf{w}\varphi_2 = 2$. More specifically,

$$\mathbf{w}\varphi_2 = \mathbf{a}\varphi_2 \cdot 5 \cdot \mathbf{b}\varphi_2 \cdot 6 \cdot \mathbf{c}\varphi_2 \cdot 6 \cdot \mathbf{d}\varphi_2 \cdot 5 \cdot \mathbf{e}\varphi_2 = 2. \tag{19.4}$$

By Lemma 19.6, the simple letter $\mathbf{h}(\mathbf{s}'_{t'})$ of \mathbf{w}' is also a simple letter in \mathbf{w} . Since $\mathbf{z}_0 = \mathbf{z}_n$ by (V), the letter $\mathbf{h}(\mathbf{s}'_{t'})$ cannot be the first or last letter of \mathbf{w} . Further, \mathbf{x} and \mathbf{y} are perfect squares of \mathbf{w} . Therefore $\mathbf{w}\varphi_2$ does not begin or end with 3, and it is a product, in some order, of more than one 5, more than one 6, and precisely one 3. Since $3 \cdot 5 = 3 \cdot 6 \cdot 5 = 5 \cdot 6 = 1$ in \mathcal{C}_8 , it is routinely shown that the only way for $\mathbf{w}\varphi_2$ to equal 2 is

$$\mathbf{w}\varphi_2 = (6 \dots 6) \cdot (5 \dots 5) \cdot 3 \cdot (6 \dots 6).$$

But this is impossible in view of (19.4).

3.2. $\text{con}(\mathbf{c}') \cap \text{con}(\mathbf{e}') \neq \emptyset$. Then there exists the least $r' \in \{q'+1, \dots, n'\}$ such that

$$\mathbf{z}'_{r'} \in \{\mathbf{z}'_{m'+1}, \dots, \mathbf{z}'_{p'-1}\} \cap \{\mathbf{z}'_{q'+1}, \dots, \mathbf{z}'_n\},$$

and there exists the greatest $h' \in \{\mathbf{z}'_{m'+1}, \dots, \mathbf{z}'_{p'-1}\}$ such that $\mathbf{z}'_{h'} = \mathbf{z}'_{r'}$. Hence

$$\begin{aligned} \mathbf{w}' &= \mathbf{a}'\mathbf{x}\mathbf{b}'\mathbf{x} \overbrace{\left(\prod_{i=m'+1}^{h'-1} (\mathbf{s}'_i\mathbf{z}'_i) \right)}^{\mathbf{c}'_1} \mathbf{s}'_{h'} \mathbf{v} \overbrace{\left(\prod_{i=h'+1}^{p'-1} (\mathbf{s}'_i\mathbf{z}'_i) \right)}^{\mathbf{c}'_2} \mathbf{s}'_{p'+1} \mathbf{y}\mathbf{d}'\mathbf{y} \\ &\quad \cdot \underbrace{\left(\prod_{i=q'+1}^{r'-1} (\mathbf{s}'_i\mathbf{z}'_i) \right)}_{\mathbf{e}'_3} \mathbf{s}'_{r'} \mathbf{v} \underbrace{\left(\prod_{i=r'+1}^{n'} (\mathbf{s}'_i\mathbf{z}'_i) \right)}_{\mathbf{e}'_4} \end{aligned}$$

where $\mathbf{v} = \mathbf{z}'_{h'} = \mathbf{z}'_{r'}$. Note that (IV) implies

(h) $\mathbf{s}'_{r'} \neq \emptyset$,

and (I) and the choice of h' and r' imply that

(i) $\mathbf{c}'_2, \mathbf{e}'_3$, and \mathbf{v} are pairwise disjoint.

If \mathbf{f} is a perfect square that is shared by \mathbf{c}'_2 and either \mathbf{a}' , \mathbf{b}' , \mathbf{c}'_1 , or \mathbf{e}'_4 , then $\mathbf{f} \neq \mathbf{v}$ by (i), so that $\mathbf{f} \sim_{\mathbf{w}'} \mathbf{v}$ by (G2). If \mathbf{g} is a perfect square that is shared by \mathbf{e}'_3 and either \mathbf{a}' , \mathbf{b}' , \mathbf{c}'_1 , or \mathbf{e}'_4 , then $\mathbf{g} \neq \mathbf{v}$ by (i), so that $\mathbf{g} \sim_{\mathbf{w}'} \mathbf{v}$ by (G2). But either of these contradicts (III). Therefore both \mathbf{f} and \mathbf{g} do not exist, whence

$$(j) \text{ con}(\mathbf{c}'_2) \cap \text{con}(\mathbf{a}'\mathbf{b}'\mathbf{c}'_1\mathbf{e}'_4) = \text{con}(\mathbf{e}'_3) \cap \text{con}(\mathbf{a}'\mathbf{b}'\mathbf{c}'_1\mathbf{e}'_4) = \emptyset.$$

Let $\varphi_3 : \mathcal{X} \rightarrow \mathcal{C}_8$ denote the substitution

$$z \mapsto \begin{cases} 5 & \text{if } z \in \text{con}(\mathbf{d}'\mathbf{y}\mathbf{e}'_3), \\ 3 & \text{if } z = \mathbf{h}(\mathbf{s}'_{r'}), \\ 6 & \text{otherwise.} \end{cases}$$

Then by (c), (d), (h), and (j),

$$\begin{aligned} \mathbf{w}'\varphi_3 &= (\mathbf{a}'\mathbf{x}\mathbf{b}'\mathbf{x}\mathbf{c}')\varphi_3 \cdot (\mathbf{y}\mathbf{d}'\mathbf{y}\mathbf{e}'_3)\varphi_3 \cdot \mathbf{s}'_{r'}\varphi_3 \cdot (\mathbf{v}\mathbf{e}'_4)\varphi_3 \\ &= (6 \dots 6) \cdot (5 \dots 5) \cdot (3 \cdot 6 \dots 6) \cdot (6 \dots 6) = 2. \end{aligned}$$

But following the same argument as in Subcase 3.1, it is impossible for $\mathbf{w}\varphi_3$ to equal 2. \square

19.4. Proof of Proposition 19.1

REMARK 19.12. Let \mathbf{w} be any word in canonical form that contains two or more distinct perfect squares. By (V), the word \mathbf{w} begins and ends with some perfect square \mathbf{z} . Hence by (IV),

$$\mathbf{w} = \mathbf{z} \prod_{i=1}^n (\mathbf{p}_i \mathbf{s}_i \mathbf{z}) = \mathbf{z} \mathbf{p}_1 \mathbf{s}_1 \mathbf{z} \dots \mathbf{p}_n \mathbf{s}_n \mathbf{z}$$

for some $n \geq 1$, where the following holds for each i :

- (Re1) \mathbf{s}_i is a nonempty maximal simple factor of \mathbf{w} ;
 - (Re2) \mathbf{p}_i is a product of some of the following: maximal simple factors of \mathbf{w} and perfect squares of \mathbf{w} different from \mathbf{z} ;
 - (Re3) if \mathbf{p}_i is nonempty, then it ends with some perfect square of \mathbf{w} .
- Note that (I), (III), and (G2) imply that:
- (Re4) $\mathbf{z}, \mathbf{p}_1, \mathbf{s}_1, \dots, \mathbf{p}_n, \mathbf{s}_n$ are pairwise disjoint;
 - (Re5) if \mathbf{x} is any perfect square of \mathbf{w} different from \mathbf{z} , then $\mathbf{x} \in_{\mathbf{w}} \mathbf{z}$.

By Lemma 19.4, the semigroup \mathcal{C}_8 has a basis of the form $\{(19.1)\} \cup \Sigma$, where Σ is some set of identities formed by connected words in canonical form. Let $\mathbf{w} \approx \mathbf{w}'$ be any identity from Σ . By Lemmas 19.6 and 19.7, the set Z of perfect squares of \mathbf{w} is equal to the set of perfect squares of \mathbf{w}' , and the set S of maximal simple factors of \mathbf{w} is equal to the set of maximal simple factors of \mathbf{w}' . The remainder of this subsection is devoted to the verification of the following statement for each $k \geq 1$:

- (†_k) if $\mathbf{w} \approx \mathbf{w}'$ is any identity in Σ that involves at most k distinct perfect squares, then the identities (19.1) imply the identity $\mathbf{w} \approx \mathbf{w}'$.

Consequently, the identities (19.1) imply every identity in Σ and so form a basis for \mathcal{C}_8 .

First consider the case when the identity $\mathbf{w} \approx \mathbf{w}'$ from Σ involves $k = 1$ perfect square \mathbf{z} . Then $\mathbf{w} = \mathbf{z} \prod_{i=1}^n (\mathbf{s}_i \mathbf{z}) = \mathbf{z} \mathbf{s}_1 \mathbf{z} \dots \mathbf{s}_n \mathbf{z}$ where $Z = \{\mathbf{z}\}$ and $S = \{\mathbf{s}_1, \dots, \mathbf{s}_n\}$. By Lemmas 19.9 and 19.10, there exists a permutation π on $\{1, \dots, n\}$ such that $\mathbf{w}' = \mathbf{z} \mathbf{s}_{1\pi} \mathbf{z} \dots \mathbf{s}_{n\pi} \mathbf{z}$. It is then easily seen that the identity (19.1b) implies the identity $\mathbf{w} \approx \mathbf{w}'$. Hence statement (†₁) holds.

Now suppose that statement (†_{k-1}) holds for some $k > 1$ and that the identity $\mathbf{w} \approx \mathbf{w}'$ from Σ contains k distinct perfect squares. By Remark 19.12, the word \mathbf{w} is of the form

$$\mathbf{w} = \mathbf{z} \mathbf{p}_1 \mathbf{s}_1 \mathbf{z} \mathbf{p}_2 \mathbf{s}_2 \mathbf{z} \dots \mathbf{p}_n \mathbf{s}_n \mathbf{z}$$

where (Re1)–(Re5) are satisfied. By (V), the word \mathbf{w}' begins and ends with the same perfect square; by Lemma 19.11 this perfect square must be \mathbf{z} . Hence it follows from Lemma 19.9 and Remark 19.12 that $\mathbf{w}' = \mathbf{z}\mathbf{p}'_1\mathbf{s}_{1\pi}\mathbf{z}\mathbf{p}'_2\mathbf{s}_{2\pi}\mathbf{z}\dots\mathbf{p}'_n\mathbf{s}_{n\pi}\mathbf{z}$ for some permutation π on $\{1, \dots, n\}$. Since the identity (19.1b) can be used to interchange any two factors $\mathbf{p}'_i\mathbf{s}_{i\pi}$ and $\mathbf{p}'_j\mathbf{s}_{j\pi}$, generality is not lost by assuming that

$$\mathbf{w}' = \mathbf{z}\mathbf{p}'_1\mathbf{s}_1\mathbf{z}\mathbf{p}'_2\mathbf{s}_2\mathbf{z}\dots\mathbf{p}'_n\mathbf{s}_n\mathbf{z}$$

to begin with. By Lemmas 19.13 and 19.14 below, the identities (19.1) imply $\mathbf{w} \approx \mathbf{w}'$.

LEMMA 19.13. *If $n = 1$, then the identities (19.1) imply the identity $\mathbf{w} \approx \mathbf{w}'$.*

Proof. Generality is not lost by writing

$$\mathbf{w} = \mathbf{z}\mathbf{p}\mathbf{s}\mathbf{z} \quad \text{and} \quad \mathbf{w}' = \mathbf{z}\mathbf{p}'\mathbf{s}\mathbf{z},$$

where \mathbf{z} , \mathbf{p} , \mathbf{p}' , and \mathbf{s} satisfy (Re1)–(Re4). It then follows that $\text{con}(\mathbf{p}) = \text{con}(\mathbf{p}')$ by Lemmas 19.6 and 19.7. If $\mathbf{p} = \mathbf{p}' = \emptyset$, then the identity $\mathbf{w} \approx \mathbf{w}'$ is trivial and so is clearly implied by the identities (19.1). Therefore suppose that the factors \mathbf{p} and \mathbf{p}' are nonempty. Note that \mathbf{p} can be written as a product $\mathbf{p} = \mathbf{a}_r\mathbf{a}_{r-1}\dots\mathbf{a}_2\mathbf{a}_1$ of pairwise disjoint words $\mathbf{a}_r, \mathbf{a}_{r-1}, \dots, \mathbf{a}_2, \mathbf{a}_1 \in \mathcal{X}^+$ where each \mathbf{a}_i is either connected or a maximal simple factor of \mathbf{w} . If \mathbf{a}_i is connected, then since $\mathbf{x} \not\sim_{\mathbf{w}} \mathbf{y}$ for any distinct perfect squares \mathbf{x} and \mathbf{y} of \mathbf{w} , it follows from (Re2) that \mathbf{a}_i begins and ends with the same perfect square of \mathbf{w} . The same applies to connected factors of the word \mathbf{p}' . Therefore in summary, the words \mathbf{p} and \mathbf{p}' can be written as

$$\mathbf{p} = \mathbf{a}_r\mathbf{a}_{r-1}\dots\mathbf{a}_2\mathbf{a}_1 \quad \text{and} \quad \mathbf{p}' = \mathbf{a}'_{r'}\mathbf{a}'_{r'-1}\dots\mathbf{a}'_2\mathbf{a}'_1$$

for some $r, r' \geq 1$, where:

- (a) \mathbf{a}_i is either a maximal simple factor of \mathbf{w} or a word that begins and ends with the same perfect square of \mathbf{w} ;
- (b) \mathbf{a}'_i is either a maximal simple factor of \mathbf{w}' or a word that begins and ends with the same perfect square of \mathbf{w}' ;
- (c) $\mathbf{a}_r, \dots, \mathbf{a}_1$ are pairwise disjoint and $\mathbf{a}'_{r'}, \dots, \mathbf{a}'_1$ are pairwise disjoint.

Note that (c) and (Re2) imply that

- (d) $\mathbf{a}_r, \dots, \mathbf{a}_1, \mathbf{s}, \mathbf{z}$ are pairwise disjoint and $\mathbf{a}'_{r'}, \dots, \mathbf{a}'_1, \mathbf{s}, \mathbf{z}$ are pairwise disjoint.

Further, since the word \mathbf{w} is in canonical form,

- (e) any two maximal simple factors of \mathbf{w} must sandwich at least one perfect square of \mathbf{w} .

First, it is shown that the identities (19.1) imply the identity $\mathbf{a}_1 \approx \mathbf{a}'_1$. To achieve this goal, observe that by (e), the word \mathbf{a}_1 is not a maximal simple factor of \mathbf{w} since it is followed by the maximal simple factor \mathbf{s} . Therefore by (a), the word \mathbf{a}_1 begins and ends with the same perfect square of \mathbf{w} . Hence \mathbf{w} is of the form

$$\mathbf{w} = \mathbf{z}\mathbf{a}_r \dots \mathbf{a}_2 \underbrace{\mathbf{z}_1\mathbf{b}_1\mathbf{z}_1}_{\mathbf{a}_1} \mathbf{s}\mathbf{z},$$

where \mathbf{z}_1 is some perfect square of \mathbf{w} . By the same argument, the word \mathbf{a}'_1 also begins and ends with the same perfect square of \mathbf{w}' ; by Lemma 19.10, this perfect square must equal \mathbf{z}_1 . Hence \mathbf{w}' is of the form

$$\mathbf{w}' = \mathbf{z}\mathbf{a}'_{r'} \dots \mathbf{a}'_2 \underbrace{\mathbf{z}_1\mathbf{b}'_1\mathbf{z}_1}_{\mathbf{a}'_1} \mathbf{s}\mathbf{z}.$$

Suppose that \mathbf{y} is a perfect square of \mathbf{w} that is a factor of \mathbf{b}_1 . If $\mathbf{y} = \mathbf{z}_1$, then \mathbf{y} is clearly a factor of \mathbf{a}'_1 . If $\mathbf{y} \neq \mathbf{z}_1$, then $\mathbf{y} \in_{\mathbf{w}} \mathbf{z}_1$ by (d), so that $\mathbf{y} \in_{\mathbf{w}'} \mathbf{z}_1$ by Lemma 19.11. Therefore in

any case, \mathbf{y} is a factor of \mathbf{a}'_1 . By symmetry, any perfect square of \mathbf{w}' that is a factor of \mathbf{a}'_1 is also a factor of \mathbf{a}_1 . Hence

(f) \mathbf{a}_1 and \mathbf{a}'_1 share the same set of perfect squares.

Suppose that \mathbf{t} is a maximal simple factor of \mathbf{w} that is a factor of \mathbf{b}_1 . Then \mathbf{t} is immediately preceded and followed by perfect squares of \mathbf{w} that occur in $\mathbf{a}_1 = \mathbf{z}_1\mathbf{b}_1\mathbf{z}_1$. More specifically, $\mathbf{y}_1\mathbf{t}\mathbf{y}_2$ is a factor of $\mathbf{a}_1 = \mathbf{z}_1\mathbf{b}_1\mathbf{z}_1$ for some perfect squares \mathbf{y}_1 and \mathbf{y}_2 . It follows from (f) and Lemmas 19.9 and 19.10 that $\mathbf{y}_1\mathbf{t}\mathbf{y}_2$ is a factor of $\mathbf{a}'_1 = \mathbf{z}_1\mathbf{b}'_1\mathbf{z}_1$, whence \mathbf{t} is a factor of \mathbf{b}'_1 . By symmetry, any maximal simple factor of \mathbf{w}' that is a factor of \mathbf{b}'_1 is also a factor of \mathbf{b}_1 . Hence

(g) \mathbf{b}_1 and \mathbf{b}'_1 share the same maximal simple factors.

It then follows from (f) and (g) that $\text{con}(\mathbf{a}_1) = \text{con}(\mathbf{a}'_1)$. Let χ_1 denote the substitution $x \mapsto \mathbf{z}_1$ for all $x \notin \text{con}(\mathbf{a}_1) = \text{con}(\mathbf{a}'_1)$. Then the deductions $\mathbf{w}\chi_1 \stackrel{(19.1a)}{\approx} \mathbf{a}_1$ and $\mathbf{w}'\chi_1 \stackrel{(19.1a)}{\approx} \mathbf{a}'_1$ imply that \mathcal{C}_8 satisfies the identity $\mathbf{a}_1 \approx \mathbf{a}'_1$. The identity $\mathbf{a}_1 \approx \mathbf{a}'_1$ involves less than k distinct perfect squares. Therefore by (\dagger_{k-1}) , the identities (19.1) imply the identity $\mathbf{a}_1 \approx \mathbf{a}'_1$.

Next, it is shown that the identities (19.1) imply the identity $\mathbf{a}_2 \approx \mathbf{a}'_2$. Based on (a), there are two cases.

Case 1: \mathbf{a}_2 is a maximal simple factor of \mathbf{w} . Then by Lemma 19.9, the factor $\mathbf{a}_2\mathbf{z}_1$ of \mathbf{w} is also a factor of \mathbf{w}' . But since $\text{con}(\mathbf{a}_1) = \text{con}(\mathbf{a}'_1)$, it follows from (d) that \mathbf{a}_2 is not a factor of \mathbf{a}'_1 . Therefore the only way for $\mathbf{a}_2\mathbf{z}_1$ to be a factor of \mathbf{w}' is when $\mathbf{a}_2 = \mathbf{a}'_2$. The identity $\mathbf{a}_2 \approx \mathbf{a}'_2$ is trivial and so is implied by the identities (19.1).

Case 2: \mathbf{a}_2 begins and ends with the same perfect square of \mathbf{w} . Then \mathbf{w} is of the form

$$\mathbf{w} = \mathbf{z}\mathbf{a}_r \dots \mathbf{a}_3 \underbrace{\mathbf{z}_2\mathbf{b}_2\mathbf{z}_2}_{\mathbf{a}_2} \mathbf{a}_1\mathbf{s}\mathbf{z},$$

where \mathbf{z}_2 is some perfect square of \mathbf{w} . If \mathbf{a}'_2 is a maximal simple factor of \mathbf{w}' , then the argument in Case 1 shows $\mathbf{a}_2 = \mathbf{a}'_2$, which is impossible. Therefore by (b), the word \mathbf{a}'_2 begins and ends with the same perfect square of \mathbf{w}' , whence

$$\mathbf{w}' = \mathbf{z}\mathbf{a}'_{r'} \dots \mathbf{a}'_3 \underbrace{\mathbf{z}'_2\mathbf{b}'_2\mathbf{z}'_2}_{\mathbf{a}'_2} \mathbf{a}'_1\mathbf{s}\mathbf{z}$$

for some perfect square \mathbf{z}'_2 of \mathbf{w}' . Suppose $\mathbf{z}_2 \neq \mathbf{z}'_2$. Since the set of perfect squares of \mathbf{w} is equal to the set of perfect squares of \mathbf{w}' , the perfect square \mathbf{z}_2 of \mathbf{w} is also a perfect square of \mathbf{w}' . If \mathbf{z}_2 is a factor of \mathbf{b}'_2 , then $\mathbf{z}_2 \in_{\mathbf{w}'} \mathbf{z}'_2$, so that $\mathbf{z}_2 \in_{\mathbf{w}} \mathbf{z}'_2$ by Lemma 19.11; but $\mathbf{z}_2 \in_{\mathbf{w}} \mathbf{z}'_2$ implies that \mathbf{z}'_2 is a factor shared by both the prefix $\mathbf{z}\mathbf{a}_r \dots \mathbf{a}_3$ and suffix $\mathbf{a}_1\mathbf{s}\mathbf{z}$ of \mathbf{w} , and this is impossible due to (d). Therefore

(h) \mathbf{z}_2 is not a factor of \mathbf{a}'_2 , but is a factor of one of $\mathbf{a}'_{r'}, \dots, \mathbf{a}'_3$.

By symmetry,

(i) \mathbf{z}'_2 is not a factor of \mathbf{a}_2 , but is a factor of one of $\mathbf{a}_r, \dots, \mathbf{a}_3$.

Let $\varphi : \mathcal{X} \rightarrow \mathcal{C}_8$ denote the substitution

$$z \mapsto \begin{cases} 5 & \text{if } z \in \text{con}(\mathbf{a}_2\mathbf{a}_1), \\ 3 & \text{if } z = \mathbf{h}(\mathbf{s}), \\ 6 & \text{otherwise.} \end{cases}$$

Then

$$\mathbf{w}\varphi = (\mathbf{z}\mathbf{a}_r \dots \mathbf{a}_3)\varphi \cdot (\mathbf{a}_2\mathbf{a}_1)\varphi \cdot \mathbf{s}\varphi \cdot \mathbf{z}\varphi = (6 \dots 6) \cdot (5 \dots 5) \cdot (3 \cdot 6 \dots 6) \cdot (6 \dots 6) = 2.$$

But (d) and (i) imply $\mathbf{z}'_2\varphi = 6$, so that (h) implies

$$\begin{aligned} (\mathbf{z}\mathbf{a}'_{r'} \dots \mathbf{a}'_3 \mathbf{z}'_2)\varphi &= \mathbf{z}\varphi \cdot (\mathbf{a}'_{r'} \dots \mathbf{a}'_3)\varphi \cdot \mathbf{z}'_2\varphi = \mathbf{z}\varphi \dots \mathbf{z}_2\varphi \dots \mathbf{z}'_2\varphi \\ &\in 6 \cdot \{5, 6\}^* \cdot 5 \cdot \{5, 6\}^* \cdot 6 = \{1\}. \end{aligned}$$

Hence the contradiction $\mathbf{w}'\varphi = 1$ is obtained. Therefore the assumption $\mathbf{z}_2 \neq \mathbf{z}'_2$ does not hold, whence

$$\mathbf{w}' = \mathbf{z}\mathbf{a}'_{r'} \dots \mathbf{a}'_3 \underbrace{\mathbf{z}_2 \mathbf{b}'_2 \mathbf{z}_2}_{\mathbf{a}'_2} \mathbf{a}'_1 \mathbf{s}\mathbf{z}.$$

It can then be shown, by the arguments used in establishing (f) and (g), that $\text{con}(\mathbf{a}_2) = \text{con}(\mathbf{a}'_2)$. Let χ_2 denote the substitution $x \mapsto \mathbf{z}_2$ for all $x \notin \text{con}(\mathbf{a}_2) = \text{con}(\mathbf{a}'_2)$. Then the deductions $\mathbf{w}\chi_2 \stackrel{(19.1a)}{\approx} \mathbf{a}_2$ and $\mathbf{w}'\chi_2 \stackrel{(19.1a)}{\approx} \mathbf{a}'_2$ imply that \mathcal{C}_8 satisfies the identity $\mathbf{a}_2 \approx \mathbf{a}'_2$. The identity $\mathbf{a}_2 \approx \mathbf{a}'_2$ involves less than k distinct perfect squares. Therefore by (\dagger_{k-1}) , the identities (19.1) imply the identity $\mathbf{a}_2 \approx \mathbf{a}'_2$.

In the two cases just considered, the identities (19.1) imply the identity $\mathbf{a}_2 \approx \mathbf{a}'_2$. The arguments in Cases 1 and 2 can be repeated to show that the identities (19.1) imply the identity $\mathbf{a}_3 \approx \mathbf{a}'_3$. This can be continued so that the identities (19.1) imply the identity $\mathbf{a}_i \approx \mathbf{a}'_i$ for all i with $r = r'$. It is then easily seen that the identities (19.1) imply $\mathbf{w} \approx \mathbf{w}'$. \square

LEMMA 19.14. *If $n > 1$, then the identities (19.1) imply the identity $\mathbf{w} \approx \mathbf{w}'$.*

Proof. Let ℓ be the least integer such that $\text{con}(\mathbf{p}_\ell) \neq \text{con}(\mathbf{p}'_\ell)$. Then generality is not lost by assuming that $x \in \text{con}(\mathbf{p}'_\ell) \setminus \text{con}(\mathbf{p}_\ell)$, so that $x \in \text{con}(\mathbf{p}_r)$ for some $r > \ell$. Let $\varphi : \mathcal{X} \rightarrow \mathcal{C}_8$ denote the substitution

$$z \mapsto \begin{cases} 5 & \text{if } z \in \text{con}(\mathbf{p}_r), \\ 3 & \text{if } z = \mathbf{h}(\mathbf{s}_r), \\ 6 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \mathbf{w}\varphi &= \left(\mathbf{z} \prod_{i=1}^{r-1} (\mathbf{p}_i \mathbf{s}_i \mathbf{z}) \right) \varphi \cdot \mathbf{p}_r \varphi \cdot \mathbf{s}_r \varphi \cdot \left(\mathbf{z} \prod_{i=1}^{r+1} (\mathbf{p}_i \mathbf{s}_i \mathbf{z}) \right) \varphi \\ &= (6 \dots 6) \cdot (5 \dots 5) \cdot (3 \cdot 6 \dots 6) \cdot (6 \dots 6) = 2. \end{aligned}$$

But $x \in \text{con}(\mathbf{p}'_\ell)$ implies

$$(\mathbf{z}\mathbf{p}'_\ell \mathbf{s}_\ell \mathbf{z})\varphi = \mathbf{z}\varphi \dots x\varphi \dots \mathbf{z}\varphi \in 6 \cdot \{5, 6\}^* \cdot 5 \cdot \{5, 6\}^* \cdot 6 = \{1\}.$$

Hence the contradiction $\mathbf{w}'\varphi = 1$ is obtained. Therefore the integer ℓ does not exist, whence $\text{con}(\mathbf{p}_i) = \text{con}(\mathbf{p}'_i)$ for all $i \in \{1, \dots, n\}$.

For each $i \in \{1, \dots, n\}$, let χ_i denote the substitution $z \mapsto \mathbf{z}$ for all $z \notin \text{con}(\mathbf{p}_i \mathbf{s}_i) = \text{con}(\mathbf{p}'_i \mathbf{s}_i)$. Then the deductions $\mathbf{w}\chi_i \stackrel{(19.1a)}{\approx} \mathbf{z}\mathbf{p}_i \mathbf{s}_i \mathbf{z}$ and $\mathbf{w}'\chi_i \stackrel{(19.1a)}{\approx} \mathbf{z}\mathbf{p}'_i \mathbf{s}_i \mathbf{z}$ imply that \mathcal{C}_8 satisfies the identity $\mathbf{z}\mathbf{p}_i \mathbf{s}_i \mathbf{z} \approx \mathbf{z}\mathbf{p}'_i \mathbf{s}_i \mathbf{z}$. By Lemma 19.13, the identities (19.1) imply $\mathbf{z}\mathbf{p}_i \mathbf{s}_i \mathbf{z} \approx \mathbf{z}\mathbf{p}'_i \mathbf{s}_i \mathbf{z}$. It is then easily seen that the identities (19.1) imply $\mathbf{w} \approx \mathbf{w}'$. \square

20. $\mathcal{D}_2, \mathcal{D}_4, \mathcal{E}_4$

This section establishes the finite basis property of the following semigroups.

\mathcal{D}_2	1 2 3 4 5 6	\mathcal{D}_4	1 2 3 4 5 6	\mathcal{E}_4	1 2 3 4 5 6
1	1 1 1 1 1 1	1	1 1 1 1 1 1	1	1 1 1 1 1 1
2	1 1 1 1 1 2	2	1 1 1 1 1 2	2	1 1 1 1 1 3
3	1 1 1 1 2 1	3	1 1 1 1 2 2	3	1 1 1 1 3 3
4	1 2 3 4 4 4	4	1 2 3 4 4 4	4	1 1 1 3 4 4
5	1 2 3 4 4 5	5	1 2 3 4 4 5	5	1 2 3 4 5 5
6	1 2 3 4 4 6	6	1 2 3 4 4 6	6	1 2 3 4 6 6

20.1. Finite basis property of \mathcal{D}_2 and \mathcal{D}_4

PROPOSITION 20.1. The variety generated by $S \in \{\mathcal{D}_2, \mathcal{D}_4\}$ is defined by the identities

$$x^2Hx \approx xHx, \tag{20.1a}$$

$$x^2y^2 \approx y^2x^2, \tag{20.1b}$$

$$xHxyKy^2 \approx xHxKy^2, \tag{20.1c}$$

$$xHyKyTx \approx yHyKyxTx, \tag{20.1d}$$

$$yHxKyTx \approx yHyKyxTx, \tag{20.1e}$$

$$xHyKyTx \approx yHxKyTx. \tag{20.1f}$$

For any word \mathbf{w} , let $F_{SL}(\mathbf{w})$ denote the set of factors of \mathbf{w} of length two that begin with a simple letter and end with the last occurrence of some non-simple letter:

$$F_{SL}(\mathbf{w}) = \{xy \in \mathcal{X}^2 \mid \mathbf{w} \in \mathcal{X}^*xy(\mathcal{X} \setminus \{y\})^*, x \in \text{sim}(\mathbf{w}), y \notin \text{sim}(\mathbf{w})\}.$$

LEMMA 20.2. Let $S \in \{\mathcal{D}_2, \mathcal{D}_4\}$. Suppose that $\mathbf{w} \approx \mathbf{w}'$ is any identity satisfied by S . Then:

- (i) $\text{con}(\mathbf{w}) = \text{con}(\mathbf{w}')$ and $\text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$;
- (ii) for any letters $x \in \text{con}(\mathbf{w}) = \text{con}(\mathbf{w}')$ and $y \in \text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$, the conditions $x \prec_{\mathbf{w}} y$ and $x \prec_{\mathbf{w}'} y$ are equivalent;
- (iii) $\mathbf{w}_{\text{sim}} = \mathbf{w}'_{\text{sim}}$;
- (iv) $F_{SL}(\mathbf{w}) = F_{SL}(\mathbf{w}')$;
- (v) $F_{SS}(\mathbf{w}) = F_{SS}(\mathbf{w}')$.

Proof. The subsemigroup $\{1, 2, 4, 6\}$ of S is isomorphic to J^1 , so that parts (i)–(iii) hold by Lemma 2.3.

(iv) Seeking a contradiction, suppose $F_{SL}(\mathbf{w}) \neq F_{SL}(\mathbf{w}')$. By symmetry, it suffices to assume that $xy \in F_{SL}(\mathbf{w}) \setminus F_{SL}(\mathbf{w}')$. Then

- (a) $\mathbf{w} = \mathbf{a}xy\mathbf{b}$ for some $\mathbf{a}, \mathbf{b} \in \mathcal{X}^*$ such that $x \notin \text{con}(\mathbf{a}\mathbf{b})$ and $y \in \text{con}(\mathbf{a}) \setminus \text{con}(\mathbf{b})$.

By part (i),

- (b) x is simple in both \mathbf{w} and \mathbf{w}' , while y is non-simple in both \mathbf{w} and \mathbf{w}' .

Since $y \not\prec_{\mathbf{w}} x$, it follows from part (ii) that

- (c) $y \not\prec_{\mathbf{w}'} x$.

Let $\varphi : \mathcal{X} \rightarrow S$ denote the substitution

$$z \mapsto \begin{cases} 3 & \text{if } z = x, \\ 5 & \text{if } z = y, \\ 6 & \text{otherwise.} \end{cases}$$

Then it follows from (a) that $\mathbf{w}\varphi \in \{5, 6\}^+ \cdot 3 \cdot 5 \cdot \{6\}^* = \{2\}$. This leads to the required contradiction, as shown in the following collectively exhaustive cases.

Case 1: xy is a factor of \mathbf{w}' . Since $xy \notin F_{SL}(\mathbf{w}')$, the y that immediately follows x is not the last occurrence; this, together with (b), imply $\mathbf{w}' = \mathbf{a}'xy\mathbf{b}'y\mathbf{c}'$ for some $\mathbf{a}', \mathbf{b}', \mathbf{c}' \in \mathcal{X}^*$ with $x \notin \text{con}(\mathbf{a}'\mathbf{b}'\mathbf{c}')$. Therefore

$$\mathbf{w}'\varphi \in \{5, 6\}^* \cdot 3 \cdot 5 \cdot \{5, 6\}^* \cdot 5 \cdot \{5, 6\}^* = \{1\}.$$

Case 2: xy is not a factor of \mathbf{w}' . Then it follows from (c) that $\mathbf{w}' = \mathbf{a}'xh\mathbf{b}'y\mathbf{c}'$ for some $\mathbf{a}', \mathbf{b}', \mathbf{c}' \in \mathcal{X}^*$ and $h \in \mathcal{X}$ such that $x \notin \text{con}(\mathbf{a}'\mathbf{b}'\mathbf{c}')$ and $x \neq h \neq y$. Hence

$$\mathbf{w}'\varphi \in \{5, 6\}^* \cdot 3 \cdot 6 \cdot \{5, 6\}^* \cdot 5 \cdot \{5, 6\}^* = \{1\}.$$

(v) Suppose $F_{SS}(\mathbf{w}) \neq F_{SS}(\mathbf{w}')$, say $xy \in F_{SS}(\mathbf{w}) \setminus F_{SS}(\mathbf{w}')$. Then it is easily seen from part (i) that $xy \in F_{SL}(y\mathbf{w}) \setminus F_{SL}(y\mathbf{w}')$. It then follows from part (iv) that S does not satisfy the identity $y\mathbf{w} \approx y\mathbf{w}'$, which is impossible. \square

For any letter $x \in \mathcal{X}$, define an x -block to be a non-simple word of the form

$$\mathbf{x} = \left(\prod_{i=1}^r (x\mathbf{s}_i) \right) x^e = x\mathbf{s}_1x\mathbf{s}_2 \dots x\mathbf{s}_rx^e, \tag{20.2}$$

where $\mathbf{s}_1, \dots, \mathbf{s}_r \in \mathcal{X}^+$, $e \in \{1, 2\}$, and $r \geq 0$ satisfy the following:

- (B1) the letters of $\mathbf{s}_1, \dots, \mathbf{s}_r$ are all simple in \mathbf{x} ;
- (B2) if $r = 0$, then $e = 2$.

- REMARK 20.3. (i) In any x -block, the letter x is the only non-simple letter.
 (ii) If $r = 0$ in (20.2), then the prefix $\prod_{i=1}^r (x\mathbf{s}_i)$ of \mathbf{x} is empty, so that $\mathbf{x} = x^2$ by (B2).

Let \mathbf{w} be any non-simple word and let x be any non-simple letter of \mathbf{w} with $\text{occ}(x, \mathbf{w}) = r \geq 2$, so that

$$\mathbf{w} = \mathbf{w}_0x\mathbf{w}_1x\mathbf{w}_2 \dots x\mathbf{w}_r$$

for some $\mathbf{w}_0, \dots, \mathbf{w}_r \in \mathcal{X}^*$ with $x \notin \text{con}(\mathbf{w}_0 \dots \mathbf{w}_r)$. Then the letter x is said to be *linked* in \mathbf{w} if $\text{con}(\mathbf{w}_1 \dots \mathbf{w}_{r-1}) \subseteq \text{sim}(\mathbf{w})$. In other words, a non-simple letter x of \mathbf{w} is linked if any other non-simple letter of \mathbf{w} is not sandwiched between any two occurrences of x . A non-simple word is *linked* if each of its non-simple letters is linked.

LEMMA 20.4. *Let \mathbf{w} be any non-simple word. Then there exists some linked word $\bar{\mathbf{w}}$ such that the identities (20.1) imply the identity $\mathbf{w} \approx \bar{\mathbf{w}}$.*

Proof. It suffices to convert \mathbf{w} , using the identities (20.1), into a linked word. Let x be the last non-simple letter of \mathbf{w} that is not linked. Then

$$\mathbf{w} = \mathbf{p}x\mathbf{q}$$

for some $\mathbf{p}, \mathbf{q} \in \mathcal{X}^*$ such that $x \in \text{con}(\mathbf{p}) \setminus \text{con}(\mathbf{q})$ and any non-simple letter of \mathbf{w} that occurs in \mathbf{q} is linked. It follows that $\mathbf{p}x$ and \mathbf{q} are disjoint words. Since the letter x is not linked and $x \in \text{con}(\mathbf{p}) \setminus \text{con}(\mathbf{q})$, some other non-simple letter y of \mathbf{w} occurring in \mathbf{p} is sandwiched between two occurrences of x , that is,

$$\mathbf{w} = \underbrace{\mathbf{a}x\mathbf{b}y\mathbf{c}}_{\mathbf{p}}x\mathbf{q} \tag{20.3}$$

for some $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathcal{X}^*$ such that $y \in \text{con}(\mathbf{ab}) \setminus \text{con}(\mathbf{cq})$. If $y \in \text{con}(\mathbf{a})$, then $\mathbf{a} = \mathbf{a}'y\mathbf{a}''$ for some $\mathbf{a}', \mathbf{a}'' \in \mathcal{X}^*$, so that

$$\mathbf{w} = \mathbf{a}'y\mathbf{a}''x\mathbf{b}y\mathbf{c}x\mathbf{q} \stackrel{(20.1e)}{\approx} \mathbf{a}'y\mathbf{a}''y\mathbf{b}y\mathbf{c}x\mathbf{q} = \mathbf{a}y\mathbf{b}y\mathbf{c}x\mathbf{q}.$$

If $y \in \text{con}(\mathbf{b})$, then $\mathbf{b} = \mathbf{b}'y\mathbf{b}''$ for some $\mathbf{b}', \mathbf{b}'' \in \mathcal{X}^*$, so that

$$\mathbf{w} = \mathbf{a}x\mathbf{b}'y\mathbf{b}''y\mathbf{c}x\mathbf{q} \stackrel{(20.1d)}{\approx} \mathbf{a}y\mathbf{b}'y\mathbf{b}''y\mathbf{c}x\mathbf{q} = \mathbf{a}y\mathbf{b}y\mathbf{c}x\mathbf{q}.$$

Therefore in any case, the two occurrences of x appearing in (20.3) no longer sandwich y . This argument can be repeated until a word of the form

$$\mathbf{w}' = \mathbf{p}'x\mathbf{s}_1x\mathbf{s}_2 \dots x\mathbf{s}_m x\mathbf{q}$$

is obtained, where $\mathbf{p}', \mathbf{s}_1, \dots, \mathbf{s}_m \in \mathcal{X}^*$ are such that $\text{con}(\mathbf{s}_1 \dots \mathbf{s}_m) \subseteq \text{sim}(\mathbf{w}')$ and $x \notin \text{con}(\mathbf{p}')$. In particular, the letter x is linked in \mathbf{w}' .

The procedure in the previous paragraph can be repeated on \mathbf{p}' to convert \mathbf{w}' into a word where all non-simple letters are linked. □

In this subsection, a non-simple word \mathbf{w} is said to be in *canonical form* if

$$\mathbf{w} = \mathbf{p}\mathbf{x}_1 \dots \mathbf{x}_m \mathbf{q}, \tag{20.4}$$

where $\mathbf{p}, \mathbf{q} \in \mathcal{X}^*$, $\mathbf{x}_1, \dots, \mathbf{x}_m \in \mathcal{X}^+$, and $m \geq 1$ satisfy the following:

- (I) \mathbf{x}_i is an x_i -block;
 - (II) the letters of \mathbf{p} and \mathbf{q} are simple in \mathbf{w} ;
 - (III) $\mathbf{p}, \mathbf{x}_1, \dots, \mathbf{x}_m, \mathbf{q}$ are pairwise disjoint;
 - (IV) if x_i^2 is a suffix of \mathbf{x}_i and $\mathbf{x}_{i+1} = x_{i+1}^2$, then x_i alphabetically precedes x_{i+1} .
- Note that x_1, \dots, x_m are precisely all non-simple letters of the word \mathbf{w} in (20.4).

LEMMA 20.5. *Let \mathbf{w} be any non-simple word. Then there exists some word $\bar{\mathbf{w}}$ in canonical form such that the identities (20.1) imply the identity $\mathbf{w} \approx \bar{\mathbf{w}}$.*

Proof. It suffices to convert \mathbf{w} , using the identities (20.1), into a word in canonical form. By Lemma 20.4, the word \mathbf{w} can be assumed linked, say

$$\mathbf{w} = \mathbf{p}\mathbf{x}_1\mathbf{t}_1\mathbf{x}_2\mathbf{t}_2 \dots \mathbf{x}_{m-1}\mathbf{t}_{m-1}\mathbf{x}_m\mathbf{q},$$

where the letters of $\mathbf{p}, \mathbf{t}_1, \dots, \mathbf{t}_{m-1}, \mathbf{q} \in \mathcal{X}^*$ are all simple in \mathbf{w} , each \mathbf{x}_i is a linked word that begins and ends with x_i , and x_1, \dots, x_m are precisely the non-simple letters of \mathbf{w} . It is clearly seen that the identities (20.1a) can be used to convert each \mathbf{x}_i into an x_i -block. If $\mathbf{t}_i \neq \emptyset$, then

$$\begin{aligned} \mathbf{w} &\stackrel{(20.1a)}{\approx} \mathbf{p}\mathbf{x}_1\mathbf{t}_1 \dots \mathbf{x}_i\mathbf{t}_i x_{i+1}^2 \mathbf{x}_{i+1} \dots \mathbf{t}_{m-1}\mathbf{x}_m\mathbf{q} \\ &\stackrel{(20.1c)}{\approx} \mathbf{p}\mathbf{x}_1\mathbf{t}_1 \dots \mathbf{x}_i x_{i+1} \mathbf{t}_i x_{i+1}^2 \mathbf{x}_{i+1} \dots \mathbf{t}_{m-1}\mathbf{x}_m\mathbf{q} \\ &\stackrel{(20.1a)}{\approx} \mathbf{p}\mathbf{x}_1\mathbf{t}_1 \dots \underbrace{\mathbf{x}_i x_{i+1} \mathbf{t}_i \mathbf{x}_{i+1}}_{\mathbf{x}'_{i+1}} \dots \mathbf{t}_{m-1}\mathbf{x}_m\mathbf{q} \end{aligned}$$

where \mathbf{x}'_{i+1} is an x_{i+1} -block. Therefore \mathbf{w} can be converted into a word of the form (20.4) with (I)–(III) satisfied.

It remains to convert \mathbf{w} into a word that satisfies (IV). Suppose that x_i^2 is a suffix of \mathbf{x}_i and $\mathbf{x}_{i+1} = x_{i+1}^2$, where x_{i+1} alphabetically precedes x_i . Then $\mathbf{x}_i = x_i\mathbf{s}_1x_i\mathbf{s}_2 \dots x_i\mathbf{s}_r x_i^2$. In what

follows, it is shown that the relabeling $x_i \leftrightarrow x_{i+1}$ can be performed on \mathbf{w} , so that (IV) is satisfied. If $r = 0$, then $\mathbf{x}_i = x_i^2$ so that

$$\mathbf{w} = \dots \underbrace{x_i^2}_{\mathbf{x}_i} \cdot \underbrace{x_{i+1}^2}_{\mathbf{x}_{i+1}} \dots \stackrel{(20.1b)}{\approx} \dots x_{i+1}^2 \cdot x_i^2 \dots$$

If $r \geq 1$, then

$$\begin{aligned} \mathbf{w} &= \dots \overbrace{x_i \mathbf{s}_1 x_i \mathbf{s}_2 \dots x_i \mathbf{s}_r x_i^2}^{\mathbf{x}_i} \cdot \overbrace{x_{i+1}^2}_{\mathbf{x}_{i+1}} \dots \\ (20.1b) \quad &\approx \dots x_i \mathbf{s}_1 x_i \mathbf{s}_2 \dots x_i \mathbf{s}_r \cdot x_{i+1}^2 x_i^2 \dots \\ (20.1a) \quad &\approx \dots x_i \mathbf{s}_1 x_i \mathbf{s}_2 \dots x_i \mathbf{s}_r x_{i+1}^r \cdot x_{i+1}^2 x_i^2 \dots \\ (20.1f) \quad &\approx \dots x_{i+1} \mathbf{s}_1 x_{i+1} \mathbf{s}_2 \dots x_{i+1} \mathbf{s}_r x_i^r \cdot x_{i+1}^2 x_i^2 \dots \\ (20.1a) \quad &\approx \dots x_{i+1} \mathbf{s}_1 x_{i+1} \mathbf{s}_2 \dots x_{i+1} \mathbf{s}_r x_i^2 \cdot x_{i+1}^2 x_i^2 \dots \\ (20.1b) \quad &\approx \dots x_{i+1} \mathbf{s}_1 x_{i+1} \mathbf{s}_2 \dots x_{i+1} \mathbf{s}_r x_{i+1}^2 \cdot x_i^2 x_i^2 \dots \\ (20.1a) \quad &\approx \dots x_{i+1} \mathbf{s}_1 x_{i+1} \mathbf{s}_2 \dots x_{i+1} \mathbf{s}_r x_{i+1}^2 \cdot x_i^2 \dots \end{aligned}$$

as required. □

LEMMA 20.6. *Let $S \in \{\mathcal{D}_2, \mathcal{D}_4\}$ and let*

$$\mathbf{w} = \mathbf{p} \mathbf{x}_1 \dots \mathbf{x}_m \mathbf{q} \quad \text{and} \quad \mathbf{w}' = \mathbf{p} \mathbf{y}_1 \dots \mathbf{y}_m \mathbf{q}$$

be any words in canonical form such that each \mathbf{x}_i is an x_i -block and each \mathbf{y}_i is a y_i -block with $\{x_1, \dots, x_m\} = \{y_1, \dots, y_m\}$. Suppose that S satisfies the identity $\mathbf{w} \approx \mathbf{w}'$. Then $(x_1, \dots, x_m) = (y_1, \dots, y_m)$.

Proof. By assumption, $x_1 \prec_{\mathbf{w}} \dots \prec_{\mathbf{w}} x_m$. Let $i \in \{1, \dots, m - 1\}$. Then it suffices to show $x_i \prec_{\mathbf{w}'} x_{i+1}$.

Case 1: \mathbf{x}_{i+1} contains some simple letter, say h , of \mathbf{w} . Then

(a) $x_{i+1} \not\prec_{\mathbf{w}} h$

and $x_i \prec_{\mathbf{w}} h$, so that $x_i \prec_{\mathbf{w}'} h$ by Lemma 20.2(ii). If $x_i \not\prec_{\mathbf{w}'} x_{i+1}$, then $x_{i+1} \prec_{\mathbf{w}'} x_i \prec_{\mathbf{w}'} h$; this contradicts Lemma 20.2(ii) in view of (a). Hence $x_i \prec_{\mathbf{w}'} x_{i+1}$.

Case 2: \mathbf{x}_{i+1} does not contain any simple letter of \mathbf{w} . Then $\mathbf{x}_{i+1} = x_{i+1}^2$ by (B2). Since \mathbf{x}_i is an x_i -block, it is of the form

$$\mathbf{x}_i = x_i \mathbf{s}_1 x_i \mathbf{s}_2 \dots x_i \mathbf{s}_r x_i^e,$$

where $e \in \{1, 2\}$ and the letters of $\mathbf{s}_1, \dots, \mathbf{s}_r \in \mathcal{X}^+$ are all simple in \mathbf{w} . There are two subcases.

2.1. $e = 1$. Then $r \geq 1$ by (B2). Let $s = \mathbf{t}(\mathbf{s}_r)$. Since $s \prec_{\mathbf{w}} x_{i+1}$ and $s x_i \in \mathbf{F}_{\text{SL}}(\mathbf{w})$,

(b) $x_{i+1} \not\prec_{\mathbf{w}'} s$

and $s x_i \in \mathbf{F}_{\text{SL}}(\mathbf{w}')$ by Lemma 20.2 parts (ii) and (iv). The letter s thus belongs to the x_i -block in \mathbf{w}' . Suppose $x_i \not\prec_{\mathbf{w}'} x_{i+1}$, so that $x_{i+1} \prec_{\mathbf{w}'} x_i$. Then within \mathbf{w}' , the x_{i+1} -block occurs before the x_i -block, so that $x_{i+1} \prec_{\mathbf{w}'} s$. In view of (b), this is impossible. Hence $x_i \prec_{\mathbf{w}'} x_{i+1}$.

2.2. $e = 2$. Then (IV) implies that

(c) x_i alphabetically precedes x_{i+1} .

Seeking a contradiction with (c), suppose $x_i \not\prec_{\mathbf{w}'} x_{i+1}$. Then $x_{i+1} \prec_{\mathbf{w}'} x_i$, whence there exist k and ℓ with $1 \leq k < \ell \leq m$ such that \mathbf{y}_k is an x_{i+1} -block (so that $y_k = x_{i+1}$) and \mathbf{y}_ℓ is an x_i -block (so that $y_\ell = x_i$), that is,

$$\mathbf{w}' = \mathbf{p}\mathbf{y}_1 \dots \mathbf{y}_k \mathbf{y}_{k+1} \dots \mathbf{y}_\ell \dots \mathbf{y}_m \mathbf{q}.$$

Suppose that the factor $\mathbf{y}_{k+1} \dots \mathbf{y}_\ell$ of \mathbf{w}' contains some simple letter h . Then $x_{i+1} = y_k \prec_{\mathbf{w}'} h$ and

(d) $x_i = y_\ell \not\prec_{\mathbf{w}'} h$,

whence $x_{i+1} \prec_{\mathbf{w}} h$ by Lemma 20.2(ii). The assumption $x_i \prec_{\mathbf{w}} x_{i+1}$ implies $x_i \prec_{\mathbf{w}} h$. But then Lemma 20.2(ii) implies $x_i \prec_{\mathbf{w}'} h$, which contradicts (d). Therefore the factor $\mathbf{y}_{k+1} \dots \mathbf{y}_\ell$ of \mathbf{w}' cannot contain any simple letter. Since each \mathbf{y}_i is a y_i -block, it follows from (B2) that

$$(\mathbf{y}_{k+1}, \dots, \mathbf{y}_\ell) = (y_{k+1}^2, \dots, y_\ell^2).$$

By (IV), the letters y_{k+1}, \dots, y_ℓ are in alphabetical order. Specifically,

(e) y_{k+1} alphabetically precedes $y_\ell = x_i$.

Now since \mathbf{y}_k is an x_{i+1} -block, it is of the form

$$\mathbf{y}_k = x_{i+1} \mathbf{t}_1 x_{i+1} \mathbf{t}_2 \dots x_{i+1} \mathbf{t}_r x_{i+1}^f,$$

where $f \in \{1, 2\}$ and the letters of $\mathbf{t}_1, \dots, \mathbf{t}_r \in \mathcal{X}^+$ are all simple in \mathbf{w}' . Suppose $f = 1$. Then $r \geq 1$ by (B2). Let $t = \mathbf{t}(\mathbf{t}_r)$. Since $t \prec_{\mathbf{w}'} y_\ell = x_i$ and $tx_{i+1} \in \mathbf{F}_{\text{SL}}(\mathbf{w}')$,

(f) $x_i \not\prec_{\mathbf{w}} t$

and $tx_{i+1} \in \mathbf{F}_{\text{SL}}(\mathbf{w})$ by Lemma 20.2 parts (ii) and (iv). Therefore the simple letter t of \mathbf{w} occurs in the x_{i+1} -block \mathbf{x}_{i+1} , whence $x_i \prec_{\mathbf{w}} t$. In view of (f), this is impossible. Hence $f = 2$. Now by (IV), the letter $x_{i+1} = y_k$ alphabetically precedes y_{k+1} . Therefore by (e), the letter x_{i+1} alphabetically precedes x_i . This is the required contradiction with (c). \square

Proof of Proposition 20.1. Let $S \in \{\mathcal{D}_2, \mathcal{D}_4\}$. It is routinely checked that S satisfies the identities (20.1). Hence it suffices to show that any identity $\mathbf{w} \approx \mathbf{w}'$ satisfied by S is implied by the identities (20.1). If either \mathbf{w} or \mathbf{w}' is a simple word, then it follows from Lemma 20.2 parts (i) and (iii) that the identity $\mathbf{w} \approx \mathbf{w}'$ is trivial and so is vacuously implied by the identities (20.1). Therefore assume that \mathbf{w} and \mathbf{w}' are both non-simple words and so by Lemma 20.5, can be chosen to be in canonical form. Since $\text{con}(\mathbf{w}) = \text{con}(\mathbf{w}')$ and $\text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$ by part (i) of Lemma 20.2, it follows from parts (iii) and (v) of the same lemma that

$$\mathbf{w} = \mathbf{p}\mathbf{x}_1 \dots \mathbf{x}_m \mathbf{q} \quad \text{and} \quad \mathbf{w}' = \mathbf{p}\mathbf{y}_1 \dots \mathbf{y}_m \mathbf{q},$$

where each \mathbf{x}_i is an x_i -block and each \mathbf{y}_i is a y_i -block with $\{x_1, \dots, x_m\} = \{y_1, \dots, y_m\}$. It then follows from Lemma 20.6 that each \mathbf{y}_i is an x_i -block. Let

$$\mathbf{p}, \mathbf{s}_1, \dots, \mathbf{s}_n, \mathbf{q} \tag{20.5}$$

be all the maximal factors of \mathbf{w} , in order of appearance, that consist of simple letters, so that $\mathbf{w}_{\text{sim}} = \mathbf{p}\mathbf{s}_1 \dots \mathbf{s}_n \mathbf{q}$. By part (v) of Lemma 20.2, the maximal factors of \mathbf{w}' , in order of appearance, that consist of simple letters coincide with the words from (20.5); by part (iii) of the same lemma, $\mathbf{w}'_{\text{sim}} = \mathbf{p}\mathbf{s}_1 \dots \mathbf{s}_n \mathbf{q}$. Hence

$$\mathbf{w} = \mathbf{p} \underbrace{x_1 \mathbf{s}_1 x_1 \mathbf{s}_2 \dots x_1 \mathbf{s}_j x_1^e}_{\mathbf{x}_1} \mathbf{x}_2 \dots \mathbf{x}_m \mathbf{q} \quad \text{and} \quad \mathbf{w}' = \mathbf{p} \underbrace{x_1 \mathbf{s}_1 x_1 \mathbf{s}_2 \dots x_1 \mathbf{s}_k x_1^f}_{\mathbf{y}_1} \mathbf{y}_2 \dots \mathbf{y}_m \mathbf{q}.$$

Suppose $j \neq k$, say $j < k$. Then $x_1 \not\prec_{\mathbf{w}'} h$ where $h = \mathbf{h}(\mathbf{s}_k)$. But \mathbf{s}_k is a factor of $\mathbf{x}_2 \dots \mathbf{x}_m$ and so $x_1 \prec_{\mathbf{w}} h$, contradicting Lemma 20.2(ii). Hence $j = k$. Let $t = \mathbf{t}(\mathbf{s}_j) = \mathbf{t}(\mathbf{s}_k)$. If $e \neq f$, say $(e, f) = (1, 2)$, then $tx_1 \in \mathbf{F}_{\text{SL}}(\mathbf{w})$ and $tx_1 \notin \mathbf{F}_{\text{SL}}(\mathbf{w}')$, contradicting Lemma 20.2(iv). Therefore $e = f$, whence $\mathbf{x}_1 = \mathbf{y}_1$. Similarly, it can be shown that $\mathbf{x}_i = \mathbf{y}_i$ for each $i > 1$, so that $\mathbf{w} = \mathbf{w}'$. Consequently, the identity $\mathbf{w} \approx \mathbf{w}'$ is implied by the identities (20.1). \square

20.2. Finite basis property of \mathcal{E}_4

PROPOSITION 20.7. *The variety generated by \mathcal{E}_4 is defined by the identities*

$$xHxKx^2 \approx xHxKx, \quad (20.6a)$$

$$xHxKx \approx xKxHx, \quad (20.6b)$$

$$xHx \cdot Ky \cdot Ty \approx xHx \cdot Ty \cdot Ky, \quad (20.6c)$$

$$x \cdot Hy \cdot Kx \cdot Ty \approx x \cdot Kx \cdot Hy \cdot Ty, \quad (20.6d)$$

$$x \cdot HyKy \cdot Tx \approx x \cdot Tx \cdot HyKy, \quad (20.6e)$$

$$xHx \cdot K_1yK_2y \cdot T_1zT_2z \approx xHx \cdot T_1zT_2z \cdot K_1yK_2y. \quad (20.6f)$$

For any word \mathbf{w} , let $F_{SN}(\mathbf{w})$ denote the set of factors of \mathbf{w} of length two that begin with a simple letter and end with a non-simple letter:

$$F_{SN}(\mathbf{w}) = \{xy \in \mathcal{X}^2 \mid \mathbf{w} \in \mathcal{X}^*xy\mathcal{X}^*, x \in \text{sim}(\mathbf{w}), y \notin \text{sim}(\mathbf{w})\}.$$

LEMMA 20.8. *Suppose that $\mathbf{w} \approx \mathbf{w}'$ is any identity satisfied by \mathcal{E}_4 . Then:*

- (i) for each $x \in \mathcal{X}$, either $\text{occ}(x, \mathbf{w}) = \text{occ}(x, \mathbf{w}') \leq 2$ or $\text{occ}(x, \mathbf{w}), \text{occ}(x, \mathbf{w}') \geq 3$;
- (ii) $h(\mathbf{w}) = h(\mathbf{w}')$;
- (iii) $F_{SN}(\mathbf{w}) = F_{SN}(\mathbf{w}')$;
- (iv) $F_{SS}(\mathbf{w}) = F_{SS}(\mathbf{w}')$.

Proof. (i) This follows from Lemma 2.1(vii) since the subsemigroup $\{1, 3, 4, 5\}$ of \mathcal{E}_4 is isomorphic to N_3^1 .

(ii) This follows from Lemma 2.1(i) since the subsemigroup $\{5, 6\}$ of \mathcal{E}_4 is isomorphic to L_2 .

(iii) Suppose $F_{SN}(\mathbf{w}) \neq F_{SN}(\mathbf{w}')$, say $xy \in F_{SN}(\mathbf{w}) \setminus F_{SN}(\mathbf{w}')$. Then it follows from part (i) that x is simple in \mathbf{w}' and y is non-simple in \mathbf{w}' , whence xy is not a factor of \mathbf{w}' . Let $\varphi: \mathcal{X} \rightarrow \mathcal{E}_4$ denote the substitution

$$z \mapsto \begin{cases} 2 & \text{if } z = x, \\ 5 & \text{if } z = y, \\ 6 & \text{otherwise.} \end{cases}$$

Then $\mathbf{w}\varphi = \dots 2 \cdot 5 \dots = 1$ and $\mathbf{w}'\varphi \in \{5, 6\}^* \cdot 2 \cdot 6 \cdot \{5, 6\}^* = \{3\}$, which is impossible. Hence $F_{SN}(\mathbf{w}) = F_{SN}(\mathbf{w}')$.

(iv) It follows from part (i) that $\text{con}(\mathbf{w}) = \text{con}(\mathbf{w}')$ and $\text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$. The identity (2.3) with $n = 3$ is not satisfied by the semigroup \mathcal{E}_4 because $5^3 \cdot 2 \cdot 5^3 \cdot 6 \cdot 5^3 \neq 5^3 \cdot 2 \cdot 6 \cdot 5^3$. Therefore $F_{SS}(\mathbf{w}) = F_{SS}(\mathbf{w}')$ by Lemma 2.10. \square

For any letter $x \in \mathcal{X}$, define an x -rigid block to be a non-simple word of the form

$$\mathbf{x} = \mathbf{s}_1x\mathbf{s}_2x \dots \mathbf{s}_{r-1}x\mathbf{s}_rx^e, \quad (20.7)$$

where $\mathbf{s}_1 \in \mathcal{X}^*$, $\mathbf{s}_2, \dots, \mathbf{s}_r \in \mathcal{X}^+$, $e \in \{1, 2, 3\}$, and $r \geq 1$ satisfy the following:

- (Ri1) the letters of $\mathbf{s}_1, \dots, \mathbf{s}_r$ are all simple in \mathbf{x} ;
- (Ri2) the letters $h(\mathbf{s}_2), \dots, h(\mathbf{s}_r)$ are in strict alphabetical order;
- (Ri3) if $r = 1$, then $e \in \{2, 3\}$;
- (Ri4) if $r = 2$, then $e \in \{1, 2\}$;
- (Ri5) if $r \geq 3$, then $e = 1$.

The x -rigid block in (20.7) is fully x -rigid if

- (Ri6) $\mathbf{s}_1 \neq \emptyset$ implies that the letters $h(\mathbf{s}_1), \dots, h(\mathbf{s}_r)$ are in strict alphabetical order.

REMARK 20.9. (i) In any x -rigid block, the letter x is the only non-simple letter.
 (ii) The x -rigid block in (20.7) is vacuously fully x -rigid if $\mathbf{s}_1 = \emptyset$.

LEMMA 20.10. Let \mathbf{h} and \mathbf{x} be any non-simple words that end with non-simple letters. Suppose that x is the only non-simple letter of \mathbf{x} and that $\mathfrak{t}(\mathbf{x}) = x$. Then:

- (i) there exists an x -rigid block $\tilde{\mathbf{x}}$ such that the identities (20.6) imply the identity $\mathbf{x} \approx \tilde{\mathbf{x}}$;
- (ii) there exists a fully x -rigid block $\tilde{\mathbf{x}}$ such that the identities (20.6) imply the identity $\mathbf{h}\mathbf{x} \approx \mathbf{h}\tilde{\mathbf{x}}$.

Proof. (i) It suffices to convert \mathbf{x} , using the identities (20.6), into an x -rigid block. By assumption, $\mathbf{x} = \mathbf{s}_1x\mathbf{s}_2x \dots \mathbf{s}_{k-1}x\mathbf{s}_kx$ for some $\mathbf{s}_1, \dots, \mathbf{s}_k \in \mathcal{X}^*$ with $\text{con}(\mathbf{s}_1 \dots \mathbf{s}_k) = \text{sim}(\mathbf{x})$. The identities (20.6b) can be used to arrange the factors $\mathbf{s}_2x, \dots, \mathbf{s}_kx$ in any order. In particular, the factors from $\mathbf{s}_2x, \dots, \mathbf{s}_kx$ with empty \mathbf{s}_i can be gathered to the right, resulting in a word of the form (20.7) with (Ri1) satisfied and $e \geq 1$. Then the identities (20.6b) can be applied to arrange the factors $\mathbf{s}_2x, \dots, \mathbf{s}_rx$ in (20.7) until (Ri2) is satisfied. Finally, the identities (20.6a) can be applied until (Ri3)–(Ri5) are satisfied.

(ii) It suffices to convert the word $\mathbf{h}\mathbf{x}$, using the identities (20.6), into a word $\mathbf{h}\tilde{\mathbf{x}}$ where $\tilde{\mathbf{x}}$ is a fully x -rigid block. By assumption, $\mathbf{h} = \mathbf{a}\mathbf{h}\mathbf{b}\mathbf{h}$ for some $\mathbf{a}, \mathbf{b} \in \mathcal{X}^*$ and $h \in \mathcal{X}$. By part (i), the identities (20.6) can be used to convert \mathbf{x} into an x -rigid block, say the one in (20.7). Hence

$$\mathbf{h}\mathbf{x} \stackrel{(20.6)}{\approx} \mathbf{a}\mathbf{h}\mathbf{b}\mathbf{h} \underbrace{\mathbf{s}_1x\mathbf{s}_2x \dots \mathbf{s}_{r-1}x\mathbf{s}_rx^e}_{\tilde{\mathbf{x}}}$$

where $\tilde{\mathbf{x}}$ satisfies (Ri1)–(Ri5). If $\mathbf{s}_1 \neq \emptyset$, then the identities $\{(20.6b), (20.6c)\}$ can be applied until (Ri6) is additionally satisfied. □

LEMMA 20.11. Let \mathbf{x}_i be any x_i -rigid block, where $i \in \{1, 2, 3\}$. Then the identities (20.6f) imply the identity $\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3 \approx \mathbf{x}_1\mathbf{x}_3\mathbf{x}_2$.

Proof. By assumption, $\mathbf{x}_i = \mathbf{a}_ix_i\mathbf{b}_ix_i$ for some $\mathbf{a}_i, \mathbf{b}_i \in \mathcal{X}^*$. Since

$$\begin{aligned} \mathbf{x}_1\mathbf{x}_2\mathbf{x}_3 &= \mathbf{a}_1x_1\mathbf{b}_1x_1 \cdot \mathbf{a}_2x_2\mathbf{b}_2x_2 \cdot \mathbf{a}_3x_3\mathbf{b}_3x_3 \\ &\stackrel{(20.6f)}{\approx} \mathbf{a}_1x_1\mathbf{b}_1x_1 \cdot \mathbf{a}_3x_3\mathbf{b}_3x_3 \cdot \mathbf{a}_2x_2\mathbf{b}_2x_2 = \mathbf{x}_1\mathbf{x}_3\mathbf{x}_2, \end{aligned}$$

the identities (20.6f) imply the identity $\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3 \approx \mathbf{x}_1\mathbf{x}_3\mathbf{x}_2$. □

In this subsection, a non-simple word \mathbf{w} is said to be in *canonical form* if

$$\mathbf{w} = \mathbf{x}_1 \dots \mathbf{x}_m\mathbf{z}, \tag{20.8}$$

where $\mathbf{x}_1, \dots, \mathbf{x}_m \in \mathcal{X}^+$ and $\mathbf{z} \in \mathcal{X}^*$ are such that:

- (I) \mathbf{x}_1 is an x_1 -rigid block;
- (II) \mathbf{x}_i is a fully x_i -rigid block for each $i \in \{2, \dots, m\}$;
- (III) the letters x_2, \dots, x_m are in strict alphabetical order;
- (IV) the letters of \mathbf{z} are simple in \mathbf{w} ;
- (V) $\mathbf{x}_1, \dots, \mathbf{x}_m, \mathbf{z}$ are pairwise disjoint.

LEMMA 20.12. Let \mathbf{w} be any non-simple word. Then there exists some word $\bar{\mathbf{w}}$ in canonical form such that the identities (20.6) imply the identity $\mathbf{w} \approx \bar{\mathbf{w}}$.

Proof. It suffices to convert \mathbf{w} , using the identities (20.6), into a word in canonical form. Consider a factorization of \mathbf{w} that displays all of its non-simple letters individually, that is,

$\mathbf{w} = \mathbf{s}_1 x_1 \mathbf{s}_2 x_2 \dots \mathbf{s}_k x_k \mathbf{z}$ where the letters x_1, \dots, x_k are non-simple in \mathbf{w} and the letters in the factors $\mathbf{s}_1, \dots, \mathbf{s}_k, \mathbf{z} \in \mathcal{X}^*$ are simple in \mathbf{w} . Each letter in the list x_1, \dots, x_k is non-simple in \mathbf{w} and thus appears at least twice in the list, whence $k \geq 2$.

Let $r \geq 2$ be such that $x_1 = x_2 = \dots = x_{r-1}$ and $x_1 \neq x_r$. Suppose that $\ell > r$ is the least integer such that $x_1 = x_\ell$. Then

$$\mathbf{w} = \left(\prod_{i=1}^{r-1} (\mathbf{s}_i x_1) \right) \underbrace{\left(\prod_{i=r}^{\ell-1} (\mathbf{s}_i x_i) \right)}_{\mathbf{a}} \mathbf{s}_\ell x_1 \underbrace{\left(\prod_{i=\ell+1}^k (\mathbf{s}_i x_i) \right)}_{\mathbf{b}} \mathbf{z}.$$

Note that the factor \mathbf{a} is nonempty because $x_r \in \text{con}(\mathbf{a})$, and that $\mathbf{t}(\mathbf{a}) = x_{\ell-1} \neq x_1$ by the minimality of ℓ . Since the letter $\mathbf{t}(\mathbf{a}) = x_{\ell-1}$ is non-simple in \mathbf{w} , either $\text{occ}(x_{\ell-1}, \mathbf{a}) \geq 2$ or $x_{\ell-1} \in \text{con}(\mathbf{b})$. Hence

$$\begin{aligned} \mathbf{w} &= \left(\prod_{i=1}^{r-1} (\mathbf{s}_i x_1) \right) \cdot \mathbf{a} \cdot (\mathbf{s}_\ell x_1) \cdot \mathbf{b} \cdot \mathbf{z} \stackrel{(20.6d), (20.6e)}{\approx} \left(\prod_{i=1}^{r-1} (\mathbf{s}_i x_1) \right) \cdot \mathbf{s}_\ell x_1 \cdot \mathbf{a} \cdot \mathbf{b} \cdot \mathbf{z} \\ &= \left(\prod_{i=1}^{r-1} (\mathbf{s}_i x_1) \right) \cdot \mathbf{s}_\ell x_1 \cdot \left(\prod_{i=r}^{\ell-1} (\mathbf{s}_i x_i) \right) \left(\prod_{i=\ell+1}^k (\mathbf{s}_i x_i) \right) \cdot \mathbf{z}, \end{aligned}$$

that is, the identities $\{(20.6d), (20.6e)\}$ can be used to move the factor $\mathbf{s}_\ell x_1$ to the left until it immediately follows the prefix $\prod_{i=1}^{r-1} (\mathbf{s}_i x_1)$.

Now $x_1 \notin \{x_r, \dots, x_{\ell-1}\}$ by the minimality of ℓ . But if $x_1 \in \{x_{\ell+1}, \dots, x_k\}$, say $x_1 = x_j$ for some least $j \in \{\ell+1, \dots, k\}$, then the procedure in the previous paragraph can be repeated to move the factor $\mathbf{s}_j x_1$ in $\prod_{i=\ell+1}^k (\mathbf{s}_i x_i)$ to the left until it immediately follows the prefix $(\prod_{i=1}^{r-1} (\mathbf{s}_i x_1)) \mathbf{s}_\ell x_1$. It is easily seen how this can be repeated until \mathbf{w} is converted into a word of the form $\mathbf{x}_1 \dots \mathbf{x}_m \mathbf{z}$ with (IV) and (V) satisfied, and each \mathbf{x}_i is a word that contains precisely one non-simple letter, say x_i , with $\mathbf{t}(\mathbf{x}_i) = x_i$. Then it follows from Lemma 20.10 that (I) and (II) are satisfied by applying the identities (20.6). By Lemma 20.11, the factors $\mathbf{x}_2, \dots, \mathbf{x}_m$ can be arranged by the identities (20.6f) in any manner. Hence (III) is satisfied. \square

Proof of Proposition 20.7. It is routinely checked that \mathcal{E}_4 satisfies the identities (20.6). Hence it suffices to show that any identity $\mathbf{w} \approx \mathbf{w}'$ satisfied by \mathcal{E}_4 is implied by the identities (20.6). If either \mathbf{w} or \mathbf{w}' is a simple word, then it follows from Lemma 20.8 parts (i) and (iv) that the identity $\mathbf{w} \approx \mathbf{w}'$ is trivial and so is vacuously implied by (20.6). Therefore assume that \mathbf{w} and \mathbf{w}' are both non-simple words and so, by Lemma 20.12, can be chosen to be in canonical form. Lemma 20.8(i) implies $\text{con}(\mathbf{w}) = \text{con}(\mathbf{w}')$ and $\text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$. Hence

$$\mathbf{w} = \mathbf{x}_1 \dots \mathbf{x}_m \mathbf{z} \quad \text{and} \quad \mathbf{w}' = \mathbf{x}'_1 \dots \mathbf{x}'_m \mathbf{z}',$$

where \mathbf{x}_1 is an x_1 -rigid block, \mathbf{x}_i is a fully x_i -rigid block for each $i \in \{2, \dots, m\}$, \mathbf{x}'_1 is an x'_1 -rigid block, and \mathbf{x}'_i is a fully x'_i -rigid block for each $i \in \{2, \dots, m\}$.

By definition, \mathbf{x}_1 and \mathbf{x}'_1 are of the form

$$\mathbf{x}_1 = \mathbf{s}_1 x_1 \mathbf{s}_2 x_1 \dots \mathbf{s}_{r-1} x_1 \mathbf{s}_r x_1^e \quad \text{and} \quad \mathbf{x}'_1 = \mathbf{s}'_1 x'_1 \mathbf{s}'_2 x'_1 \dots \mathbf{s}'_{r'-1} x'_1 \mathbf{s}'_{r'} (x'_1)^{e'}.$$

Then Lemma 20.8 parts (ii)–(iv) imply $\mathbf{s}_1 x_1 = \mathbf{s}'_1 x'_1$, and parts (iii) and (iv) of the same lemma imply $\{\mathbf{s}_2, \dots, \mathbf{s}_r\} = \{\mathbf{s}'_2, \dots, \mathbf{s}'_{r'}\}$. Since \mathbf{x}_1 and \mathbf{x}'_1 are x_1 -rigid blocks, it follows from (Ri2) that $(\mathbf{s}_2, \dots, \mathbf{s}_r) = (\mathbf{s}'_2, \dots, \mathbf{s}'_{r'})$. It is then easily shown by (Ri3)–(Ri5) and Lemma 20.8(i) that $e = e'$. Hence $\mathbf{x}_1 = \mathbf{x}'_1$. Now since \mathbf{w} and \mathbf{w}' share the same set of non-simple letters, it follows from (III) that $x_i = x'_i$ for all $i \in \{2, \dots, m\}$.

Now consider any $i \in \{2, \dots, m\}$. Then \mathbf{x}_i and \mathbf{x}'_i are of the form

$$\mathbf{x}_i = \mathbf{s}_1 x_i \mathbf{s}_2 x_i \dots \mathbf{s}_{r-1} x_i \mathbf{s}_r x_i^e \quad \text{and} \quad \mathbf{x}'_i = \mathbf{s}'_1 x_i \mathbf{s}'_2 x_i \dots \mathbf{s}'_{r'-1} x_i \mathbf{s}'_r x_i^{e'}$$

Lemma 20.8 parts (iii) and (iv) imply $\{\mathbf{s}_1, \dots, \mathbf{s}_r\} = \{\mathbf{s}'_1, \dots, \mathbf{s}'_{r'}\}$. Since \mathbf{x}_i and \mathbf{x}'_i are fully x_i -rigid, it follows from (Ri2) and (Ri6) that $(\mathbf{s}_1, \dots, \mathbf{s}_r) = (\mathbf{s}'_1, \dots, \mathbf{s}'_{r'})$. It is then easily shown by (Ri3)–(Ri5) and Lemma 20.8(i) that $e = e'$. Therefore $\mathbf{x}_i = \mathbf{x}'_i$. Since $i \in \{2, \dots, m\}$ is arbitrary, $\mathbf{x}_1 \dots \mathbf{x}_m = \mathbf{x}'_1 \dots \mathbf{x}'_m$. Finally, $\mathbf{z} = \mathbf{z}'$ by Lemma 20.8(iv). Consequently, the identity $\mathbf{w} \approx \mathbf{w}'$ is trivial and is implied by the identities (20.6). \square

21. $\mathcal{D}_6, \mathcal{D}_7, \mathcal{D}_8, \mathcal{D}_9, \mathcal{E}_2, \mathcal{E}_3$

This section establishes the finite basis property of the following semigroups.

\mathcal{D}_6	1 2 3 4 5 6	\mathcal{D}_7	1 2 3 4 5 6	\mathcal{D}_8	1 2 3 4 5 6
1	1 1 1 1 1 1	1	1 1 1 1 1 1	1	1 1 1 1 1 1
2	1 1 1 1 1 2	2	1 1 1 1 1 2	2	1 1 1 1 1 2
3	1 1 1 1 4 1	3	1 1 1 1 4 1	3	1 1 1 1 4 2
4	4 4 4 4 4 4	4	4 4 4 4 4 4	4	4 4 4 4 4 4
5	5 5 5 5 5 5	5	5 5 5 5 5 5	5	5 5 5 5 5 5
6	1 2 3 4 1 6	6	1 2 3 4 4 6	6	1 2 3 4 1 6
\mathcal{D}_9	1 2 3 4 5 6	\mathcal{E}_2	1 2 3 4 5 6	\mathcal{E}_3	1 2 3 4 5 6
1	1 1 1 1 1 1	1	1 1 1 1 1 1	1	1 1 1 1 1 1
2	1 1 1 1 1 2	2	1 1 1 1 1 2	2	1 1 1 1 1 2
3	1 1 1 1 4 2	3	3 3 3 3 3 3	3	3 3 3 3 3 3
4	4 4 4 4 4 4	4	3 3 3 3 1 3	4	3 3 3 3 1 3
5	5 5 5 5 5 5	5	5 5 5 5 5 5	5	5 5 5 5 5 5
6	1 2 3 4 4 6	6	1 2 3 4 1 6	6	1 2 3 4 3 6

PROPOSITION 21.1. *The variety generated by any $S \in \{\mathcal{D}_6, \mathcal{D}_7, \mathcal{D}_8, \mathcal{D}_9, \mathcal{E}_2, \mathcal{E}_3\}$ is defined by the identities*

$$xHx^2 \approx xHx, \tag{21.1a}$$

$$xHxKy^2x \approx xHxKy^2, \tag{21.1b}$$

$$xHyKxTy \approx xHyKyTx. \tag{21.1c}$$

The proof of Proposition 21.1 is given in §21.3.

21.1. *Identities satisfied by $\mathcal{D}_6, \mathcal{D}_7, \mathcal{D}_8, \mathcal{D}_9, \mathcal{E}_2$, and \mathcal{E}_3*

LEMMA 21.2. *The identities (21.1) imply the identity*

$$xabhx \approx xaxhxb \tag{21.2}$$

for any $\mathbf{a}, \mathbf{b} \in \mathcal{X}^+$ such that $\text{con}(\mathbf{b}) \subseteq \text{con}(x\mathbf{a})$.

Proof. This is proved by induction on $|\mathbf{b}|$. If $|\mathbf{b}| = 1$, say $\mathbf{b} = y \in \text{con}(x\mathbf{a})$, then

$$xabhx = xayhx \stackrel{(21.1a)}{\approx} xayhxx \stackrel{(21.1c)}{\approx} xaxhxy = xaxhxb.$$

Assume that the identities (21.1) imply the identity (21.2) whenever $1 \leq |\mathbf{b}| < n$. If $|\mathbf{b}| = n$, say $\mathbf{b} = \mathbf{b}'\mathbf{b}''$ for some $\mathbf{b}', \mathbf{b}'' \in \mathcal{X}^+$ with $|\mathbf{b}'| + |\mathbf{b}''| = n$, then

$$\begin{aligned} x\mathbf{a}b\mathbf{h}x &= x\mathbf{a}\mathbf{b}'\mathbf{b}''\mathbf{h}x \stackrel{(21.1)}{\approx} x\mathbf{a}\mathbf{b}'x\mathbf{h}x\mathbf{b}'' \stackrel{(21.1)}{\approx} x\mathbf{a}x\mathbf{h}x\mathbf{b}'\mathbf{b}'' \\ &\stackrel{(21.1a)}{\approx} x\mathbf{a}x\mathbf{h}x\mathbf{b}'\mathbf{b}'' = x\mathbf{a}x\mathbf{h}x\mathbf{b}, \end{aligned} \tag{21.3}$$

where the two deductions in (21.3) hold by induction hypothesis. □

For any word \mathbf{w} , let $F_{11}(\mathbf{w})$ denote the set of factors of \mathbf{w} of length two that consist of two first occurrences in \mathbf{w} :

$$F_{11}(\mathbf{w}) = \{xy \in \mathcal{X}^2 \mid \mathbf{w} \in (\mathcal{X} \setminus \{x, y\})^* xy \mathcal{X}^* \text{ and } x \neq y\}.$$

LEMMA 21.3. *Let $S \in \{\mathcal{D}_6, \mathcal{D}_7, \mathcal{D}_8, \mathcal{D}_9, \mathcal{E}_2, \mathcal{E}_3\}$. Suppose that $\mathbf{w} \approx \mathbf{w}'$ is any identity satisfied by S . Then:*

- (i) $\text{ini}(\mathbf{w}) = \text{ini}(\mathbf{w}')$;
- (ii) $\text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$;
- (iii) $\text{t}(\mathbf{w}) \in \text{sim}(\mathbf{w})$ if and only if $\text{t}(\mathbf{w}') \in \text{sim}(\mathbf{w}')$;
- (iv) $F_{11}(\mathbf{w}) = F_{11}(\mathbf{w}')$.

Proof. (i) The subsemigroup $\{1, 4, 6\}$ of $S' \in \{\mathcal{D}_6, \mathcal{D}_7, \mathcal{D}_8, \mathcal{D}_9\}$ and the subsemigroup $\{1, 3, 6\}$ of $S'' \in \{\mathcal{E}_2, \mathcal{E}_3\}$ are isomorphic to L_2^1 . Therefore the result holds by Lemma 2.1(iii).

(ii) This holds by Lemma 2.1(v) since the subsemigroup $\{1, 2, 6\}$ of S is isomorphic to N_2^1 .

(iii) This follows from Lemma 2.2 since the subsemigroup $\{1, 3, 6\}$ of $S' \in \{\mathcal{D}_6, \mathcal{D}_7\}$, the subsemigroup $\{3, 4, 6\}$ of $S'' \in \{\mathcal{E}_2, \mathcal{E}_3\}$, and the divisor $\{1, 2, 3, 6\}/\{1, 2\}$ of $S''' \in \{\mathcal{D}_8, \mathcal{D}_9\}$ are isomorphic to J .

(iv) Suppose $F_{11}(\mathbf{w}) \neq F_{11}(\mathbf{w}')$, say with $xy \in F_{11}(\mathbf{w}) \setminus F_{11}(\mathbf{w}')$. Then $\mathbf{w} = \mathbf{a}xy\mathbf{b}$ for some $\mathbf{a}, \mathbf{b} \in \mathcal{X}^*$ such that $x, y \notin \text{con}(\mathbf{a})$. Since $\text{ini}(\mathbf{w}') = \text{ini}(\mathbf{w})$ by part (i) and $\text{ini}(\mathbf{w}) = \dots xy \dots$, it follows that $\mathbf{w}' = \mathbf{c}x\mathbf{d}y\mathbf{e}$ for some $\mathbf{c}, \mathbf{e} \in \mathcal{X}^*$ and $\mathbf{d} \in \mathcal{X}^+$ such that $x \notin \text{con}(\mathbf{c})$ and $y \notin \text{con}(\mathbf{cd})$. Let φ_1 and φ_2 denote the following substitutions into $S' \in \{\mathcal{D}_6, \mathcal{D}_7, \mathcal{D}_8, \mathcal{D}_9\}$ and $S'' \in \{\mathcal{E}_2, \mathcal{E}_3\}$, respectively:

$$z \mapsto \begin{cases} 3 & \text{if } z = x, \\ 5 & \text{if } z = y, \\ 6 & \text{otherwise;} \end{cases} \quad z \mapsto \begin{cases} 4 & \text{if } z = x, \\ 5 & \text{if } z = y, \\ 6 & \text{otherwise.} \end{cases}$$

Then the following contradictions are obtained:

$$\begin{aligned} \mathbf{w}\varphi_1 \in \{6\}^* \cdot 3 \cdot 5 \cdot \mathbf{b}\varphi_1 = \{4\} \quad \text{and} \quad \mathbf{w}'\varphi_1 \in \{6\}^* \cdot 3 \cdot \{3, 6\}^+ \cdot 5 \cdot \mathbf{e}\varphi_1 = \{1\}; \\ \mathbf{w}\varphi_2 \in \{6\}^* \cdot 4 \cdot 5 \cdot \mathbf{b}\varphi_2 = \{1\} \quad \text{and} \quad \mathbf{w}'\varphi_2 \in \{6\}^* \cdot 4 \cdot \{4, 6\}^+ \cdot 5 \cdot \mathbf{e}\varphi_2 = \{3\}. \end{aligned}$$

Hence $F_{11}(\mathbf{w}) = F_{11}(\mathbf{w}')$. □

21.2. A canonical form

Any non-simple word \mathbf{w} can be written in the form $\mathbf{w} = \mathbf{p}\mathbf{u}\mathbf{q}$, where $\mathbf{p}, \mathbf{q} \in \mathcal{X}^*$ and $\mathbf{u} \in \mathcal{X}^+$ are such that $\text{con}(\mathbf{p}\mathbf{q}) \subseteq \text{sim}(\mathbf{w})$ and $\text{h}(\mathbf{u}), \text{t}(\mathbf{u}) \notin \text{sim}(\mathbf{w})$. Generality is not lost by assuming $\text{ini}(\mathbf{u}) = x_1 \dots x_m$ for some $m \geq 1$, so that $\text{ini}(\mathbf{w}) = \mathbf{p}x_1 \dots x_m \mathbf{q}$. In this section, such a word \mathbf{w} is said to be in *canonical form* if

$$\mathbf{w} = \mathbf{p} \left(\prod_{i=1}^m (x_i \mathbf{x}_i) \right) \mathbf{q}, \tag{21.4}$$

where all of the following are satisfied:

- (I) $\mathbf{x}_1, \dots, \mathbf{x}_{m-1} \in \{\emptyset, x_1\}$;
- (II) $\mathbf{x}_m = x_1^{e_1} \dots x_m^{e_m}$ for some $e_1, \dots, e_m \in \{0, 1\}$ with $(e_1, \dots, e_m) \neq (0, \dots, 0)$;
- (III) if $(\mathbf{x}_1, \dots, \mathbf{x}_{m-1}) \neq (\emptyset, \dots, \emptyset)$ and $(e_2, \dots, e_m) \neq (0, \dots, 0)$, then $e_1 = 0$.

REMARK 21.4. Consider the word \mathbf{w} in (21.4).

- (i) If $m = 1$, then $\mathbf{w} = \mathbf{p}x_1\mathbf{x}_1\mathbf{q} = \mathbf{p}x_1^2\mathbf{q}$ by (II).
- (ii) The assumption $x_1 = \mathbf{h}(\mathbf{u}) \notin \text{sim}(\mathbf{w})$ implies

$$2 \leq \text{occ}(x_1, \mathbf{w}) = 1 + |\mathbf{x}_1 \dots \mathbf{x}_{m-1}| + e_1.$$

Therefore if $(\mathbf{x}_1, \dots, \mathbf{x}_{m-1}) = (\emptyset, \dots, \emptyset)$, then $e_1 = 1$.

LEMMA 21.5. *Let \mathbf{w} be any non-simple word. Then there exists some word $\bar{\mathbf{w}}$ in canonical form such that the identities (21.1) imply the identity $\mathbf{w} \approx \bar{\mathbf{w}}$.*

Proof. By Lemma 21.2, it suffices to convert \mathbf{w} , using the identities $\{(21.1), (21.2)\}$, into a word in canonical form. As observed in the beginning of this subsection, the word \mathbf{w} can be written as $\mathbf{w} = \mathbf{p}\mathbf{u}\mathbf{q}$ where $\mathbf{p}, \mathbf{q} \in \mathcal{X}^*$ and $\mathbf{u} \in \mathcal{X}^+$ are such that $\text{con}(\mathbf{p}\mathbf{q}) \subseteq \text{sim}(\mathbf{w})$ and $\mathbf{h}(\mathbf{u}), \mathbf{t}(\mathbf{u}) \notin \text{sim}(\mathbf{w})$. Let $\text{ini}(\mathbf{u}) = x_1 \dots x_m$. Then \mathbf{u} can be written as $\mathbf{u} = \prod_{i=1}^m (x_i \mathbf{w}_i)$, where $\mathbf{w}_i \in \{x_1, \dots, x_i\}^*$ and $\mathbf{w}_m \neq \emptyset$. The letter $t = \mathbf{t}(\mathbf{w}_m)$ and any other letter in \mathbf{w}_m are clearly non-simple in \mathbf{w} . By assumption, the letter $x_1 = \mathbf{h}(\mathbf{u})$ is also non-simple in \mathbf{w} . Therefore

$$\begin{aligned} \mathbf{w} &\stackrel{(21.1a)}{\approx} \mathbf{p} \left(\prod_{i=1}^{m-1} (x_i \mathbf{w}_i) \right) x_m \mathbf{w}_m t^2 \mathbf{q} \stackrel{(21.1b)}{\approx} \mathbf{p} \left(\prod_{i=1}^{m-1} (x_i \mathbf{w}_i) \right) x_m \mathbf{w}_m t^2 x_1 \mathbf{q} \\ &\stackrel{(21.1a)}{\approx} \mathbf{p} \left(\prod_{i=1}^{m-1} (x_i \mathbf{w}_i) \right) x_m \mathbf{w}_m x_1 \mathbf{q} \stackrel{(21.1c)}{\approx} \mathbf{p} \left(\prod_{i=1}^{m-1} (x_i \mathbf{w}_i) \right) x_m (x_1 \mathbf{w}_m) \mathbf{q}. \end{aligned}$$

Hence generality is not lost by assuming $\mathbf{h}(\mathbf{w}_m) = x_1$ to begin with, so that $\mathbf{w}_m = x_1 \mathbf{w}'_m$ for some $\mathbf{w}'_m \in \{x_1, \dots, x_m\}^*$.

If $\mathbf{w}_1 \neq \emptyset$, then since $\mathbf{w}_1 \in \{x_1\}^*$, the identity $x^3 \approx x^2$ from (21.1a) can be used to reduce the factor $x_1 \mathbf{w}_1$ of \mathbf{w} to $x_1 x_1$. In other words, the word \mathbf{w}_1 can be converted into $x_1 = \mathbf{x}_1$. Now suppose $\mathbf{w}_k \neq \emptyset$ for some $k \in \{2, \dots, m-1\}$. Since $\text{con}(\mathbf{w}_k) \subseteq \text{con}(\left(\prod_{i=1}^{k-1} (x_i \mathbf{w}_i)\right) x_k)$ and $\mathbf{h}(\mathbf{w}_m) = x_1$, it follows from Lemma 21.2 that

$$\begin{aligned} \mathbf{w} &= \mathbf{p}x_1\mathbf{w}_1 \left(\prod_{i=2}^{k-1} (x_i \mathbf{w}_i) \right) x_k \mathbf{w}_k \left(\prod_{i=k+1}^{m-1} (x_i \mathbf{w}_i) \right) x_m \overbrace{x_1 \mathbf{w}'_m}^{\mathbf{w}_m} \mathbf{q} \\ &\stackrel{(21.2)}{\approx} \mathbf{p}x_1\mathbf{w}_1 \left(\prod_{i=2}^{k-1} (x_i \mathbf{w}_i) \right) x_k x_1 \left(\prod_{i=k+1}^{m-1} (x_i \mathbf{w}_i) \right) x_m x_1 \mathbf{w}_k \mathbf{w}'_m \mathbf{q}, \end{aligned}$$

that is, the identities (21.2) can be used to simultaneously replace the factor \mathbf{w}_k by $x_1 = \mathbf{x}_i$ and insert a copy of \mathbf{w}_k between the x_1 and \mathbf{w}'_m in \mathbf{w}_m . It is easily seen how this can be repeated until \mathbf{w} is converted into the word

$$\mathbf{w} = \mathbf{p} \left(\prod_{i=1}^m (x_i \mathbf{x}_i) \right) x_m \underbrace{x_1 \mathbf{w}_2 \dots \mathbf{w}_{m-1} \mathbf{w}'_m}_{\mathbf{w}''_m} \mathbf{q}$$

where $\mathbf{x}_i \in \{\emptyset, x_1\}$. Since the letters of \mathbf{w}''_m are non-first occurrences in \mathbf{w} , the identities (21.1c) can be used to rearrange them into $\mathbf{x}_m = x_1^{e_1} \dots x_m^{e_m}$, where $e_1, \dots, e_m \geq 0$ are such that

$(e_1, \dots, e_m) \neq (0, \dots, 0)$. The identities (21.1a) can then be applied to reduce any nonzero exponent e_i to 1. Hence (I) and (II) are satisfied. Since x_1 is a letter of \mathbf{w}''_m , it is also a letter of \mathbf{x}_m . Therefore $e_1 = 1$.

Suppose $\mathbf{x}_k \neq \emptyset$ for some $k < m$ and $e_\ell \neq 0$ for some $\ell > 1$. Then $\mathbf{x}_k = x_1$ and $\mathbf{x}_m = x_1 \mathbf{a} x_\ell \mathbf{b}$, where $\mathbf{a} = \prod_{i=2}^{\ell-1} x_i^{e_i}$ and $\mathbf{b} = \prod_{i=\ell+1}^m x_i^{e_i}$. Since the letters of \mathbf{x}_m are non-first occurrences in \mathbf{w} ,

$$\begin{aligned} \mathbf{w} &= \mathbf{p} x_1 \mathbf{x}_1 \dots x_k \mathbf{x}_k \dots x_m \overbrace{x_1 \mathbf{a} x_\ell \mathbf{b}}^{\mathbf{x}_m} \mathbf{q} \\ (21.1a) \quad &\approx \mathbf{p} x_1 \mathbf{x}_1 \dots x_k \mathbf{x}_k \dots x_m x_1 \mathbf{a} x_\ell^2 \mathbf{b} \mathbf{q} \\ (21.1c) \quad &\approx \mathbf{p} x_1 \mathbf{x}_1 \dots x_k \mathbf{x}_k \dots x_m \mathbf{a} x_\ell^2 x_1 \mathbf{b} \mathbf{q} \\ (21.1b) \quad &\approx \mathbf{p} x_1 \mathbf{x}_1 \dots x_k \mathbf{x}_k \dots x_m \mathbf{a} x_\ell^2 \mathbf{b} \mathbf{q} \quad \text{since } \mathbf{x}_k = x_1 \\ (21.1a) \quad &\approx \mathbf{p} x_1 \mathbf{x}_1 \dots x_k \mathbf{x}_k \dots x_m \mathbf{a} x_\ell \mathbf{b} \mathbf{q}. \end{aligned}$$

The first letter x_1 of \mathbf{w}_m is thus eliminated, resulting in $e_1 = 0$. Hence (III) is satisfied. \square

21.3. Proof of Proposition 21.1

Let $S \in \{\mathcal{D}_6, \mathcal{D}_7, \mathcal{D}_8, \mathcal{D}_9, \mathcal{E}_2, \mathcal{E}_3\}$. It is routinely checked that S satisfies the identities (21.1). Hence it suffices to show that any identity $\mathbf{w} \approx \mathbf{w}'$ satisfied by S is implied by the identities (21.1). If either \mathbf{w} or \mathbf{w}' is a simple word, then it follows from Lemma 21.3 parts (i) and (ii) that the identity $\mathbf{w} \approx \mathbf{w}'$ is trivial and so is vacuously implied by the identities (21.1). Therefore assume that \mathbf{w} and \mathbf{w}' are both non-simple words and so by Lemma 21.5, can be chosen to be in canonical form. Hence

$$\mathbf{w} = \mathbf{p} \left(\prod_{i=1}^m (x_i \mathbf{x}_i) \right) \mathbf{q} \quad \text{and} \quad \mathbf{w}' = \mathbf{p}' \left(\prod_{i=1}^{m'} (x'_i \mathbf{x}'_i) \right) \mathbf{q}'.$$

Then by Lemma 21.3(i),

(a) $\mathbf{p} x_1 \dots x_m \mathbf{q} = \text{ini}(\mathbf{w}) = \text{ini}(\mathbf{w}') = \mathbf{p}' x'_1 \dots x'_{m'} \mathbf{q}'$.

Since $x_1 \notin \text{sim}(\mathbf{w})$ by assumption and $\text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$ by Lemma 21.3(ii), the letter x_1 is non-simple in \mathbf{w}' , so that $x_1 \notin \text{con}(\mathbf{p}')$. By a symmetrical argument, $x'_1 \notin \text{sim}(\mathbf{p})$. It then follows from (a) that

(b) $\mathbf{p} = \mathbf{p}'$.

If either \mathbf{q} or \mathbf{q}' is empty, then $\mathbf{q} = \mathbf{q}' = \emptyset$ by Lemma 21.3(iii). Therefore suppose that \mathbf{q} and \mathbf{q}' are nonempty. Seeking a contradiction, suppose $\mathbf{q} \neq \mathbf{q}'$. Then by (a), either \mathbf{q} is a proper suffix of \mathbf{q}' or \mathbf{q}' is a proper suffix of \mathbf{q} . By symmetry, it suffices to assume that \mathbf{q} is a proper suffix of \mathbf{q}' . It then follows from (a) that $x_m \mathbf{q}$ is a suffix of \mathbf{q}' , whence $x_m \in \text{sim}(\mathbf{w}')$ and $x_m \mathbf{h}(\mathbf{q}) \in F_{11}(\mathbf{q}')$. However, $\mathbf{x}_m \neq \emptyset$ by (II) so that $x_m \mathbf{h}(\mathbf{q}) \notin F_{11}(\mathbf{q})$, and this contradicts Lemma 21.3(iv). Therefore

(c) $\mathbf{q} = \mathbf{q}'$.

It now follows from (a)–(c) that $m = m'$ and $x_i = x'_i$ for all i . If $\mathbf{x}_i \neq \mathbf{x}'_i$ for some $i < m$, say $\mathbf{x}_i = \emptyset$ and $\mathbf{x}'_i = x_1$, then $x_i x_{i+1} \in F_{11}(\mathbf{w}) \setminus F_{11}(\mathbf{w}')$, contradicting Lemma 21.3(iv). Therefore $\mathbf{x}_i = \mathbf{x}'_i$ for all $i < m$, whence

$$\mathbf{w} = \mathbf{p} \mathbf{z} \underbrace{x_1^{e_1} x_2^{e_2} \dots x_m^{e_m}}_{\mathbf{x}_m} \mathbf{q} \quad \text{and} \quad \mathbf{w}' = \mathbf{p} \mathbf{z} \underbrace{x_1^{e'_1} x_2^{e'_2} \dots x_m^{e'_m}}_{\mathbf{x}'_m} \mathbf{q}$$

with $\mathbf{z} = \prod_{i=1}^{m-1} (x_i \mathbf{x}_i)$. Now (I) and (II) imply $\text{occ}(x_i, \mathbf{w}) = 1 + e_i$ and $\text{occ}(x_i, \mathbf{w}') = 1 + e'_i$ for each $i > 1$. Hence $(e_2, \dots, e_m) = (e'_2, \dots, e'_m)$ by Lemma 21.3(ii). In each of the following two

cases, it is shown that $e_1 = e'_1$. Consequently, the identity $\mathbf{w} \approx \mathbf{w}'$ is trivial and is implied by the identities (21.1).

Case 1: $(e_2, \dots, e_m) = (0, \dots, 0)$. Then $e_1 = 1 = e'_1$ by (II).

Case 2: $(e_2, \dots, e_m) \neq (0, \dots, 0)$. If $(\mathbf{x}_1, \dots, \mathbf{x}_{m-1}) \neq (\emptyset, \dots, \emptyset)$, then $e_1 = 0 = e'_1$ by (III). If $(\mathbf{x}_1, \dots, \mathbf{x}_{m-1}) = (\emptyset, \dots, \emptyset)$, then $e_1 = 1 = e'_1$ by Remark 21.4(ii).

22. $\mathcal{E}_5, \mathcal{E}_6, \mathcal{E}_7$

This section establishes the finite basis property of the following semigroups.

\mathcal{E}_5	1 2 3 4 5 6	\mathcal{E}_6	1 2 3 4 5 6	\mathcal{E}_7	1 2 3 4 5 6
1	1 1 1 1 1 1	1	1 1 1 1 1 1	1	1 1 1 1 1 1
2	1 1 1 1 1 3	2	1 1 1 1 1 3	2	1 1 1 1 1 3
3	1 1 1 1 3 3	3	1 1 1 1 3 3	3	3 3 3 3 3 3
4	1 2 3 4 4 4	4	4 4 4 4 4 4	4	1 2 3 4 5 1
5	1 2 3 4 5 5	5	1 2 3 4 5 5	5	1 2 3 4 5 3
6	1 2 3 4 6 6	6	1 2 3 4 6 6	6	6 6 6 6 6 6

22.1. Finite basis property of \mathcal{E}_5

PROPOSITION 22.1. The variety generated by \mathcal{E}_5 is defined by the identities

$$x^3 \approx x^2, \quad x^2yx \approx xyx, \quad xyx^2 \approx xyx, \tag{22.1a}$$

$$h^2xyx \approx h^2yx^2, \tag{22.1b}$$

$$h^2x^2y^2 \approx h^2y^2x^2. \tag{22.1c}$$

For any word \mathbf{w} , let $F_{SN}(\mathbf{w})$ denote the set of factors of \mathbf{w} of length two that begin with a simple letter and end with a non-simple letter:

$$F_{SN}(\mathbf{w}) = \{xy \in \mathcal{X}^2 \mid \mathbf{w} \in \mathcal{X}^*xy\mathcal{X}^*, x \in \text{sim}(\mathbf{w}), y \notin \text{sim}(\mathbf{w})\}.$$

LEMMA 22.2. Suppose that $\mathbf{w} \approx \mathbf{w}'$ is any identity satisfied by \mathcal{E}_5 . Then:

- (i) $h(\mathbf{w}) = h(\mathbf{w}')$;
- (ii) $\text{con}(\mathbf{w}) = \text{con}(\mathbf{w}')$ and $\text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$;
- (iii) for any letters $x \in \text{con}(\mathbf{w}) = \text{con}(\mathbf{w}')$ and $y \in \text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$, the conditions $x \prec_{\mathbf{w}} y$ and $x \prec_{\mathbf{w}'} y$ are equivalent;
- (iv) $\mathbf{w}_{\text{sim}} = \mathbf{w}'_{\text{sim}}$;
- (v) $F_{SN}(\mathbf{w}) = F_{SN}(\mathbf{w}')$.

Proof. Parts (i)–(iv) follow from Lemmas 2.1(i) and 2.3 since the subsemigroups $\{5, 6\}$ and $\{1, 3, 4, 5\}$ of \mathcal{E}_5 are isomorphic to L_2 and J^1 , respectively.

(v) Suppose $F_{SN}(\mathbf{w}) \neq F_{SN}(\mathbf{w}')$, say $xy \in F_{SN}(\mathbf{w}) \setminus F_{SN}(\mathbf{w}')$. Then $x \in \text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$ by part (ii). Therefore $\mathbf{w} = \mathbf{a}xy\mathbf{b}$ and $\mathbf{w}' = \mathbf{a}'x\mathbf{b}'$ for some $\mathbf{a}, \mathbf{b}, \mathbf{a}', \mathbf{b}' \in \mathcal{X}^*$ such that $h(\mathbf{b}') \neq y$ and $x \notin \text{con}(\mathbf{a}\mathbf{b}\mathbf{a}'\mathbf{b}')$. Let $\varphi : \mathcal{X} \rightarrow \mathcal{E}_5$ denote the substitution

$$z \mapsto \begin{cases} 2 & \text{if } z = x, \\ 6 & \text{if } z = y, \\ 5 & \text{otherwise.} \end{cases}$$

Then $5 \cdot \mathbf{a}\varphi = 5 \cdot \mathbf{a}'\varphi = \mathbf{b}'\varphi \cdot 5 = 5$ and $6 \cdot \mathbf{b}\varphi \cdot 5 = 6$, so that

$$5 \cdot \mathbf{w}\varphi \cdot 5 = (5 \cdot \mathbf{a}\varphi) \cdot 2 \cdot (6 \cdot \mathbf{b}\varphi \cdot 5) = 5 \cdot 2 \cdot 6 = 3$$

and

$$5 \cdot \mathbf{w}'\varphi \cdot 5 = (5 \cdot \mathbf{a}'\varphi) \cdot 2 \cdot (\mathbf{b}'\varphi \cdot 5) = 5 \cdot 2 \cdot 5 = 1,$$

which is impossible. □

In this subsection, a non-simple word \mathbf{w} is said to be in *canonical form* if

$$\mathbf{w} = \prod_{i=1}^m (\mathbf{s}_i \mathbf{w}_i), \tag{22.2}$$

where all of the following are satisfied:

- (I) the letters of $\mathbf{s}_1 \in \mathcal{X}^*$ and $\mathbf{s}_2, \dots, \mathbf{s}_m \in \mathcal{X}^+$ are simple in \mathbf{w} ;
- (II) the letters of $\mathbf{w}_1, \dots, \mathbf{w}_{m-1} \in \mathcal{X}^+$ and $\mathbf{w}_m \in \mathcal{X}^*$ are non-simple in \mathbf{w} ;
- (III) $\text{con}(\mathbf{w}_1) \supseteq \dots \supseteq \text{con}(\mathbf{w}_m)$.

LEMMA 22.3. *Let \mathbf{w} be any non-simple word. Then there exists some word $\bar{\mathbf{w}}$ in canonical form such that the identities (22.1) imply the identity $\mathbf{w} \approx \bar{\mathbf{w}}$.*

Proof. It suffices to convert \mathbf{w} , using the identities (22.1), into a word in canonical form. It is easily seen that \mathbf{w} can be written as a word of the form (22.2) with (I) and (II) satisfied. If $x \in \text{con}(\mathbf{w}_{i+1}) \setminus \text{con}(\mathbf{w}_i)$, say $\mathbf{w}_{i+1} = \mathbf{a}x\mathbf{b}$ for some $\mathbf{a}, \mathbf{b} \in \mathcal{X}^*$, then the identities (22.1) can be used to convert \mathbf{w}_i into $\mathbf{w}_i x$:

$$\begin{aligned} \mathbf{w} &= \dots \mathbf{s}_i \mathbf{w}_i \cdot \mathbf{s}_{i+1} \mathbf{w}_{i+1} \dots \stackrel{(22.1a)}{\approx} \dots \mathbf{s}_i \mathbf{w}_i (\mathbf{t}(\mathbf{w}_i))^2 \cdot \mathbf{s}_{i+1} \mathbf{a} x^2 \mathbf{b} \dots \\ &\stackrel{(22.1b)}{\approx} \dots \mathbf{s}_i \mathbf{w}_i (\mathbf{t}(\mathbf{w}_i))^2 x \cdot \mathbf{s}_{i+1} \mathbf{a} x \mathbf{b} \dots \stackrel{(22.1a)}{\approx} \dots \mathbf{s}_i (\mathbf{w}_i x) \cdot \mathbf{s}_{i+1} \mathbf{w}_{i+1} \dots \end{aligned}$$

Hence (III) is satisfied by applying the identities (22.1). □

LEMMA 22.4. *Suppose that \mathcal{E}_5 satisfies an identity $\mathbf{w} \approx \mathbf{w}'$, where*

$$\mathbf{w} = \prod_{i=1}^m (\mathbf{s}_i \mathbf{w}_i) \quad \text{and} \quad \mathbf{w}' = \prod_{i=1}^m (\mathbf{s}_i \mathbf{w}'_i)$$

are in canonical form. Then $\text{con}(\mathbf{w}_i) = \text{con}(\mathbf{w}'_i)$ for all i .

Proof. First note that $\bigcup_{i=1}^m \text{con}(\mathbf{w}_i) = \bigcup_{i=1}^m \text{con}(\mathbf{w}'_i)$ holds by Lemma 22.2(ii). Suppose $\text{con}(\mathbf{w}_1) \neq \text{con}(\mathbf{w}'_1)$, say $x \in \text{con}(\mathbf{w}_1) \setminus \text{con}(\mathbf{w}'_1)$. Then by (II), the letter x is non-simple in \mathbf{w} and so by Lemma 22.2(ii), it is also non-simple in \mathbf{w}' . By (II), there exists some $i > 1$ such that $x \in \text{con}(\mathbf{w}'_i)$. But then (III) implies $x \in \text{con}(\mathbf{w}'_1)$, contradicting the assumption. Therefore $\text{con}(\mathbf{w}_1) = \text{con}(\mathbf{w}'_1)$.

Now suppose $\text{con}(\mathbf{w}_k) \neq \text{con}(\mathbf{w}'_k)$ for some $k > 1$, say $x \in \text{con}(\mathbf{w}_k) \setminus \text{con}(\mathbf{w}'_k)$. Then (III) implies $x \in \text{con}(\mathbf{w}_1 \dots \mathbf{w}_k)$ and $x \notin \text{con}(\mathbf{w}'_k \dots \mathbf{w}'_m)$, whence $x \not\prec_{\mathbf{w}} h(\mathbf{s}_k)$ and $x \prec_{\mathbf{w}'} h(\mathbf{s}_k)$. But this violates Lemma 22.2(iii). □

Proof of Proposition 22.1. It is routinely checked that \mathcal{E}_5 satisfies the identities (22.1). Hence it suffices to show that any identity $\mathbf{w} \approx \mathbf{w}'$ satisfied by \mathcal{E}_5 is implied by the identities (22.1). If either \mathbf{w} or \mathbf{w}' is a simple word, then it follows from Lemma 22.2 that the identity $\mathbf{w} \approx \mathbf{w}'$

is trivial and so is vacuously implied by (22.1). Therefore assume that \mathbf{w} and \mathbf{w}' are both non-simple words and so by Lemma 22.3, can be chosen to be in canonical form. Hence

$$\mathbf{w} = \prod_{i=1}^m (\mathbf{s}_i \mathbf{w}_i) \quad \text{and} \quad \mathbf{w}' = \prod_{i=1}^{m'} (\mathbf{s}'_i \mathbf{w}'_i).$$

Since $\mathbf{s}_1 \dots \mathbf{s}_m = \mathbf{w}_{\text{sim}} = \mathbf{w}'_{\text{sim}} = \mathbf{s}'_1 \dots \mathbf{s}'_{m'}$ by Lemma 22.2 part (iv), it follows from parts (i) and (v) of the same lemma that $m = m'$, $\mathbf{s}_i = \mathbf{s}'_i$, and $h(\mathbf{w}_i) = h(\mathbf{w}'_i)$ for all i . Therefore $\mathbf{w}' = \prod_{i=1}^m (\mathbf{s}_i \mathbf{w}'_i)$. Now Lemma 22.4 implies $\text{con}(\mathbf{w}_i) = \text{con}(\mathbf{w}'_i)$ for all i . Hence it can be assumed that $h_i = h(\mathbf{w}_i) = h(\mathbf{w}'_i)$ and $\text{con}(\mathbf{w}_i) = \text{con}(\mathbf{w}'_i) = \{h_i, x_{i,1}, \dots, x_{i,r_i}\}$. Let φ denote the substitution $x \mapsto x^2$ for all $x \in \mathcal{X}$. Then it follows from (II) that

$$\begin{aligned} \mathbf{w} &\stackrel{(22.1a)}{\approx} \prod_{i=1}^m (\mathbf{s}_i (\mathbf{w}_i \varphi)) \stackrel{(22.1c)}{\approx} \prod_{i=1}^m (\mathbf{s}_i h_i^{e_{i,0}} x_{i,1}^{e_{i,1}} \dots x_{i,r_i}^{e_{i,r_i}}) \quad \text{for some } e_{i,j} \in \{2, 4, 6, \dots\} \\ &\stackrel{(22.1a)}{\approx} \prod_{i=1}^m (\mathbf{s}_i h_i^2 x_{i,1}^2 \dots x_{i,r_i}^2). \end{aligned}$$

Similarly, the deduction $\mathbf{w}' \stackrel{(22.1)}{\approx} \prod_{i=1}^m (\mathbf{s}_i h_i^2 x_{i,1}^2 \dots x_{i,r_i}^2)$ also holds. Consequently, the identity $\mathbf{w} \approx \mathbf{w}'$ is implied by the identities (22.1). □

22.2. Finite basis property of \mathcal{E}_6

PROPOSITION 22.5. The variety generated by \mathcal{E}_6 is defined by the identities

$$x^3 \approx x^2, \quad x^2yx \approx xyx, \quad xyx^2 \approx xyx, \tag{22.3a}$$

$$xHy^2x \approx x^2Hy^2. \tag{22.3b}$$

LEMMA 22.6. Suppose that $\mathbf{w} \approx \mathbf{w}'$ is any identity satisfied by \mathcal{E}_6 . Then $\text{ini}(\mathbf{w}) = \text{ini}(\mathbf{w}')$ and $\text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$.

Proof. This follows from Lemma 2.1 since the subsemigroups $\{1, 4, 5\}$ and $\{1, 3, 5\}$ of \mathcal{E}_6 are isomorphic to L_2^1 and N_2^1 , respectively. □

In this subsection, a word \mathbf{w} with $\text{ini}(\mathbf{w}) = x_1 \dots x_m$ is said to be in *canonical* form if

$$\mathbf{w} = \prod_{i=1}^m (x_i^{e_i} \mathbf{w}_i), \tag{22.4}$$

where all of the following are satisfied:

- (I) $\mathbf{w}_i \in \{\emptyset, x_1, \dots, x_{i-1}\}$;
- (II) $e_1, \dots, e_m \in \{1, 2\}$;
- (III) if $x_i \notin \text{sim}(\mathbf{w})$, then $\mathbf{w}_i = \emptyset$;
- (IV) if $x_i \notin \text{sim}(\mathbf{w})$ and $x_i = \mathbf{w}_j$ for some $j > i$, then $e_i = 1$.

Note that (IV) implies that the word \mathbf{w} in (22.4) does not contain a factor of the form $x^2\mathbf{a}x$ where $\mathbf{a} \in \mathcal{X}^+$.

LEMMA 22.7. Let \mathbf{w} be any word. Then there exists some word $\overline{\mathbf{w}}$ in canonical form such that the identities (22.3) imply the identity $\mathbf{w} \approx \overline{\mathbf{w}}$.

Proof. It suffices to convert \mathbf{w} , using the identities (22.3), into a word in canonical form. Suppose that x and y are distinct non-simple letters of \mathbf{w} such that $\mathbf{w} = \mathbf{a}x\mathbf{b}y\mathbf{x}c$ for some $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathcal{X}^*$ with $y \in \text{con}(\mathbf{abc})$. Then

$$\mathbf{w} \stackrel{(22.3a)}{\approx} \mathbf{a}x\mathbf{b}y^2\mathbf{x}c \stackrel{(22.3b)}{\approx} \mathbf{a}x^2\mathbf{b}y^2\mathbf{c} \stackrel{(22.3a)}{\approx} \mathbf{a}x^2\mathbf{b}y\mathbf{c},$$

that is, using the identities (22.3), any non-first occurrence of x in \mathbf{w} that immediately follows another non-simple letter can be gathered with the first x in \mathbf{w} . It follows that \mathbf{w} can be converted into a word of the form (22.4) with (I) and (III) satisfied. Then (II) and (IV) are satisfied by applying the identities (22.3a). \square

LEMMA 22.8. *Suppose that \mathcal{E}_6 satisfies an identity $\mathbf{w} \approx \mathbf{w}'$, where*

$$\mathbf{w} = \prod_{i=1}^m (x_i^{e_i} \mathbf{w}_i) \quad \text{and} \quad \mathbf{w}' = \prod_{i=1}^m (x_i^{e'_i} \mathbf{w}'_i)$$

are in canonical form. Then $e_i = e'_i$ for all i .

Proof. Since $\text{occ}(x_m, \mathbf{w}) = e_m$ and $\text{occ}(x_m, \mathbf{w}') = e'_m$ by (I), it follows from (II) and Lemma 22.6 that $e_m = e'_m$. Suppose $e_j \neq e'_j$ for some $j < m$, say $(e_j, e'_j) = (1, 2)$. Then $x_j \notin \text{sim}(\mathbf{w}')$, so that $x_j \notin \text{sim}(\mathbf{w})$ by Lemma 22.6. The second x_j in \mathbf{w} coincides with \mathbf{w}_k for some $k > j$, so that $\mathbf{w}_k \neq \emptyset$. Hence $x_k \in \text{sim}(\mathbf{w})$ by (III), and

$$\mathbf{w} = \mathbf{a}x_j^{e_j} \mathbf{b}x_k \mathbf{w}_k \mathbf{c} = \mathbf{a}x_j \mathbf{b}x_k x_j \mathbf{c}$$

for some $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathcal{X}^*$ with $x_j \notin \text{con}(\mathbf{ab})$ and $x_k \notin \text{con}(\mathbf{abc})$. By Lemma 22.6, the letter x_k is also simple in \mathbf{w}' . Since $e'_j = 2$, it follows from (I) and (IV) that $\text{occ}(x_j, \mathbf{w}') = 2$, whence

$$\mathbf{w}' = \mathbf{a}'x_j^{e'_j} \mathbf{b}'x_k \mathbf{c}' = \mathbf{a}'x_j^2 \mathbf{b}'x_k \mathbf{c}'$$

for some $\mathbf{a}', \mathbf{b}', \mathbf{c}' \in \mathcal{X}^*$ with $x_j, x_k \notin \text{con}(\mathbf{a}'\mathbf{b}'\mathbf{c}')$. Let $\varphi : \mathcal{X} \rightarrow \mathcal{E}_6$ denote the substitution

$$z \mapsto \begin{cases} 5 & \text{if } z = x_j, \\ 2 & \text{if } z = x_k, \\ 6 & \text{otherwise.} \end{cases}$$

Note that $x_j, x_k \notin \text{con}(\mathbf{aba}'\mathbf{b}'\mathbf{c}')$, so $6 \cdot \mathbf{a}\varphi = 6 \cdot \mathbf{a}'\varphi = \mathbf{c}'\varphi \cdot 6 = 6$ and $5 \cdot \mathbf{b}\varphi = 5^2 \cdot \mathbf{b}'\varphi = 5$. Hence

$$6 \cdot \mathbf{w}\varphi \cdot 6 = (6 \cdot \mathbf{a}\varphi) \cdot (5 \cdot \mathbf{b}\varphi) \cdot (2 \cdot 5) \cdot \mathbf{c}\varphi \cdot 6 = 6 \cdot 5 \cdot 1 \cdot \mathbf{c}\varphi \cdot 6 = 1$$

and

$$6 \cdot \mathbf{w}'\varphi \cdot 6 = (6 \cdot \mathbf{a}'\varphi) \cdot (5^2 \cdot \mathbf{b}'\varphi) \cdot 2 \cdot (\mathbf{c}'\varphi \cdot 6) = 6 \cdot 5 \cdot 2 \cdot 6 = 3,$$

which is a contradiction. Consequently, j does not exist, so that $e_i = e'_i$ for all i . \square

Proof of Proposition 22.5. It is routinely checked that \mathcal{E}_6 satisfies the identities (22.3). Hence it suffices to show that any identity $\mathbf{w} \approx \mathbf{w}'$ satisfied by \mathcal{E}_6 is implied by the identities (22.3). By Lemma 22.7, the words \mathbf{w} and \mathbf{w}' can be assumed to be in canonical form. Since $\text{ini}(\mathbf{w}) = \text{ini}(\mathbf{w}')$ by Lemma 22.6, it follows from Lemma 22.8 that

$$\mathbf{w} = \prod_{i=1}^m (x_i^{e_i} \mathbf{w}_i) \quad \text{and} \quad \mathbf{w}' = \prod_{i=1}^m (x_i^{e'_i} \mathbf{w}'_i).$$

Suppose that ℓ is the least integer such that $\mathbf{w}_\ell \neq \mathbf{w}'_\ell$. Then $(\mathbf{w}_\ell, \mathbf{w}'_\ell) \neq (\emptyset, \emptyset)$, say $\mathbf{w}_\ell \neq \emptyset$. It follows from (I) that $\mathbf{w}_\ell = x_k$ for some $k < \ell$, whence $x_\ell \in \text{sim}(\mathbf{w})$ and $e_\ell = 1$ by (III). Thus

$$\mathbf{w} = \mathbf{p}x_\ell^{e_\ell}\mathbf{w}_\ell\mathbf{q} = \mathbf{p}x_\ell x_k \mathbf{q}$$

where $\mathbf{p} = \prod_{i=1}^{\ell-1} (x_i^{e_i} \mathbf{w}_i)$ and $\mathbf{q} = \prod_{i=\ell+1}^m (x_i^{e_i} \mathbf{w}_i)$, and

$$\mathbf{w}' = \mathbf{p}x_\ell^{e_\ell}\mathbf{w}'_\ell\mathbf{q}' = \mathbf{p}x_\ell \mathbf{w}'_\ell \mathbf{q}'$$

where $\mathbf{q}' = \prod_{i=\ell+1}^m (x_i^{e_i} \mathbf{w}'_i)$ and $\mathbf{w}'_\ell \in \{\emptyset, x_1, \dots, x_{\ell-1}\} \setminus \{x_k\}$. Further, $\text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$ by Lemma 22.6, so that $x_\ell \notin \text{con}(\mathbf{p}\mathbf{q}\mathbf{w}'_\ell\mathbf{q}')$. Let $\varphi : \mathcal{X} \rightarrow \mathcal{E}_6$ denote the substitution

$$z \mapsto \begin{cases} 5 & \text{if } z = x_k, \\ 2 & \text{if } z = x_\ell, \\ 6 & \text{otherwise.} \end{cases}$$

Then $6 \cdot \mathbf{w}\varphi \cdot 6 = (6 \cdot \mathbf{p}\varphi) \cdot 2 \cdot (5 \cdot \mathbf{q}\varphi \cdot 6) = 6 \cdot 2 \cdot 5 = 1$. Note that if $\mathbf{q}' \neq \emptyset$, then $\text{h}(\mathbf{q}') = x_{\ell+1}$ and $x_\ell \notin \text{con}(\mathbf{q}')$, so that $\mathbf{q}'\varphi \cdot 6 = 6$. Therefore

$$\begin{aligned} 6 \cdot \mathbf{w}'\varphi \cdot 6 &= \begin{cases} (6 \cdot \mathbf{p}\varphi) \cdot 2 \cdot 6 \cdot (\mathbf{q}'\varphi \cdot 6) & \text{if } \mathbf{w}'_\ell \neq \emptyset, \\ (6 \cdot \mathbf{p}\varphi) \cdot 2 \cdot (\mathbf{q}'\varphi \cdot 6) & \text{if } \mathbf{w}'_\ell = \emptyset, \end{cases} \\ &= 6 \cdot 2 \cdot 6 = 3, \end{aligned}$$

and the contradiction $6 \cdot \mathbf{w}\varphi \cdot 6 \neq 6 \cdot \mathbf{w}'\varphi \cdot 6$ is established. Hence the integer ℓ does not exist. Consequently, the identity $\mathbf{w} \approx \mathbf{w}'$ is trivial and is implied by the identities (22.3). \square

22.3. Finite basis property of \mathcal{E}_7

PROPOSITION 22.9. The variety generated by \mathcal{E}_7 is defined by the identities

$$x\text{H}x^2 \approx x\text{H}x, \quad x\text{H}y\text{K}x\text{y} \approx x\text{H}y\text{K}y, \quad x\text{H}y\text{K}y\text{x} \approx x\text{H}y\text{K}x. \tag{22.5}$$

In this subsection, a word \mathbf{w} with $\text{ini}(\mathbf{w}) = x_1 \dots x_m$ is said to be in canonical form if

$$\mathbf{w} = \prod_{i=1}^m (x_i \mathbf{w}_i), \tag{22.6}$$

where $\mathbf{w}_i \in \{\emptyset, x_1, \dots, x_i\}$.

LEMMA 22.10. Let \mathbf{w} be any word. Then there exists some word $\bar{\mathbf{w}}$ in canonical form such that the identities (22.5) imply the identity $\mathbf{w} \approx \bar{\mathbf{w}}$.

Proof. It suffices to convert \mathbf{w} , using the identities (22.5), into a word in canonical form. Suppose $\text{ini}(\mathbf{w}) = x_1 \dots x_m$. Then \mathbf{w} can be written in the form (22.6) with $\mathbf{w}_i \in \{x_1, \dots, x_i\}^*$. Since any letter in \mathbf{w}_i is a non-first occurrence in \mathbf{w} , the identities (22.5) can be used to eliminate all except the last letter of \mathbf{w}_i . Therefore $\mathbf{w}_i \in \{\emptyset, x_1, \dots, x_i\}$. \square

Proof of Proposition 22.9. It is routinely checked that \mathcal{E}_7 satisfies the identities (22.5). Hence it suffices to show that any identity $\mathbf{w} \approx \mathbf{w}'$ satisfied by \mathcal{E}_7 is implied by the identities (22.5). By Lemma 22.10, the words \mathbf{w} and \mathbf{w}' can be assumed to be in canonical form. Now the subsemigroup $\{1, 3, 5\}$ of \mathcal{E}_7 is isomorphic to L_2^1 , so that $\text{ini}(\mathbf{w}) = \text{ini}(\mathbf{w}')$ by Lemmas 2.1(iii). Hence

$$\mathbf{w} = \prod_{i=1}^m (x_i \mathbf{w}_i) \quad \text{and} \quad \mathbf{w}' = \prod_{i=1}^m (x_i \mathbf{w}'_i).$$

Suppose that ℓ is the least integer such that $\mathbf{w}_\ell \neq \mathbf{w}'_\ell$. Then

$$\mathbf{w} = \mathbf{p}x_\ell\mathbf{w}_\ell\mathbf{q} \quad \text{and} \quad \mathbf{w}' = \mathbf{p}x_\ell\mathbf{w}'_\ell\mathbf{q}'$$

where $\mathbf{p} = \prod_{i=1}^{\ell-1}(x_i\mathbf{w}_i) = \prod_{i=1}^{\ell-1}(x_i\mathbf{w}'_i)$, $\mathbf{q} = \prod_{i=\ell+1}^m(x_i\mathbf{w}_i)$, and $\mathbf{q}' = \prod_{i=\ell+1}^m(x_i\mathbf{w}'_i)$. By symmetry, there are three cases.

Case 1: $(\mathbf{w}_\ell, \mathbf{w}'_\ell) = (\emptyset, x_\ell)$. Let $\varphi_1 : \mathcal{X} \rightarrow \mathcal{E}_7$ denote the substitution

$$z \mapsto \begin{cases} 2 & \text{if } z = x_\ell, \\ 6 & \text{if } z = x_{\ell+1}, \\ 4 & \text{otherwise.} \end{cases}$$

Then $x_\ell, x_{\ell+1} \notin \text{con}(\mathbf{p})$, so that $4 \cdot \mathbf{p}\varphi_1 = 4$. Hence

$$4 \cdot \mathbf{w}\varphi_1 \cdot 6 = (4 \cdot \mathbf{p}\varphi_1) \cdot 2 \cdot (\mathbf{q}\varphi_1 \cdot 6) = 4 \cdot 2 \cdot 6 = 3$$

and

$$4 \cdot \mathbf{w}'\varphi_1 \cdot 6 = (4 \cdot \mathbf{p}\varphi_1) \cdot 2 \cdot 2 \cdot (\mathbf{q}'\varphi_1 \cdot 6) = 4 \cdot 2 \cdot 2 \cdot 6 = 1.$$

Case 2: $(\mathbf{w}_\ell, \mathbf{w}'_\ell) = (\emptyset, x_k)$ with $k < \ell$. Let $\varphi_2 : \mathcal{X} \rightarrow \mathcal{E}_7$ denote the substitution

$$z \mapsto \begin{cases} 5 & \text{if } z = x_k, \\ 6 & \text{if } z = x_{\ell+1}, \\ 4 & \text{otherwise.} \end{cases}$$

Then $x_{\ell+1} \notin \text{con}(\mathbf{p})$, so that $4 \cdot \mathbf{p}\varphi_2 \in \{4, 5\}$. Hence

$$4 \cdot \mathbf{w}\varphi_2 \cdot 6 = (4 \cdot \mathbf{p}\varphi_2) \cdot 4 \cdot (\mathbf{q}\varphi_2 \cdot 6) \in \{4, 5\} \cdot 4 \cdot 6 = \{1\}$$

and

$$4 \cdot \mathbf{w}'\varphi_2 \cdot 6 = (4 \cdot \mathbf{p}\varphi_2) \cdot 4 \cdot 5 \cdot (\mathbf{q}'\varphi_2 \cdot 6) \in \{4, 5\} \cdot 4 \cdot 5 \cdot 6 = \{3\}.$$

Case 3: $(\mathbf{w}_\ell, \mathbf{w}'_\ell) = (x_j, x_k)$ with $j < k \leq \ell$. Then since $x_{\ell+1} \notin \text{con}(\mathbf{p})$, it follows that $(\mathbf{p}x_\ell)\varphi_2 \in \{4, 5\} \cdot 4 = \{4\}$. Hence

$$\mathbf{w}\varphi_2 \cdot 6 = (\mathbf{p}x_\ell)\varphi_2 \cdot 4 \cdot (\mathbf{q}\varphi_2 \cdot 6) = 4 \cdot 4 \cdot 6 = 1$$

and

$$\mathbf{w}'\varphi_2 \cdot 6 = (\mathbf{p}x_\ell)\varphi_2 \cdot 5 \cdot (\mathbf{q}'\varphi_2 \cdot 6) = 4 \cdot 5 \cdot 6 = 3.$$

Since all of the above cases are impossible, the integer ℓ does not exist. Consequently, the identity $\mathbf{w} \approx \mathbf{w}'$ is trivial and so is implied by the identities (22.5). \square

23. $\mathcal{E}_8, \mathcal{E}_9, \mathcal{E}_{10}, \mathcal{F}_5$

This section establishes the finite basis property of the following semigroups.

\mathcal{E}_8	1 2 3 4 5 6	\mathcal{E}_9	1 2 3 4 5 6	\mathcal{E}_{10}	1 2 3 4 5 6	\mathcal{F}_5	1 2 3 4 5 6
1	1 1 1 1 1 1	1	1 1 1 1 1 1	1	1 1 1 1 1 1	1	1 1 1 1 1 1
2	1 1 1 1 1 3	2	1 1 1 1 1 3	2	1 1 1 1 2 2	2	1 1 1 1 2 2
3	3 3 3 3 3 3	3	3 3 3 3 3 3	3	1 1 1 1 3 3	3	3 3 3 3 3 3
4	1 2 3 4 5 1	4	1 2 3 4 5 3	4	4 4 4 4 4 4	4	3 3 3 3 4 4
5	1 2 3 5 4 1	5	1 2 3 5 4 3	5	1 1 3 1 5 6	5	1 2 1 1 5 6
6	6 6 6 6 6 6	6	6 6 6 6 6 6	6	1 3 3 1 5 6	6	1 2 1 2 5 6

23.1. Finite basis property of \mathcal{E}_8 and \mathcal{E}_9

PROPOSITION 23.1. The variety generated by $S \in \{\mathcal{E}_8, \mathcal{E}_9\}$ is defined by the identities

$$x^2yx \approx x^3y^3, \tag{23.1a}$$

$$xHx^3 \approx xHx, \tag{23.1b}$$

$$xHyx^2 \approx xHy^3, \tag{23.1c}$$

$$xHy^3x \approx xHyx, \tag{23.1d}$$

$$xHyKxy \approx xHyKyx, \tag{23.1e}$$

$$yHxyKz^2 \approx yHx^2Kz^2xy. \tag{23.1f}$$

LEMMA 23.2. Let $S \in \{\mathcal{E}_8, \mathcal{E}_9\}$. Suppose that $\mathbf{w} \approx \mathbf{w}'$ is any identity satisfied by S . Then $\text{ini}(\mathbf{w}) = \text{ini}(\mathbf{w}')$ and $\text{occ}(x, \mathbf{w}) \equiv \text{occ}(x, \mathbf{w}') \pmod{2}$ for all $x \in \mathcal{X}$.

Proof. This follows from Lemma 2.1 since the subsemigroups $\{1, 3, 4\}$ and $\{4, 5\}$ of S are isomorphic to L_2^1 and \mathbb{Z}_2 , respectively. □

LEMMA 23.3. Let $S \in \{\mathcal{E}_8, \mathcal{E}_9\}$. Then S does not satisfy the identity

$$x^2y^3z^2 \approx x^2yz^2. \tag{23.2}$$

Proof. This holds because $4^2 \cdot 2^3 \cdot 6^2 = 1 \neq 3 = 4^2 \cdot 2 \cdot 6^2$ in S . □

In this subsection, a non-simple word \mathbf{w} with $\text{ini}(\mathbf{w}) = x_0 \dots x_m$ is said to be in *canonical form* if

$$\mathbf{w} = x_0^{e_0} \dots x_r^{e_r} \cdot x_0^{f_0} \dots x_{r-1}^{f_{r-1}} \cdot x_{r+1} \dots x_m \tag{23.3}$$

for some $r \in \{0, \dots, m\}$ such that all of the following are satisfied:

- (I) $e_0, \dots, e_r \in \{1, 2, 3\}$;
- (II) $f_0, \dots, f_{r-1} \in \{0, 1\}$;
- (III) if $e_i \geq 2$ and $i \in \{0, \dots, r-1\}$, then $f_i = 0$;
- (IV) if $e_r = 1$, then $(f_0, \dots, f_{r-1}) \neq (0, \dots, 0)$;
- (V) if $r \geq 1$ and $(f_0, \dots, f_{r-1}) \neq (0, \dots, 0)$, then $e_r \leq 2$.

REMARK 23.4. (i) The word \mathbf{w} in (23.3) satisfies $\text{occ}(x, \mathbf{w}) \leq 3$ for all $x \in \mathcal{X}$.
 (ii) If $r = 0$, then the word \mathbf{w} in (23.3) is $x_0^{e_0} x_1 \dots x_m$.

LEMMA 23.5. Let \mathbf{w} be any non-simple word. Then there exists some word $\bar{\mathbf{w}}$ in canonical form such that the identities (23.1) imply the identity $\mathbf{w} \approx \bar{\mathbf{w}}$.

Proof. It suffices to convert \mathbf{w} , using the identities (23.1), into a word in canonical form. Suppose $\text{ini}(\mathbf{w}) = x_0 \dots x_m$. Then \mathbf{w} can be written as a word of the form $\mathbf{w} = \prod_{i=0}^m (x_i \mathbf{w}_i)$ with $\mathbf{w}_i \in \{x_0, \dots, x_i\}^*$. Let r be the largest integer for which $\mathbf{w}_r \neq \emptyset$, so that $\mathbf{w}_j = \emptyset$ for all $j > r$. Then $\mathbf{w} = (\prod_{i=0}^r (x_i \mathbf{w}_i)) (\prod_{i=r+1}^m x_i)$. Since the letters of \mathbf{w}_i are non-first occurrences, the identities (23.1e) can be used to arrange them, within \mathbf{w}_i , in any order. Hence it can be assumed that

$$\mathbf{w} = x_0^{e_0} \cdot x_1^{e_1} \mathbf{w}'_1 \dots x_r^{e_r} \mathbf{w}'_r \cdot x_{r+1} \dots x_m$$

where $e_0, \dots, e_r \geq 1$ and $\mathbf{w}'_i \in \{x_0, \dots, x_{i-1}\}^*$. The assumption $\mathbf{w}_r \neq \emptyset$ implies that either $e_r \geq 2$ or $\mathbf{w}'_r \neq \emptyset$. Therefore

$$\mathbf{w} \stackrel{(23.1b)}{\approx} \begin{cases} x_0^{e_0} \cdot x_1^{e_1} \mathbf{w}'_1 \dots x_{r-1}^{e_{r-1}} \mathbf{w}'_{r-1} \cdot x_r^{e_r} \mathbf{w}'_r x_r^2 \cdot x_{r+1} \dots x_m & \text{if } \mathbf{w}'_r = \emptyset, \\ x_0^{e_0} \cdot x_1^{e_1} \mathbf{w}'_1 \dots x_{r-1}^{e_{r-1}} \mathbf{w}'_{r-1} \cdot x_r^{e_r} \mathbf{w}'_r (\mathbf{t}(\mathbf{w}'_r))^2 \cdot x_{r+1} \dots x_m & \text{if } \mathbf{w}'_r \neq \emptyset. \end{cases}$$

In either case, there exists some $z \in \{x_0, \dots, x_r\}$ such that

$$\mathbf{w} \stackrel{(23.1b)}{\approx} x_0^{e_0} \cdot x_1^{e_1} \mathbf{w}'_1 \dots x_{r-1}^{e_{r-1}} \mathbf{w}'_{r-1} \cdot x_r^{e_r} \mathbf{w}'_r z^2 \cdot x_{r+1} \dots x_m.$$

Suppose $\mathbf{w}'_k \neq \emptyset$ for some $k < r$. Then $\mathbf{w}'_k = y_1 y_2 \dots y_p$ for some $y_i \in \{x_0, \dots, x_{k-1}\}$, whence

$$\mathbf{w} = \mathbf{p} \cdot x_k^{e_k} y_1 y_2 \dots y_p \cdot \mathbf{q} \cdot x_r^{e_r} \mathbf{w}'_r \cdot x_{r+1} \dots x_m$$

where $\mathbf{p} = x_0^{e_0} \prod_{i=1}^{k-1} (x_i^{e_i} \mathbf{w}'_i)$ and $\mathbf{q} = \prod_{i=k+1}^{r-1} (x_i^{e_i} \mathbf{w}'_i)$. Since the prefix \mathbf{p} contains the letters y_1, \dots, y_p ,

$$\begin{aligned} \mathbf{w} &\stackrel{(23.1b)}{\approx} \mathbf{p} \cdot x_k^{e_k} y_1 y_2 \dots y_p \cdot \mathbf{q} \cdot x_r^{e_r} \mathbf{w}'_r z^2 \cdot x_{r+1} \dots x_m \\ &\stackrel{(23.1f)}{\approx} \mathbf{p} \cdot x_k^{e_k+1} y_2 y_3 \dots y_p \cdot \mathbf{q} \cdot x_r^{e_r} \mathbf{w}'_r z^2 \cdot x_k y_1 \cdot x_{r+1} \dots x_m \\ &\stackrel{(23.1f)}{\approx} \mathbf{p} \cdot x_k^{e_k+2} y_3 y_4 \dots y_p \cdot \mathbf{q} \cdot x_r^{e_r} \mathbf{w}'_r z^2 \cdot x_k y_2 \cdot x_k y_1 \cdot x_{r+1} \dots x_m \\ &\vdots \\ &\stackrel{(23.1f)}{\approx} \mathbf{p} \cdot x_k^{e_k+p} \cdot \mathbf{q} \cdot x_r^{e_r} \mathbf{w}'_r z^2 \cdot x_k y_p \cdot x_k y_{p-1} \dots x_k y_1 \cdot x_{r+1} \dots x_m \\ &\stackrel{(23.1e)}{\approx} \mathbf{p} \cdot x_k^{e_k+p} \cdot \mathbf{q} \cdot x_r^{e_r} \mathbf{w}'_r z^2 \cdot x_k^p y_1 y_2 \dots y_p \cdot x_{r+1} \dots x_m \\ &\stackrel{(23.1b)}{\approx} \mathbf{p} \cdot x_k^{e_k+p} \cdot \mathbf{q} \cdot x_r^{e_r} \mathbf{w}'_r \cdot x_k^p y_1 y_2 \dots y_p \cdot x_{r+1} \dots x_m \\ &= \mathbf{p} \cdot x_k^{e_k+|\mathbf{w}'_k|} \cdot \mathbf{q} \cdot x_r^{e_r} \mathbf{w}'_r x_k^{|\mathbf{w}'_k|} \mathbf{w}'_k \cdot x_{r+1} \dots x_m. \end{aligned}$$

Hence the identities (23.1) can be used to move the factor \mathbf{w}'_k to the right until it immediately precedes the letter x_{r+1} (while simultaneously introducing two copies of $x_k^{|\mathbf{w}'_k|}$ to the word), that is,

$$\begin{aligned} \mathbf{w} &= \mathbf{p} \cdot x_k^{e_k} \mathbf{w}'_k \cdot \mathbf{q} \cdot x_r^{e_r} \mathbf{w}'_r \cdot x_{r+1} \dots x_m \\ &\stackrel{(23.1)}{\approx} \mathbf{p} \cdot x_k^{e_k+|\mathbf{w}'_k|} \cdot \mathbf{q} \cdot x_r^{e_r} \mathbf{w}'_r x_k^{|\mathbf{w}'_k|} \mathbf{w}'_k \cdot x_{r+1} \dots x_m. \end{aligned}$$

The above deduction holds vacuously if $\mathbf{w}'_k = \emptyset$. Since $k < r$ is arbitrary, this deduction can be repeated on every $\mathbf{w}'_1, \dots, \mathbf{w}'_{r-1}$, resulting in

$$\mathbf{w} \stackrel{(23.1)}{\approx} x_0^{e_0} \left(\prod_{i=1}^{r-1} x_i^{e_i+|\mathbf{w}'_i|} \right) x_r^{e_r} \mathbf{w}'_r \left(\prod_{i=1}^{r-1} (x_i^{|\mathbf{w}'_i|} \mathbf{w}'_i) \right) x_{r+1} \dots x_m.$$

Since the letters in $\mathbf{w}'_r (\prod_{i=1}^{r-1} (x_i^{|\mathbf{w}'_i|} \mathbf{w}'_i))$ are non-first occurrences in \mathbf{w} , the identities (23.1e) can be used to order them. Hence \mathbf{w} can be converted by the identities (23.1) into the word in (23.3). As observed earlier, either $e_r \geq 2$ or $\mathbf{w}'_r \neq \emptyset$. Hence (IV) is satisfied. The identities $\{(23.1b), (23.1c)\}$ can be used to reduce the exponents f_{r-1}, \dots, f_0 to numbers in $\{0, 1\}$, so that (II) is satisfied.

Suppose $e_i \geq 2$ and $f_i = 1$ for some $i < r$. Then

$$\mathbf{w} = x_0^{e_0} \cdot \mathbf{a} \cdot x_i^{e_i} x_{i+1}^{e_{i+1}} \dots x_r^{e_r} \cdot x_0^{f_0} \dots x_{i-1}^{f_{i-1}} x_i \cdot \mathbf{b}$$

where $\mathbf{a} = x_1^{e_1} \dots x_{i-1}^{e_{i-1}}$ and $\mathbf{b} = x_{i+1}^{f_{i+1}} \dots x_{r-1}^{f_{r-1}} \cdot x_{r+1} \dots x_m$, whence

$$\begin{aligned}
 \mathbf{w} &\stackrel{(23.1e)}{\approx} x_0^{e_0} \cdot \mathbf{a} \cdot (x_i^{e_i} x_{i+1}^{e_{i+1}} \dots x_r^{e_r} x_i) \cdot x_0^{f_0} \dots x_{i-1}^{f_{i-1}} \cdot \mathbf{b} \\
 &\stackrel{(23.1a)}{\approx} x_0^{e_0} \cdot \mathbf{a} \cdot x_i^{e_i+1} (x_{i+1}^{e_{i+1}} \dots x_r^{e_r})^3 \cdot x_0^{f_0} \dots x_{i-1}^{f_{i-1}} \cdot \mathbf{b} \\
 &= x_0^{e_0} \cdot \mathbf{a} \cdot x_i^{e_i+1} x_{i+1}^{e_{i+1}} \dots x_{r-1}^{e_{r-1}} x_r^{e_r} (x_{i+1}^{e_{i+1}} \dots x_{r-1}^{e_{r-1}} x_r^{e_r})^2 \cdot x_0^{f_0} \dots x_{i-1}^{f_{i-1}} \cdot \mathbf{b} \\
 &\stackrel{(23.1e)}{\approx} x_0^{e_0} \cdot \mathbf{a} \cdot x_i^{e_i+1} x_{i+1}^{e_{i+1}} \dots x_{r-1}^{e_{r-1}} x_r^{e_r} x_{i+1}^{2e_{i+1}} \dots x_{r-1}^{2e_{r-1}} x_r^{2e_r} \cdot x_0^{f_0} \dots x_{i-1}^{f_{i-1}} \cdot \mathbf{b} \\
 &\stackrel{(23.1b)}{\approx} x_0^{e_0} \cdot \mathbf{a} \cdot x_i^{e_i+1} x_{i+1}^{e_{i+1}} \dots x_{r-1}^{e_{r-1}} x_r^{e_r} x_{i+1}^2 \dots x_{r-1}^2 x_r^2 \cdot x_0^{f_0} \dots x_{i-1}^{f_{i-1}} \cdot \mathbf{b} \\
 &\stackrel{(23.1c)}{\approx} x_0^{e_0} \cdot \mathbf{a} \cdot x_i^{e_i+1} x_{i+1}^{e_{i+1}} \dots x_{r-1}^{e_{r-1}} x_r^{e_r+2(r-i)} x_0^{f_0} \dots x_{i-1}^{f_{i-1}} \cdot \mathbf{b} \\
 &\stackrel{(23.1b)}{\approx} x_0^{e_0} \cdot \mathbf{a} \cdot x_i^{e_i+1} x_{i+1}^{e_{i+1}} \dots x_{r-1}^{e_{r-1}} x_r^{e_r+2} x_0^{f_0} \dots x_{i-1}^{f_{i-1}} \cdot \mathbf{b}.
 \end{aligned}$$

Therefore the identities (23.1) can be used to reduce the exponent f_i from 1 to 0. Since i is arbitrary, this argument can be repeated until (III) is satisfied. It is then easily shown that (I) is satisfied by applying the identities (23.1b). Finally, (V) is satisfied by applying the identities (23.1d). □

LEMMA 23.6. *Let $S \in \{\mathcal{E}_8, \mathcal{E}_9\}$. Suppose that $\mathbf{w} \approx \mathbf{w}'$ is any nontrivial identity satisfied by S . Then the words \mathbf{w} and \mathbf{w}' are both non-simple.*

Proof. By Lemma 23.2, it can be assumed that $\text{ini}(\mathbf{w}) = \text{ini}(\mathbf{w}') = x_0 \dots x_m$. Therefore if the words \mathbf{w} and \mathbf{w}' are both simple, then the identity $\mathbf{w} \approx \mathbf{w}'$ is contradictorily trivial. Hence it suffices to assume that \mathbf{w} is non-simple. By Lemma 23.5, there exists a word $\bar{\mathbf{w}}$ in canonical form such that the identities (23.1) imply the identity $\mathbf{w} \approx \bar{\mathbf{w}}$. It is routinely checked that S satisfies the identities (23.1), whence S satisfies $\bar{\mathbf{w}} \approx \mathbf{w}'$.

Suppose that the word \mathbf{w}' is simple. Then $\mathbf{w}' = \text{ini}(\mathbf{w}') = x_0 \dots x_m$. Let x_i be any non-simple letter in $\bar{\mathbf{w}}$. Since $\text{occ}(x_i, \bar{\mathbf{w}}) \equiv \text{occ}(x_i, \mathbf{w}') \pmod{2}$ by Lemma 23.2, it follows from (I)–(III) that $\text{occ}(x_i, \bar{\mathbf{w}}) = 3$ and x_i^3 is a factor of $\bar{\mathbf{w}}$. Let φ denote the substitution $x_i \mapsto x_i^3$. The deduction $\bar{\mathbf{w}}\varphi \stackrel{(23.1b)}{\approx} \bar{\mathbf{w}}$ is easily verified. Since

$$x_0 \dots x_m = \mathbf{w}' \stackrel{S}{\approx} \bar{\mathbf{w}} \stackrel{(23.1b)}{\approx} \bar{\mathbf{w}}\varphi \stackrel{S}{\approx} \mathbf{w}'\varphi = x_0 \dots x_{i-1} x_i^3 x_{i+1} \dots x_m,$$

the semigroup S satisfies the identity $x_0 \dots x_m \approx x_0 \dots x_{i-1} x_i^3 x_{i+1} \dots x_m$. It is then easily shown that S satisfies the identity (23.2), contradicting Lemma 23.3. Consequently, \mathbf{w}' cannot be simple. □

LEMMA 23.7. *Let $S \in \{\mathcal{E}_8, \mathcal{E}_9\}$. Suppose that S satisfies an identity $\mathbf{w} \approx \mathbf{w}'$, where*

$$\mathbf{w} = x_0^{e_0} \dots x_r^{e_r} \cdot x_0^{f_0} \dots x_{r-1}^{f_{r-1}} \cdot x_{r+1} \dots x_m$$

and

$$\mathbf{w}' = x_0^{e'_0} \dots x_{r'}^{e'_{r'}} \cdot x_0^{f'_0} \dots x_{r'-1}^{f'_{r'-1}} \cdot x_{r'+1} \dots x_m$$

are in canonical form. Then $r = r'$.

Proof. Seeking a contradiction, suppose $r > r'$. Then the letter x_r occurs in the suffix $x_{r'+1} \dots x_m$ of \mathbf{w}' , so that $\text{occ}(x_r, \mathbf{w}') = 1$. Hence $e_r = \text{occ}(x_r, \mathbf{w}) \in \{1, 3\}$ by Lemma 23.2

and Remark 23.4(i). Let φ denote the following substitution into S :

$$z \mapsto \begin{cases} 4 & \text{if } z \in \{x_0, \dots, x_{r-1}\}, \\ 2 & \text{if } z = x_r, \\ 6 & \text{if } z \in \{x_{r+1}, \dots, x_m\}. \end{cases}$$

Then $4 \cdot \mathbf{w}'\varphi \cdot 6 = 4 \cdot 2 \cdot 6 = 3$ and

$$4 \cdot \mathbf{w}\varphi \cdot 6 = 4 \cdot x_r^{e_r} \varphi \cdot (x_0^{f_0} \dots x_{r-1}^{f_{r-1}}) \varphi \cdot 6.$$

If $e_r = 3$, then $4 \cdot \mathbf{w}\varphi \cdot 6 = 4 \cdot 2^3 \dots 6 = 1$. If $e_r = 1$, then $(f_0, \dots, f_{r-1}) \neq (0, \dots, 0)$ by (IV), so that $4 \cdot \mathbf{w}\varphi \cdot 6 = 4 \cdot 2 \cdot 4 \cdot 6 = 1$. In either case, the contradiction $4 \cdot \mathbf{w}\varphi \cdot 6 \neq 4 \cdot \mathbf{w}'\varphi \cdot 6$ is obtained. \square

LEMMA 23.8. *Let $S \in \{\mathcal{E}_8, \mathcal{E}_9\}$. Suppose that S satisfies an identity $\mathbf{w} \approx \mathbf{w}'$, where*

$$\mathbf{w} = x_0^{e_0} \dots x_r^{e_r} \cdot x_0^{f_0} \dots x_{r-1}^{f_{r-1}} \cdot x_{r+1} \dots x_m$$

and

$$\mathbf{w}' = x_0^{e'_0} \dots x_r^{e'_r} \cdot x_0^{f'_0} \dots x_{r-1}^{f'_{r-1}} \cdot x_{r+1} \dots x_m$$

are in canonical form. Then:

- (i) $(f_0, \dots, f_{r-1}) = (f'_0, \dots, f'_{r-1})$;
- (ii) $(e_0, \dots, e_r) = (e'_0, \dots, e'_r)$.

Consequently, $\mathbf{w} = \mathbf{w}'$.

Proof. (i) Seeking a contradiction, suppose $f_i > f'_i$ for some $i \in \{0, \dots, r-1\}$. Then $(f_i, f'_i) = (1, 0)$ by (II), so that $e_i = 1$ by (I) and (III). Since $\text{occ}(x_i, \mathbf{w}) = e_i + f_i = 2$, it follows from Remark 23.4(i) and Lemma 23.2 that $2 = \text{occ}(x_i, \mathbf{w}') = e'_i + f'_i = e'_i$. Let $\varphi : \mathcal{X} \rightarrow S$ denote the substitution

$$z \mapsto \begin{cases} 4 & \text{if } z \in \{x_0, \dots, x_{i-1}\}, \\ 2 & \text{if } z = x_i, \\ 6 & \text{otherwise.} \end{cases}$$

Then $e_i = 1$ and $e'_i = 2$ imply that

$$4 \cdot \mathbf{w}\varphi \cdot 6 = 4 \cdot (x_0^{e_0} \dots x_{i-1}^{e_{i-1}}) \varphi \cdot 2 \cdot (x_{i+1}^{e_{i+1}} \dots x_r^{e_r}) \varphi \dots = 4 \cdot 2 \cdot 6 \dots = 3$$

and

$$4 \cdot \mathbf{w}'\varphi \cdot 6 = 4 \cdot (x_0^{e'_0} \dots x_{i-1}^{e'_{i-1}}) \varphi \cdot 2^2 \cdot (x_{i+1}^{e'_{i+1}} \dots x_r^{e'_r}) \varphi \dots = 4 \cdot 1 \cdot 6 \dots = 1,$$

whence the contradiction $\mathbf{w}\varphi \neq \mathbf{w}'\varphi$ is deduced.

- (ii) Seeking a contradiction, suppose $e_i > e'_i$ for some $i \in \{0, \dots, r\}$.

Case 1: $e_r > e'_r$. Since $\text{occ}(x_r, \mathbf{w}) = e_r$ and $\text{occ}(x_r, \mathbf{w}') = e'_r$, it follows from (I) and Lemma 23.2 that $(e_r, e'_r) = (3, 1)$. If both $r \geq 1$ and $(f_0, \dots, f_{r-1}) \neq (0, \dots, 0)$ hold, then the contradiction $e_r \leq 2$ follows from (V). Therefore either $r = 0$ or $(f_0, \dots, f_{r-1}) = (0, \dots, 0)$, whence the identity $\mathbf{w} \approx \mathbf{w}'$ is either

$$x_0^3 x_1 \dots x_m \approx x_0 x_1 \dots x_m \quad \text{or} \quad x_0 \dots x_{r-1} x_r^3 x_{r+1} \dots x_m \approx x_0 \dots x_m.$$

In both cases, S satisfies the identity (23.2), which is impossible by Lemma 23.3.

Case 2: $e_i > e'_i$ for some $i < r$. Since $f_i = f'_i$ by part (i) with $\text{occ}(x_i, \mathbf{w}) = e_i + f_i$ and $\text{occ}(x_i, \mathbf{w}') = e'_i + f'_i$, it follows from (I) and Lemma 23.2 that $(e_i, e'_i) = (3, 1)$. Let $\varphi : \mathcal{X} \rightarrow S$ denote the substitution given in part (i). Then

$$4 \cdot \mathbf{w}\varphi \cdot 6 = 4 \cdot (x_0^{e_0} \dots x_{i-1}^{e_{i-1}})\varphi \cdot 2^3 \cdot (x_{i+1}^{e_{i+1}} \dots x_r^{e_r})\varphi \dots = 4 \cdot 1 \cdot 6 \dots = 1$$

and

$$4 \cdot \mathbf{w}'\varphi \cdot 6 = 4 \cdot (x_0^{e'_0} \dots x_{i-1}^{e'_{i-1}})\varphi \cdot 2 \cdot (x_{i+1}^{e'_{i+1}} \dots x_r^{e'_r})\varphi \dots = 4 \cdot 2 \cdot 6 \dots = 3,$$

which is impossible. □

Proof of Proposition 23.1. Let $S \in \{\mathcal{E}_8, \mathcal{E}_9\}$. It is routinely checked that S satisfies the identities (23.1). Hence it suffices to show that any nontrivial identity $\mathbf{w} \approx \mathbf{w}'$ satisfied by S is implied by the identities (23.1). By Lemma 23.6, the words \mathbf{w} and \mathbf{w}' are non-simple. Therefore by Lemma 23.5, there exist words $\bar{\mathbf{w}}$ and $\bar{\mathbf{w}}'$ in canonical form such that the identities (23.1) imply the identities $\mathbf{w} \approx \bar{\mathbf{w}}$ and $\mathbf{w}' \approx \bar{\mathbf{w}}'$. Since $\text{ini}(\bar{\mathbf{w}}) = \text{ini}(\bar{\mathbf{w}}')$ by Lemma 23.2,

$$\bar{\mathbf{w}} = x_0^{e_0} \dots x_r^{e_r} \cdot x_0^{f_0} \dots x_{r-1}^{f_{r-1}} \cdot x_{r+1} \dots x_m$$

and

$$\bar{\mathbf{w}}' = x_0^{e'_0} \dots x_{r'}^{e'_{r'}} \cdot x_0^{f'_0} \dots x_{r'-1}^{f'_{r'-1}} \cdot x_{r'+1} \dots x_m.$$

Then $r = r'$ by Lemma 23.7, and $\bar{\mathbf{w}} = \bar{\mathbf{w}}'$ by Lemma 23.8. Consequently, the identities (23.1) imply the identity $\mathbf{w} \approx \mathbf{w}'$. □

23.2. Finite basis property of \mathcal{E}_{10} and \mathcal{F}_5

PROPOSITION 23.9. *The variety generated by $S \in \{\mathcal{E}_{10}, \mathcal{F}_5\}$ is defined by the identities*

$$x^3 \approx x^2, \quad x^2yx \approx xyx, \quad xyx^2 \approx xyx, \tag{23.4a}$$

$$hx^2ky^2tz^2 \approx hy^2tx^2kz^2. \tag{23.4b}$$

LEMMA 23.10. *For all $\mathbf{h}, \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{X}^+$ such that $\mathbf{h}(\mathbf{x}), \mathbf{h}(\mathbf{y}), \mathbf{h}(\mathbf{z}) \notin \text{sim}(\mathbf{hxyz})$, the identities (23.4) imply the identity*

$$\mathbf{hxyz} \approx \mathbf{hyxz}. \tag{23.5}$$

Proof. If $x = \mathbf{h}(\mathbf{x})$, $y = \mathbf{h}(\mathbf{y})$, and $z = \mathbf{h}(\mathbf{z})$ are non-simple letters of \mathbf{hxyz} , then

$$\mathbf{hxyz} \stackrel{(23.4a)}{\approx} \mathbf{hx^2xy^2yz^2z} \stackrel{(23.4b)}{\approx} \mathbf{hy^2yx^2xz^2z} \stackrel{(23.4a)}{\approx} \mathbf{hyxz}.$$

Hence the identities (23.4) imply the identity (23.5). □

For any word \mathbf{w} , let $F_{NS}(\mathbf{w})$ denote the set of factors of \mathbf{w} of length two that begin with a non-simple letter and end with a simple letter:

$$F_{NS}(\mathbf{w}) = \{xy \in \mathcal{X}^2 \mid \mathbf{w} \in \mathcal{X}^*xy\mathcal{X}^*, x \notin \text{sim}(\mathbf{w}), y \in \text{sim}(\mathbf{w})\}.$$

LEMMA 23.11. *Let $S \in \{\mathcal{E}_{10}, \mathcal{F}_5\}$. Suppose that $\mathbf{w} \approx \mathbf{w}'$ is any identity satisfied by S . Then:*

- (i) $\text{con}(\mathbf{w}) = \text{con}(\mathbf{w}')$, $\text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$, $\mathbf{h}(\mathbf{w}) = \mathbf{h}(\mathbf{w}')$, and $\mathbf{t}(\mathbf{w}) = \mathbf{t}(\mathbf{w}')$;
- (ii) $F_{NS}(\mathbf{w}) = F_{NS}(\mathbf{w}')$;
- (iii) $F_{SS}(\mathbf{w}) = F_{SS}(\mathbf{w}')$.

Proof. (i) This follows from Lemma 2.1 since the subsemigroups $\{1, 4\}$, $\{5, 6\}$, and $\{1, 3, 5\}$ of \mathcal{E}_{10} are isomorphic to L_2 , R_2 , and N_2^1 , respectively, while the subsemigroups $\{1, 3\}$, $\{5, 6\}$, and $\{1, 2, 5\}$ of \mathcal{F}_5 are isomorphic to L_2 , R_2 , and N_2^1 , respectively.

(ii) Suppose $F_{NS}(\mathbf{w}) \neq F_{NS}(\mathbf{w}')$, say $xy \in F_{NS}(\mathbf{w}) \setminus F_{NS}(\mathbf{w}')$. Then it follows from part (i) that x is non-simple in \mathbf{w}' and y is simple in \mathbf{w}' , whence xy is not a factor of \mathbf{w}' . Let φ_1 and φ_2 denote the following substitutions into \mathcal{E}_{10} and \mathcal{F}_5 , respectively:

$$z \mapsto \begin{cases} 5 & \text{if } z = x, \\ 2 & \text{if } z = y, \\ 6 & \text{otherwise;} \end{cases} \quad z \mapsto \begin{cases} 5 & \text{if } z = x, \\ 4 & \text{if } z = y, \\ 6 & \text{otherwise.} \end{cases}$$

Then the following contradictions are obtained:

$$\begin{aligned} \mathbf{w}\varphi_1 &\in \{5, 6\}^* \cdot 5 \cdot 2 \cdot \{5, 6\}^* = \{1\} & \text{and} & & \mathbf{w}'\varphi_1 &\in \{5, 6\}^* \cdot 6 \cdot 2 \cdot \{5, 6\}^* = \{3\}; \\ \mathbf{w}\varphi_2 &\in \{5, 6\}^* \cdot 5 \cdot 4 \cdot \{5, 6\}^* = \{1\} & \text{and} & & \mathbf{w}'\varphi_2 &\in \{5, 6\}^* \cdot 6 \cdot 4 \cdot \{5, 6\}^* = \{2\}. \end{aligned}$$

Hence $F_{NS}(\mathbf{w}) = F_{NS}(\mathbf{w}')$.

(iii) Suppose $F_{SS}(\mathbf{w}) \neq F_{SS}(\mathbf{w}')$, say $xy \in F_{SS}(\mathbf{w}) \setminus F_{SS}(\mathbf{w}')$. Then $x, y \in \text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$ by part (i) so that $xy \in F_{NS}(x\mathbf{w}) \setminus F_{NS}(x\mathbf{w}')$. Therefore by part (ii), the semigroup S does not satisfy the identity $x\mathbf{w} \approx x\mathbf{w}'$, but this is impossible. \square

For any non-simple word \mathbf{w} , define $\vec{\mathbf{w}} = y_1^2 \dots y_k^2$ where y_1, \dots, y_k are precisely all the distinct non-simple letters of \mathbf{w} listed in alphabetical order. In this subsection, a non-simple word \mathbf{w} is said to be in *canonical form* if it can be written as

$$\mathbf{w} = \mathbf{w}_0 \cdot \vec{\mathbf{w}} \cdot x_1 \mathbf{w}_1 \dots x_m \mathbf{w}_m, \tag{23.6}$$

where $\mathbf{w}_0, \dots, \mathbf{w}_m \in \mathcal{X}^+$ and $x_1, \dots, x_m \in \mathcal{X}$ satisfy the following:

- (I) the letters of $\mathbf{w}_0, \dots, \mathbf{w}_m$ are precisely all simple letters of \mathbf{w} ;
- (II) the letters x_1, \dots, x_m are non-simple in \mathbf{w} ;
- (III) the letters x_1, \dots, x_{m-1} are in alphabetical order.

Note that the letters x_1, \dots, x_m need not be distinct, but it follows from (I) that the words $x_1 \mathbf{w}_1, \dots, x_m \mathbf{w}_m$ are distinct.

LEMMA 23.12. *Let \mathbf{w} be any non-simple word such that $h(\mathbf{w}), t(\mathbf{w}) \in \text{sim}(\mathbf{w})$. Then there exists some word $\vec{\mathbf{w}}$ in canonical form such that the identities (23.4) imply the identity $\mathbf{w} \approx \vec{\mathbf{w}}$.*

Proof. By Lemma 23.10, it suffices to convert \mathbf{w} , using the identities $\{(23.4), (23.5)\}$, into a word in canonical form. Consider a factorization of \mathbf{w} that displays all of its non-simple letters individually, that is, $\mathbf{w} = \mathbf{w}_0 x_1 \mathbf{w}_1 \dots x_r \mathbf{w}_r$ where the letters x_1, \dots, x_r are non-simple in \mathbf{w} and the letters in the factors $\mathbf{w}_0, \dots, \mathbf{w}_r \in \mathcal{X}^*$ are simple in \mathbf{w} . The assumption $h(\mathbf{w}), t(\mathbf{w}) \in \text{sim}(\mathbf{w})$ implies $\mathbf{w}_0, \mathbf{w}_r \neq \emptyset$. Further, each letter in the list x_1, \dots, x_r is non-simple in \mathbf{w} and thus appears at least twice in the list, whence $r \geq 2$.

Now the words $x_1 \mathbf{w}_1, \dots, x_r \mathbf{w}_r$ begin with non-simple letters of \mathbf{w} . Therefore by applying the identity (23.5) sufficiently many times, the factors

$$x_1 \mathbf{w}_1, \dots, x_{r-1} \mathbf{w}_{r-1} \tag{23.7}$$

of \mathbf{w} can be rearranged in any manner. In particular, the factors from (23.7) with nonempty \mathbf{w}_i can be gathered to the right, resulting in a word of the form

$$\mathbf{w}_0 \left(\prod_{i \in I_1} x_i \right) \left(\prod_{i \in I_2} (x_i \mathbf{w}_i) \right) x_r \mathbf{w}_r$$

for some I_1 and I_2 such that $I_1 \cup I_2 = \{1, \dots, r-1\}$ and $I_1 \cap I_2 = \emptyset$. Hence

$$\begin{aligned} \mathbf{w} &\stackrel{(23.5)}{\approx} \mathbf{w}_0 \left(\prod_{i \in I_1} x_i \right) \left(\prod_{i \in I_2} (x_i \mathbf{w}_i) \right) x_r \mathbf{w}_r \stackrel{(23.4a)}{\approx} \mathbf{w}_0 \left(\prod_{i \in I_1} x_i^2 \right) \left(\prod_{i \in I_2} (x_i^2 x_i \mathbf{w}_i) \right) x_r^2 x_r \mathbf{w}_r \\ &\stackrel{(23.5)}{\approx} \mathbf{w}_0 \left(\prod_{i=1}^r x_i^2 \right) \left(\prod_{i \in I_2} (x_i \mathbf{w}_i) \right) x_r \mathbf{w}_r. \end{aligned}$$

The identities $\{(23.4), (23.5)\}$ can be used to convert the factor $\prod_{i=1}^r x_i^2$ of the latter word into $\bar{\mathbf{w}}$, resulting in a word of the form (23.6) with (I) and (II) satisfied. The identities (23.5) can then be used to rearrange the factors $x_i \mathbf{w}_i$ in $\prod_{i \in I_2} (x_i \mathbf{w}_i)$ until (III) is satisfied. \square

Let $\mathbf{X}_{(23.4)}$ denote the variety defined by the identities (23.4).

LEMMA 23.13. *Suppose that $\mathbf{w}, \mathbf{w}' \in \mathcal{X}^+$ and $x, y \in \mathcal{X}$ are such that $\text{con}(\mathbf{w}) = \text{con}(\mathbf{w}')$, $\text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$, $h = h(\mathbf{w}) = h(\mathbf{w}')$, $t = t(\mathbf{w}) = t(\mathbf{w}')$, and $x, y \notin \text{con}(\mathbf{w}) = \text{con}(\mathbf{w}')$. Then $\mathbf{X}_{(23.4)}\{\mathbf{w} \approx \mathbf{w}'\} = \mathbf{X}_{(23.4)}\{\sigma\}$ where the identity*

$$\sigma \text{ is } \begin{cases} x\mathbf{w} \approx x\mathbf{w}' & \text{if } h \notin \text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}') \text{ and } t \in \text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}'), \\ \mathbf{w}y \approx \mathbf{w}'y & \text{if } h \in \text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}') \text{ and } t \notin \text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}'), \\ x\mathbf{w}y \approx x\mathbf{w}'y & \text{if } h, t \notin \text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}'). \end{cases}$$

Proof. It suffices to consider the case when $h \notin \text{sim}(\mathbf{w})$ and $t \in \text{sim}(\mathbf{w})$ since the other two cases can be established similarly. Let σ denote the identity $x\mathbf{w} \approx x\mathbf{w}'$. Then the inclusion $\mathbf{X}_{(23.4)}\{\mathbf{w} \approx \mathbf{w}'\} \subseteq \mathbf{X}_{(23.4)}\{\sigma\}$ holds trivially. Conversely, since the words \mathbf{w} and \mathbf{w}' begin with the non-simple letter h ,

$$\mathbf{w} \stackrel{(23.4a)}{\approx} h\mathbf{w} \stackrel{\sigma}{\approx} h\mathbf{w}' \stackrel{(23.4a)}{\approx} \mathbf{w}'.$$

Hence the inclusion $\mathbf{X}_{(23.4)}\{\sigma\} \subseteq \mathbf{X}_{(23.4)}\{\mathbf{w} \approx \mathbf{w}'\}$ holds. \square

Proof of Proposition 23.9. Let $S \in \{\mathcal{L}_{10}, \mathcal{F}_5\}$. It is routinely verified that S satisfies the identities (23.4). Therefore there exists some set Σ of identities satisfied by S such that $\{(23.4)\} \cup \Sigma$ is a basis for S . If $\mathbf{w} \approx \mathbf{w}'$ is an identity in Σ such that either \mathbf{w} or \mathbf{w}' is a simple word, then it follows from Lemma 23.11(i) that the identity $\mathbf{w} \approx \mathbf{w}'$ is trivial. Hence generality is not lost by assuming that all identities in Σ are formed by non-simple words.

Let $\mathbf{w} \approx \mathbf{w}'$ be any identity from Σ . By Lemma 23.11(i),

(a) $\text{con}(\mathbf{w}) = \text{con}(\mathbf{w}')$, $\text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$, $h(\mathbf{w}) = h(\mathbf{w}')$, and $t(\mathbf{w}) = t(\mathbf{w}')$.

By Lemma 23.13, the words forming the identities in Σ can be chosen to begin and end with simple letters. Therefore by Lemma 23.12,

(b) the words forming the identities in Σ can be chosen to be in canonical form.

It then follows from (a) and (b) that \mathbf{w} and \mathbf{w}' are of the form

$$\mathbf{w} = \mathbf{w}_0 \cdot \bar{\mathbf{w}} \cdot x_1 \mathbf{w}_1 \dots x_m \mathbf{w}_m \quad \text{and} \quad \mathbf{w}' = \mathbf{w}'_0 \cdot \bar{\mathbf{w}} \cdot y_1 \mathbf{w}'_1 \dots y_n \mathbf{w}'_n,$$

where $\text{con}(\mathbf{w}_0 \dots \mathbf{w}_m) = \text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}') = \text{con}(\mathbf{w}'_0 \dots \mathbf{w}'_n)$, $h(\mathbf{w}_0) = h(\mathbf{w}'_0)$, $t(\mathbf{w}_m) = t(\mathbf{w}'_n)$, and $x_1, \dots, x_m, y_1, \dots, y_n \in \text{con}(\bar{\mathbf{w}})$. Then by Lemma 23.11:

(c) $\mathbf{w}_0 = \mathbf{w}'_0$;

(d) $x_m \mathbf{w}_m = y_n \mathbf{w}'_n$;

(e) $\{x_1 \mathbf{w}_1, \dots, x_{m-1} \mathbf{w}_{m-1}\} = \{y_1 \mathbf{w}'_1, \dots, y_{n-1} \mathbf{w}'_{n-1}\}$.

Therefore $(x_1 \mathbf{w}_1, \dots, x_{m-1} \mathbf{w}_{m-1}) = (y_1 \mathbf{w}'_1, \dots, y_{n-1} \mathbf{w}'_{n-1})$ by (c)–(e) and (III), whence the identity $\mathbf{w} \approx \mathbf{w}'$ is trivial. Since the identity $\mathbf{w} \approx \mathbf{w}'$ is arbitrary in Σ , the identities (23.4) constitute a basis for S . \square

24. $\mathcal{F}_1, \mathcal{F}_2$

This section establishes the finite basis property of the following semigroups.

\mathcal{F}_1	1 2 3 4 5 6	\mathcal{F}_2	1 2 3 4 5 6
1	1 1 1 1 1 1	1	1 1 1 1 1 1
2	1 1 1 1 2 2	2	1 1 1 1 2 2
3	1 1 1 1 3 3	3	1 1 2 1 3 3
4	4 4 4 4 4 4	4	4 4 4 4 4 4
5	1 1 3 4 5 6	5	1 2 3 4 5 6
6	1 3 3 4 5 6	6	1 2 3 4 5 6

24.1. Finite basis property of \mathcal{F}_1

PROPOSITION 24.1. The variety generated by \mathcal{F}_1 is defined by the identities

$$x^3 \approx x^2, \quad x^2yx \approx xyx, \quad xyx^2 \approx xyx, \tag{24.1a}$$

$$xyxz^2 \approx x^2yz^2. \tag{24.1b}$$

LEMMA 24.2. Suppose that $\mathbf{w} \approx \mathbf{w}'$ is any identity satisfied by \mathcal{F}_1 . Then $\text{ini}(\mathbf{w}) = \text{ini}(\mathbf{w}')$, $\text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$, and $\text{t}(\mathbf{w}) = \text{t}(\mathbf{w}')$.

Proof. This follows from Lemma 2.1 since the subsemigroups $\{1, 4, 6\}$, $\{1, 3, 5\}$, and $\{5, 6\}$ of \mathcal{F}_1 are isomorphic to L_2^1 , N_2^1 , and R_2 , respectively. \square

In this subsection, a word \mathbf{w} with $\text{ini}(\mathbf{w}) = x_1 \dots x_m$ is said to be in *canonical form* if

$$\mathbf{w} = \prod_{i=1}^m (x_i^{e_i} \mathbf{w}_i), \tag{24.2}$$

where all of the following are satisfied:

- (I) $\mathbf{w}_i \in \{\emptyset, x_1, \dots, x_{i-1}\}$;
- (II) $e_1, \dots, e_m \in \{1, 2\}$;
- (III) if $x_i \notin \text{sim}(\mathbf{w})$, then $\mathbf{w}_{i-1} = \emptyset$;
- (IV) if $x_i \notin \text{sim}(\mathbf{w})$ and $x_i = \mathbf{w}_j$ for some $j > i$, then $e_i = 1$.

Note that (I) and (IV) imply that the word \mathbf{w} in (24.2) does not contain a factor of the form $x^2\mathbf{a}x$ where $\mathbf{a} \in \mathcal{X}^*$.

LEMMA 24.3. Let \mathbf{w} be any word. Then there exists some word $\bar{\mathbf{w}}$ in canonical form such that the identities (24.1) imply the identity $\mathbf{w} \approx \bar{\mathbf{w}}$.

Proof. It suffices to convert \mathbf{w} , using the identities (24.1), into a word in canonical form. Suppose that x and y are non-simple letters of \mathbf{w} such that $\mathbf{w} = \mathbf{a}x\mathbf{b}xy\mathbf{c}$ for some $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathcal{X}^*$ with $y \in \text{con}(\mathbf{a}\mathbf{b}\mathbf{c})$. Then

$$\mathbf{w} \stackrel{(24.1a)}{\approx} \mathbf{a}x\mathbf{b}xy^2\mathbf{c} \stackrel{(24.1b)}{\approx} \mathbf{a}x^2\mathbf{b}y^2\mathbf{c} \stackrel{(24.1a)}{\approx} \mathbf{a}x^2\mathbf{b}y\mathbf{c},$$

that is, using the identities (24.1), any non-first occurrence of x in \mathbf{w} that immediately precedes another non-simple letter can be gathered with the first x in \mathbf{w} . It follows that \mathbf{w} can be converted into a word of the form (24.2) with (I) and (III) satisfied. Then (II) and (IV) are satisfied by applying the identities (24.1a). \square

LEMMA 24.4. Suppose that \mathcal{F}_1 satisfies an identity $\mathbf{w} \approx \mathbf{w}'$, where

$$\mathbf{w} = \prod_{i=1}^m (x_i^{e_i} \mathbf{w}_i) \quad \text{and} \quad \mathbf{w}' = \prod_{i=1}^m (x_i^{e'_i} \mathbf{w}'_i)$$

are words in canonical form. Then $e_i = e'_i$ for all i .

Proof. Observe that $\text{occ}(x_m, \mathbf{w}) = e_m$ and $\text{occ}(x_m, \mathbf{w}') = e'_m$ by (I) and that

(a) $\text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$

by Lemma 24.2. Therefore $e_m = e'_m$ by (II). Seeking a contradiction, suppose $e_r \neq e'_r$ for some $r < m$, say $(e_r, e'_r) = (1, 2)$. Then $x_r \notin \text{sim}(\mathbf{w}')$, whence $x_r \notin \text{sim}(\mathbf{w})$ by (a). Since \mathbf{w} is in canonical form, the second x_r in \mathbf{w} coincides with one of $\mathbf{w}_{r+1}, \dots, \mathbf{w}_m$, say $x_r = \mathbf{w}_\ell$. Then $x_r \neq x_\ell$ by (I), and the word \mathbf{w} can be written as

$$\mathbf{w} = \mathbf{p}x_r\mathbf{w}_r\mathbf{q}x_\ell^{e_\ell} \underbrace{x_r}_{\mathbf{w}_\ell} \mathbf{u},$$

where $\mathbf{p} = \prod_{i=1}^{r-1} (x_i^{e_i} \mathbf{w}_i)$, $\mathbf{q} = \prod_{i=r+1}^{\ell-1} (x_i^{e_i} \mathbf{w}_i)$, and $\mathbf{u} = \prod_{i=\ell+1}^m (x_i^{e_i} \mathbf{w}_i)$ with

(b) $x_r \notin \text{con}(\mathbf{p}\mathbf{w}_r\mathbf{q}x_\ell)$.

As for the word \mathbf{w}' , since $e'_r = 2$, it can be written as

$$\mathbf{w}' = \mathbf{p}'x_r^2\mathbf{w}'_r\mathbf{q}'x_\ell^{e'_\ell}\mathbf{w}'_\ell\mathbf{u}'$$

where $\mathbf{p}' = \prod_{i=1}^{r-1} (x_i^{e'_i} \mathbf{w}'_i)$, $\mathbf{q}' = \prod_{i=r+1}^{\ell-1} (x_i^{e'_i} \mathbf{w}'_i)$, and $\mathbf{u}' = \prod_{i=\ell+1}^m (x_i^{e'_i} \mathbf{w}'_i)$ with $x_r \notin \text{con}(\mathbf{p}')$ and $x_\ell \notin \text{con}(\mathbf{p}'x_r\mathbf{w}'_r\mathbf{q}')$. In fact, it follows from (I) and (IV) that

(c) $x_r \notin \text{con}(\mathbf{p}'\mathbf{w}'_r\mathbf{q}'x_\ell\mathbf{w}'_\ell\mathbf{u}')$.

In particular, $t(\mathbf{w}') \neq x_r$. Since $t(\mathbf{w}) = t(\mathbf{w}')$ by Lemma 24.2, it follows that $\mathbf{u} \neq \emptyset$, whence

(d) $h(\mathbf{u}) = x_{\ell+1} \in \text{sim}(\mathbf{w})$

by (III), and

(e) $h(\mathbf{u}') = x_{\ell+1} \in \text{sim}(\mathbf{w}')$

by (a). Let $\varphi : \mathcal{X} \rightarrow \mathcal{F}_1$ denote the substitution

$$z \mapsto \begin{cases} 6 & \text{if } z = x_r, \\ 2 & \text{if } z = x_{\ell+1}, \\ 5 & \text{otherwise.} \end{cases}$$

Then $\mathbf{u}\varphi = 2$ and $\mathbf{u}'\varphi = 2$ by (d) and (e). Since $x_r, x_{\ell+1} \notin \text{con}(\mathbf{p}\mathbf{w}_r\mathbf{q}x_\ell)$ by (b) and (d), it follows that $5 \cdot \mathbf{p}\varphi = (\mathbf{w}_r\mathbf{q}x_\ell^{e_\ell})\varphi = 5$. Thus

$$5 \cdot \mathbf{w}\varphi = (5 \cdot \mathbf{p}\varphi) \cdot x_r\varphi \cdot (\mathbf{w}_r\mathbf{q}x_\ell^{e_\ell})\varphi \cdot x_r\varphi \cdot \mathbf{u}\varphi = 5 \cdot 6 \cdot 5 \cdot 6 \cdot 2 = 3.$$

Similarly, $x_r, x_{\ell+1} \notin \text{con}(\mathbf{p}'\mathbf{w}'_r\mathbf{q}'x_\ell\mathbf{w}'_\ell)$ by (c) and (e), so $5 \cdot \mathbf{p}'\varphi = (\mathbf{w}'_r\mathbf{q}'x_\ell^{e'_\ell}\mathbf{w}'_\ell)\varphi = 5$. Hence

$$5 \cdot \mathbf{w}'\varphi = (5 \cdot \mathbf{p}'\varphi) \cdot x_r^2\varphi \cdot (\mathbf{w}'_r\mathbf{q}'x_\ell^{e'_\ell}\mathbf{w}'_\ell)\varphi \cdot \mathbf{u}'\varphi = 5 \cdot 6 \cdot 5 \cdot 2 = 1.$$

Consequently, the contradiction $5 \cdot \mathbf{w}\varphi \neq 5 \cdot \mathbf{w}'\varphi$ is established. □

Proof of Proposition 24.1. It is routinely checked that \mathcal{F}_1 satisfies the identities (24.1). Hence it suffices to show that any identity $\mathbf{w} \approx \mathbf{w}'$ satisfied by \mathcal{F}_1 is implied by the identities (24.1). By Lemma 24.3, the words \mathbf{w} and \mathbf{w}' can be assumed to be in canonical form. By Lemmas 24.2 and 24.4,

$$\mathbf{w} = \prod_{i=1}^m (x_i^{e_i} \mathbf{w}_i) \quad \text{and} \quad \mathbf{w}' = \prod_{i=1}^m (x_i^{e'_i} \mathbf{w}'_i)$$

with $\mathbf{w}_m = \mathbf{w}'_m$. Let $\ell < m$ be the least integer such that $\mathbf{w}_\ell \neq \mathbf{w}'_\ell$. Then the words \mathbf{w} and \mathbf{w}' share the prefix $\mathbf{p} = \prod_{i=1}^{\ell-1} (x_i^{e_i} \mathbf{w}_i) = \prod_{i=1}^{\ell-1} (x_i^{e_i} \mathbf{w}'_i)$. Since $(\mathbf{w}_\ell, \mathbf{w}'_\ell) \neq (\emptyset, \emptyset)$, generality is not lost by assuming $\mathbf{w}_\ell \neq \emptyset$. Then (I) implies $\mathbf{w}_\ell = x_k$ for some $k \in \{0, \dots, \ell - 1\}$ and

(a) $\mathbf{w}'_\ell \in \{\emptyset, x_0, \dots, x_{\ell-1}\} \setminus \{x_k\}$,

whence

$$\mathbf{w} = \mathbf{p} x_\ell^{e_\ell} \underbrace{x_k}_{\mathbf{w}_\ell} x_{\ell+1}^{e_{\ell+1}} \mathbf{q} \quad \text{and} \quad \mathbf{w}' = \mathbf{p} x_\ell^{e_\ell} \mathbf{w}'_\ell x_{\ell+1}^{e_{\ell+1}} \mathbf{q}'$$

where $\mathbf{q} = \mathbf{w}_{\ell+1} \prod_{i=\ell+2}^m (x_i^{e_i} \mathbf{w}_i)$ and $\mathbf{q}' = \mathbf{w}'_{\ell+1} \prod_{i=\ell+2}^m (x_i^{e_i} \mathbf{w}'_i)$. The conditions $\mathbf{w}_\ell \neq \emptyset$ and (III) imply $x_{\ell+1} \in \text{sim}(\mathbf{w})$; since $\text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$ by Lemma 24.2,

(b) $x_{\ell+1} \in \text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$ and $e_{\ell+1} = 1$.

Let $\varphi : \mathcal{X} \rightarrow \mathcal{F}_1$ denote the substitution

$$z \mapsto \begin{cases} 6 & \text{if } z = x_k, \\ 2 & \text{if } z = x_{\ell+1}, \\ 5 & \text{otherwise.} \end{cases}$$

Since $x_{\ell+1} \notin \text{con}(\mathbf{p} x_\ell \mathbf{q})$ by (b), it follows that $(\mathbf{p} x_\ell^{e_\ell})\varphi = 5$ and $\mathbf{q}\varphi \in \{\emptyset, 5, 6\}$. Thus

$$\mathbf{w}\varphi = (\mathbf{p} x_\ell^{e_\ell})\varphi \cdot x_k \varphi \cdot x_{\ell+1} \varphi \cdot \mathbf{q}\varphi = 5 \cdot 6 \cdot 2 \cdot \mathbf{q}\varphi = 3.$$

On the other hand, $\mathbf{w}'_\ell \varphi \in \{\emptyset, 5\}$ by (a), so that

$$\mathbf{w}'\varphi = (\mathbf{p} x_\ell^{e_\ell})\varphi \cdot \mathbf{w}'_\ell \varphi \cdot x_{\ell+1} \varphi \cdot \mathbf{q}'\varphi = 5 \cdot \mathbf{w}'_\ell \varphi \cdot 2 \cdot \mathbf{q}'\varphi = 1.$$

The contradiction $\mathbf{w}\varphi \neq \mathbf{w}'\varphi$ implies that the integer ℓ does not exist. Consequently, $\mathbf{w}_i = \mathbf{w}'_i$ for all i , whence the identity $\mathbf{w} \approx \mathbf{w}'$ is trivial and is implied by the identities (24.1). \square

24.2. Finite basis property of \mathcal{F}_2

PROPOSITION 24.5. The variety generated by \mathcal{F}_2 is defined by the identities

$$x^4 \approx x^3, \tag{24.3a}$$

$$x^3 y x \approx x^2 y x, \tag{24.3b}$$

$$x y x z \approx x^2 y z. \tag{24.3c}$$

In this subsection, a word \mathbf{w} is said to be in *canonical form* if

$$\mathbf{w} = x_1^{e_1} \dots x_m^{e_m} y,$$

where all of the following are satisfied:

- (I) $e_1, \dots, e_m \in \{1, 2, 3\}$;
- (II) $x_1, \dots, x_m \in \mathcal{X}$ are distinct;
- (III) either (a) $y = \emptyset$ or (b) $y = x_\ell$ for some $\ell < m$ with $e_\ell \in \{1, 2\}$.

LEMMA 24.6. Let \mathbf{w} be any word. Then there exists some word $\bar{\mathbf{w}}$ in canonical form such that the identities (24.3) imply the identity $\mathbf{w} \approx \bar{\mathbf{w}}$.

Proof. It suffices to convert \mathbf{w} , using the identities (24.3), into a word in canonical form. This is easily achieved if $|\text{con}(\mathbf{w})| = 1$. Hence assume $|\text{con}(\mathbf{w})| \geq 2$. Then $\mathbf{w} = \mathbf{p}y$ for some

$\mathbf{p} \in \mathcal{X}^+$ with $y = \mathbf{t}(\mathbf{w})$. If a letter x occurs more than once in \mathbf{p} , then the identity (24.3c) can be used to gather any non-first x in \mathbf{p} with the first x in \mathbf{p} :

$$\mathbf{w} = \underbrace{\mathbf{axbxc}}_{\mathbf{p}} y \stackrel{(24.3c)}{\approx} \mathbf{ax}^2\mathbf{bcy}.$$

Hence the identity (24.3c) can be used to convert \mathbf{w} into a word of the form $\mathbf{w}' = x_1^{e_1} \dots x_m^{e_m} y$ that satisfies (II). If $y = x_m$, then $\mathbf{w}' = x_1^{e_1} \dots x_{m-1}^{e_{m-1}} x_m^{e_m+1}$ and the identity (24.3a) can be used to reduce the exponents $e_1, \dots, e_{m-1}, e_m + 1$ in \mathbf{w}' to numbers in $\{1, 2, 3\}$, whence (I) and (IIIa) are satisfied. If $y = x_\ell$ for some $\ell < m$, then (I) and (IIIb) are satisfied by applying the identities $\{(24.3a), (24.3b)\}$. \square

Proof of Proposition 24.5. It is routinely checked that \mathcal{F}_2 satisfies the identities (24.3). Therefore it suffices to show that any identity $\mathbf{w} \approx \mathbf{w}'$ satisfied by \mathcal{F}_2 is implied by the identities (24.3). By Lemma 24.6, the words \mathbf{w} and \mathbf{w}' can be assumed to be in canonical form. Since the subsemigroups $\{1, 4, 6\}$ and $\{5, 6\}$ of \mathcal{F}_2 are isomorphic to L_2^1 and R_2 , respectively, it follows from Lemma 2.1 that $\mathbf{w} = x_1^{e_1} \dots x_m^{e_m} y$ and $\mathbf{w}' = x_1^{e'_1} \dots x_m^{e'_m} y$. Since the subsemigroup $\{1, 2, 3, 5\}$ of \mathcal{F}_2 is isomorphic to N_3^1 , it follows from (I), (III), and Lemma 2.1(vii) that $e_i = e'_i$ for all i . Hence the identity $\mathbf{w} \approx \mathbf{w}'$ is trivial and so is implied by the identities (24.3). \square

25. $\mathcal{F}_3, \mathcal{F}_4$

This section establishes the finite basis property of the following semigroups.

\mathcal{F}_3	1	2	3	4	5	6	\mathcal{F}_4	1	2	3	4	5	6
1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	1	1	1	2	2	2	1	1	1	1	2	2
3	1	2	3	1	1	1	3	1	2	3	1	2	2
4	4	4	4	4	4	4	4	4	4	4	4	4	4
5	1	1	1	4	5	6	5	1	1	1	4	5	6
6	4	4	4	1	6	5	6	4	4	4	1	6	5

Let \mathbf{W} denote the variety generated by \overleftarrow{J} and O , where \overleftarrow{J} is the dual semigroup of J and O is the semigroup from Remark 6.6.

LEMMA 25.1. *The variety \mathbf{W} is defined by the identities*

$$xy^3 \approx xy, \tag{25.1a}$$

$$x^3y \approx xyx^2, \tag{25.1b}$$

$$xyxy \approx xy^2x. \tag{25.1c}$$

Proof. Let $\mathbf{X}_{(25.1)}$ denote the variety defined by the identities (25.1). It is routinely checked that $\overleftarrow{J}, O \in \mathbf{X}_{(25.1)}$. Results from Edmunds *et al.* [9, § 6] (specifically, the proofs of Lemmas 6.3, 6.4, and 6.9–6.11) imply that any proper subvariety of $\mathbf{X}_{(25.1)}$ must satisfy either $x^3y \approx xy$ or $xyx \approx x^2y$; it is routinely verified that these two identities are not satisfied by \overleftarrow{J} and O , respectively. Consequently, $\mathbf{W} = \mathbf{X}_{(25.1)}$. \square

A finite basis for the variety $\mathbf{A}_0 \vee \mathbf{W}$ is established in § 25.1. It is then routinely checked that $\mathcal{F}_4 \in \mathbf{A}_0 \vee \mathbf{W}$. But since the subsemigroups $\{1, 2, 3, 5\}$, $\{1, 2, 5\}$, and $\{1, 4, 5, 6\}$ of \mathcal{F}_4

are isomorphic to A_0 , \overleftarrow{J} , and O , respectively, the semigroup \mathcal{F}_4 generates the variety $\mathbf{A}_0 \vee \mathbf{W}$ and so is finitely based.

Similarly, a finite basis for the variety $\mathbf{B}_0 \vee \mathbf{W}$ is established in §25.2, from which the condition $\mathcal{F}_3 \in \mathbf{B}_0 \vee \mathbf{W}$ can be routinely checked. The subsemigroups $\{1, 2, 3, 5\}$, $\{1, 2, 5\}$, and $\{1, 4, 5, 6\}$ of \mathcal{F}_3 are isomorphic to B_0 , \overleftarrow{J} , and O , respectively, whence \mathcal{F}_3 generates the variety $\mathbf{B}_0 \vee \mathbf{W}$ and is finitely based.

25.1. A basis for $\mathbf{A}_0 \vee \mathbf{W}$

PROPOSITION 25.2. The variety $\mathbf{A}_0 \vee \mathbf{W}$ is defined by the identities

$$xHx^2Kx \approx xHKx, \tag{25.2a}$$

$$xHyKy^2Tx \approx xHyKTx, \tag{25.2b}$$

$$xHyKx^2Ty \approx xHyKTyx^2, \tag{25.2c}$$

$$xHyKxy \approx xHyKyx. \tag{25.2d}$$

LEMMA 25.3. Suppose that \mathbf{w} is any connected word. Then there exists some word $\tilde{\mathbf{w}}$ with $\text{con}(\mathbf{w}) = \text{con}(\tilde{\mathbf{w}})$ and $\text{h}(\mathbf{w}) = \text{h}(\tilde{\mathbf{w}}) = \text{t}(\tilde{\mathbf{w}})$ such that the identities (25.2) imply $\mathbf{w} \approx \tilde{\mathbf{w}}$.

Proof. If $\text{h}(\mathbf{w}) = \text{t}(\mathbf{w})$, then the lemma holds vacuously with $\tilde{\mathbf{w}} = \mathbf{w}$. Therefore assume $\text{h}(\mathbf{w}) \neq \text{t}(\mathbf{w})$. Since \mathbf{w} is connected, there exists a sequence $\text{h}(\mathbf{w}) = x_1, x_2, \dots, x_m = \text{t}(\mathbf{w})$ of non-simple letters of \mathbf{w} occurring in an overlapping pattern such that the first x_{i+1} occurs between two occurrences of x_i and the last x_{i+1} occurs to the right of all occurrences of x_i . This sequence can be chosen to have minimal length among all such sequences, so that

$$\mathbf{w} = x_1 \mathbf{p}_1 x_2 \mathbf{p}_2 x_1 \mathbf{q}_1 x_3 \mathbf{p}_3 x_2 \mathbf{q}_2 x_4 \mathbf{p}_4 x_3 \mathbf{q}_3 \dots \mathbf{q}_{m-1} x_m$$

for some $\mathbf{p}_1, \dots, \mathbf{p}_m, \mathbf{q}_1, \dots, \mathbf{q}_{m-1} \in \mathcal{X}^*$. (Note that \mathbf{p}_i follows the first x_i while \mathbf{q}_i follows the last x_i .) Then

$$\begin{aligned} \mathbf{w} &= (x_1 \mathbf{p}_1 x_2 \mathbf{p}_2 x_1) \mathbf{q}_1 x_3 \mathbf{p}_3 x_2 \mathbf{q}_2 x_4 \mathbf{p}_4 x_3 \mathbf{q}_3 \dots \mathbf{q}_{m-1} x_m \\ (25.2a) \quad &\approx (x_1 \mathbf{p}_1 x_2 \mathbf{p}_2 x_1^3 \mathbf{q}_1 x_3 \mathbf{p}_3 x_2) \mathbf{q}_2 x_4 \mathbf{p}_4 x_3 \mathbf{q}_3 \dots \mathbf{q}_{m-1} x_m \\ (25.2c) \quad &\approx (x_1 \mathbf{p}_1 x_2 \mathbf{p}_2 x_1 \mathbf{q}_1 x_3 \mathbf{p}_3 x_2 x_1^2 \mathbf{q}_2 x_4 \mathbf{p}_4 x_3) \mathbf{q}_3 \dots \mathbf{q}_{m-1} x_m \\ (25.2c) \quad &\approx x_1 \mathbf{p}_1 x_2 \mathbf{p}_2 x_1 \mathbf{q}_1 x_3 \mathbf{p}_3 x_2 \mathbf{q}_2 x_4 \mathbf{p}_4 x_3 x_1^2 \mathbf{q}_3 \dots \mathbf{q}_{m-1} x_m \\ &\vdots \\ (25.2c) \quad &\approx \underbrace{x_1 \mathbf{p}_1 x_2 \mathbf{p}_2 x_1 \mathbf{q}_1 x_3 \mathbf{p}_3 x_2 \mathbf{q}_2 x_4 \mathbf{p}_4 x_3 \mathbf{q}_3 \dots \mathbf{q}_{m-1} x_m x_1^2}_{\tilde{\mathbf{w}}}, \end{aligned}$$

where the word $\tilde{\mathbf{w}}$ satisfies the required properties. □

Let $\mathbf{X}_{(25.2)}$ denote the variety defined by the identities (25.2).

LEMMA 25.4. Let $\mathbf{w} \approx \mathbf{w}'$ be any identity satisfied by A_0 , \overleftarrow{J} , and O . Then

$$\mathbf{X}_{(25.2)}\{\mathbf{w} \approx \mathbf{w}'\} = \mathbf{X}_{(25.2)}\{\mathbf{w}_i \approx \mathbf{w}'_i \mid 1 \leq i \leq m\}$$

for some identities $\mathbf{w}_i \approx \mathbf{w}'_i$ such that $\text{h}(\mathbf{w}_i) = \text{t}(\mathbf{w}_i) = \text{h}(\mathbf{w}'_i) = \text{t}(\mathbf{w}'_i)$.

Proof. Since A_0 satisfies $\mathbf{w} \approx \mathbf{w}'$, it follows from Lee and Volkov [21, Proposition 3.2(i)] that

$$\mathbf{w} = \mathbf{s}_0 \prod_{i=1}^m (\mathbf{w}_i \mathbf{s}_i) \quad \text{and} \quad \mathbf{w}' = \mathbf{s}_0 \prod_{i=1}^m (\mathbf{w}'_i \mathbf{s}_i),$$

where:

- (a) the letters of $\mathbf{s}_0, \mathbf{s}_m \in \mathcal{X}^*$ and $\mathbf{s}_1, \dots, \mathbf{s}_{m-1} \in \mathcal{X}^+$ are simple in \mathbf{w} and \mathbf{w}' ;
- (b) $\mathbf{w}_1, \dots, \mathbf{w}_m, \mathbf{w}'_1, \dots, \mathbf{w}'_m \in \mathcal{X}^+$ are connected;
- (c) $\mathbf{s}_0, \mathbf{w}_1, \mathbf{s}_1, \dots, \mathbf{w}_m, \mathbf{s}_m$ are pairwise disjoint;
- (d) $\mathbf{s}_0, \mathbf{w}'_1, \mathbf{s}_1, \dots, \mathbf{w}'_m, \mathbf{s}_m$ are pairwise disjoint;
- (e) $\text{con}(\mathbf{w}_i) = \text{con}(\mathbf{w}'_i)$ and A_0 satisfies $\mathbf{w}_i \approx \mathbf{w}'_i$ for all i .

It is easily seen that the inclusion $\mathbf{X}_{(25.2)}\{\mathbf{w}_i \approx \mathbf{w}'_i \mid 1 \leq i \leq m\} \subseteq \mathbf{X}_{(25.2)}\{\mathbf{w} \approx \mathbf{w}'\}$ holds.

Consider any $i \in \{1, \dots, m\}$. The subsemigroup $\{a, ba, 1\}$ of O is isomorphic to L_2^1 . Therefore $\text{ini}(\mathbf{w}_i) = \text{ini}(\mathbf{w}'_i)$ by Lemma 2.1(iii); in particular, $h(\mathbf{w}_i) = h(\mathbf{w}'_i)$. Further, by Lemma 25.3, the words \mathbf{w}_i and \mathbf{w}'_i can be chosen to satisfy $h(\mathbf{w}_i) = t(\mathbf{w}_i)$ and $h(\mathbf{w}'_i) = t(\mathbf{w}'_i)$. For convenience, let $h_i = h(\mathbf{w}_i) = t(\mathbf{w}_i) = h(\mathbf{w}'_i) = t(\mathbf{w}'_i)$. Let φ denote the substitution $x \mapsto h_i^2$ for all $x \notin \text{con}(\mathbf{w}_i) = \text{con}(\mathbf{w}'_i)$. Then the deductions $\mathbf{w}\varphi \stackrel{(25.2a)}{\approx} \mathbf{w}_i$ and $\mathbf{w}'\varphi \stackrel{(25.2a)}{\approx} \mathbf{w}'_i$ hold, so that the identities $\{(25.2), \mathbf{w} \approx \mathbf{w}'\}$ imply $\mathbf{w}_i \approx \mathbf{w}'_i$. Since $i \in \{1, \dots, m\}$ is arbitrary, the inclusion $\mathbf{X}_{(25.2)}\{\mathbf{w} \approx \mathbf{w}'\} \subseteq \mathbf{X}_{(25.2)}\{\mathbf{w}_i \approx \mathbf{w}'_i \mid 1 \leq i \leq m\}$ holds. \square

Proof of Proposition 25.2. It is routinely checked that A_0, \overleftarrow{J} , and O satisfy the identities (25.2). Hence $\mathbf{A}_0 \vee \mathbf{W} = \mathbf{X}_{(25.2)}\Sigma$ for some set Σ of identities. By Lemma 25.4, each identity $\mathbf{w} \approx \mathbf{w}'$ in Σ can be chosen to satisfy $h(\mathbf{w}) = t(\mathbf{w}) = h(\mathbf{w}') = t(\mathbf{w}')$. Let $\mathbf{w} \approx \mathbf{w}'$ be any such identity with $h = h(\mathbf{w}) = t(\mathbf{w}) = h(\mathbf{w}') = t(\mathbf{w}')$. Since \overleftarrow{J} and O satisfy the identity $\mathbf{w} \approx \mathbf{w}'$, it follows from Lemma 25.1 and Birkhoff’s completeness theorem of equational logic [3] that there exists a deduction sequence

$$\mathbf{w} = \mathbf{z}_0 \Rightarrow \mathbf{z}_1 \Rightarrow \dots \Rightarrow \mathbf{z}_r = \mathbf{w}', \tag{25.3}$$

where each deduction $\mathbf{z}_i \Rightarrow \mathbf{z}_{i+1}$ involves an identity from (25.1), that is, there exist $\mathbf{e}_i, \mathbf{f}_i \in \mathcal{X}^*$, an identity $\mathbf{u}_i \approx \mathbf{v}_i$ from (25.1), and an endomorphism φ_i of \mathcal{X}^+ such that $\mathbf{z}_i = \mathbf{e}_i(\mathbf{u}_i\varphi_i)\mathbf{f}_i$ and $\mathbf{z}_{i+1} = \mathbf{e}_i(\mathbf{v}_i\varphi_i)\mathbf{f}_i$. In what follows, it is shown that each deduction $\mathbf{z}_i \Rightarrow \mathbf{z}_{i+1}$ in (25.3) can be replaced by a deduction sequence that involves only identities from (25.2), whence the variety $\mathbf{X}_{(25.2)}$ satisfies the identity $\mathbf{w} \approx \mathbf{w}'$. Since the identity $\mathbf{w} \approx \mathbf{w}'$ is arbitrary in Σ , it follows that $\mathbf{X}_{(25.2)} = \mathbf{X}_{(25.2)}\Sigma = \mathbf{A}_0 \vee \mathbf{W}$.

Multiplying each word in (25.3) by h^2 on both the left and the right results in

$$h^2\mathbf{w}h^2 = h^2\mathbf{z}_0h^2 \Rightarrow h^2\mathbf{z}_1h^2 \Rightarrow \dots \Rightarrow h^2\mathbf{z}_rh^2 = h^2\mathbf{w}'h^2.$$

Since the identities $h^2\mathbf{w}h^2 \approx \mathbf{w}$ and $h^2\mathbf{w}'h^2 \approx \mathbf{w}'$ are implied by the identities (25.2a) of $\mathbf{X}_{(25.2)}$, generality is not lost if every word in (25.3) is assumed to begin and end with the letter h . Let i be any fixed number from $\{0, \dots, r-1\}$.

Case 1: $\mathbf{z}_i \Rightarrow \mathbf{z}_{i+1}$ involves (25.1a). Then $\{\mathbf{z}_i, \mathbf{z}_{i+1}\} = \{\mathbf{eab}^3\mathbf{f}, \mathbf{eabf}\}$ for some $\mathbf{a}, \mathbf{b} \in \mathcal{X}^+$ and $\mathbf{e}, \mathbf{f} \in \mathcal{X}^*$. Since

$$\mathbf{eab}^3\mathbf{f} \stackrel{(25.2a)}{\approx} h^2\mathbf{eab}^3\mathbf{f}h^2 \stackrel{(25.2b)}{\approx} h^2\mathbf{eabf}h^2 \stackrel{(25.2a)}{\approx} \mathbf{eabf},$$

the deduction $\mathbf{z}_i \Rightarrow \mathbf{z}_{i+1}$ can be replaced by a deduction sequence that involves identities from (25.2).

Case 2: $\mathbf{z}_i \Rightarrow \mathbf{z}_{i+1}$ involves (25.1b). Then $\{\mathbf{z}_i, \mathbf{z}_{i+1}\} = \{\mathbf{ea}^3\mathbf{bf}, \mathbf{eaba}^2\mathbf{f}\}$ for some $\mathbf{a}, \mathbf{b} \in \mathcal{X}^+$ and $\mathbf{e}, \mathbf{f} \in \mathcal{X}^*$. Since

$$\mathbf{ea}^3\mathbf{bf} \stackrel{(25.2a)}{\approx} h^2\mathbf{ea}^3\mathbf{bf}h^2 \stackrel{(25.2b)}{\approx} h^2\mathbf{eabf}h^2 \stackrel{(25.2b)}{\approx} h^2\mathbf{eaba}^2\mathbf{fh}^2 \stackrel{(25.2a)}{\approx} \mathbf{eaba}^2\mathbf{f},$$

the deduction $\mathbf{z}_i \Rightarrow \mathbf{z}_{i+1}$ can be replaced by a deduction sequence that involves identities from (25.2).

Case 3: $\mathbf{z}_i \Rightarrow \mathbf{z}_{i+1}$ involves (25.1c). Since the identity (25.1c) belongs to (25.2d), the deduction $\mathbf{z}_i \Rightarrow \mathbf{z}_{i+1}$ can be replaced by a deduction that involves an identity from (25.2). \square

25.2. A basis for $\mathbf{B}_0 \vee \mathbf{W}$

PROPOSITION 25.5. *The variety $\mathbf{B}_0 \vee \mathbf{W}$ is defined by the identities*

$$x\mathbf{H}x^2\mathbf{K}x \approx x\mathbf{H}\mathbf{K}x, \quad (25.4a)$$

$$x\mathbf{H}y\mathbf{K}y^2\mathbf{T}x \approx x\mathbf{H}y\mathbf{K}\mathbf{T}x, \quad (25.4b)$$

$$x\mathbf{H}y\mathbf{K}x^2\mathbf{T}y \approx x\mathbf{H}y\mathbf{K}\mathbf{T}yx^2, \quad (25.4c)$$

$$x\mathbf{H}y\mathbf{K}xy \approx x\mathbf{H}y\mathbf{K}yx, \quad (25.4d)$$

$$x^2y^2x^2y^2 \approx x^2y^2. \quad (25.4e)$$

LEMMA 25.6. *Let $\mathbf{w} = \mathbf{w}_1 \dots \mathbf{w}_m$ where $\mathbf{w}_1, \dots, \mathbf{w}_m$ are pairwise disjoint connected words. Then there exists some word $\tilde{\mathbf{w}}$ with $\text{con}(\mathbf{w}) = \text{con}(\tilde{\mathbf{w}})$ and $\mathbf{h}(\mathbf{w}) = \mathbf{h}(\tilde{\mathbf{w}}) = \mathbf{t}(\tilde{\mathbf{w}})$ such that the identities (25.4) imply the identity $\mathbf{w} \approx \tilde{\mathbf{w}}$.*

Proof. It suffices to verify the lemma for $m = 2$ since the general case can be obtained by induction. Since the word \mathbf{w}_1 is connected, the letter $t = \mathbf{t}(\mathbf{w}_1)$ occurs at least twice in \mathbf{w}_1 . Hence $\mathbf{w}_1 = \mathbf{a}t\mathbf{b}t$ for some $\mathbf{a}, \mathbf{b} \in \mathcal{X}^*$. Similarly, the letter $h = \mathbf{h}(\mathbf{w}_2)$ occurs at least twice in \mathbf{w}_2 , so that $\mathbf{w}_2 = h\mathbf{c}h\mathbf{d}$ for some $\mathbf{c}, \mathbf{d} \in \mathcal{X}^*$. Then $\mathbf{w}_1h^2t^2\mathbf{w}_2$ is a connected word such that

$$\mathbf{w}_1h^2t^2\mathbf{w}_2 \stackrel{(25.4a)}{\approx} \mathbf{w}_1t^2h^2t^2h^2\mathbf{w}_2 \stackrel{(25.4e)}{\approx} \mathbf{w}_1t^2h^2\mathbf{w}_2 \stackrel{(25.4a)}{\approx} \mathbf{w}_1\mathbf{w}_2.$$

Since the identities (25.2) belong to (25.4), the proof of Lemma 25.3 can be repeated to convert the connected word $\mathbf{w}_1h^2t^2\mathbf{w}_2$, using the identities (25.4), into a word $\tilde{\mathbf{w}}$ with $\text{con}(\mathbf{w}) = \text{con}(\tilde{\mathbf{w}})$ and $\mathbf{h}(\mathbf{w}) = \mathbf{h}(\tilde{\mathbf{w}}) = \mathbf{t}(\tilde{\mathbf{w}})$. \square

Let $\mathbf{X}_{(25.4)}$ denote the variety defined by the identities (25.4).

LEMMA 25.7. *Let $\mathbf{w} \approx \mathbf{w}'$ be any identity satisfied by B_0 , \overleftarrow{J} , and O . Then*

$$\mathbf{X}_{(25.4)}\{\mathbf{w} \approx \mathbf{w}'\} = \mathbf{X}_{(25.4)}\{\mathbf{w}_i \approx \mathbf{w}'_i \mid 1 \leq i \leq m\}$$

for some identities $\mathbf{w}_i \approx \mathbf{w}'_i$ such that $\mathbf{h}(\mathbf{w}_i) = \mathbf{t}(\mathbf{w}_i) = \mathbf{h}(\mathbf{w}'_i) = \mathbf{t}(\mathbf{w}'_i)$.

Proof. Since B_0 satisfies $\mathbf{w} \approx \mathbf{w}'$, it follows from Lee and Volkov [21, Proposition 3.2(ii)] that

$$\mathbf{w} = \mathbf{s}_0 \prod_{i=1}^m (\mathbf{w}_i \mathbf{s}_i) \quad \text{and} \quad \mathbf{w}' = \mathbf{s}_0 \prod_{i=1}^m (\mathbf{w}'_i \mathbf{s}_i),$$

where the following holds:

- (a) the letters of $\mathbf{s}_0, \mathbf{s}_m \in \mathcal{X}^*$ and $\mathbf{s}_1, \dots, \mathbf{s}_{m-1} \in \mathcal{X}^+$ are simple in \mathbf{w} and \mathbf{w}' ;
- (b) each $\mathbf{w}_1, \dots, \mathbf{w}_m, \mathbf{w}'_1, \dots, \mathbf{w}'_m \in \mathcal{X}^+$ is a product of pairwise disjoint connected words;
- (c) $\mathbf{s}_0, \mathbf{w}_1, \mathbf{s}_1, \dots, \mathbf{w}_m, \mathbf{s}_m$ are pairwise disjoint;
- (d) $\mathbf{s}_0, \mathbf{w}'_1, \mathbf{s}_1, \dots, \mathbf{w}'_m, \mathbf{s}_m$ are pairwise disjoint;
- (e) $\text{con}(\mathbf{w}_i) = \text{con}(\mathbf{w}'_i)$ and B_0 satisfies $\mathbf{w}_i \approx \mathbf{w}'_i$ for all i .

By Lemma 25.6, each word in (b) can be replaced with a connected word. Since the identities (25.2) belong to (25.4), the proof of Lemma 25.4 can be repeated to establish the present lemma. □

Proof of Proposition 25.5. It is routinely shown that B_0, \overleftarrow{J} , and O satisfy the identities $\{(25.2), (25.4)\}$. Therefore $\mathbf{B}_0 \vee \mathbf{W} = \mathbf{X}_{(25.4)}\Sigma$ for some set Σ of identities. By Lemma 25.7, each identity $\mathbf{w} \approx \mathbf{w}'$ in Σ can be chosen to satisfy $h(\mathbf{w}) = t(\mathbf{w}) = h(\mathbf{w}') = t(\mathbf{w}')$. Let $\mathbf{w} \approx \mathbf{w}'$ be any such identity with $h = h(\mathbf{w}) = t(\mathbf{w}) = h(\mathbf{w}') = t(\mathbf{w}')$. Then the proof of Proposition 25.2 can be repeated to show that the identity $\mathbf{w} \approx \mathbf{w}'$ is deducible from the identities (25.2). Since the identities (25.2) belong to (25.4), the identity $\mathbf{w} \approx \mathbf{w}'$ is also deducible from the identities (25.4). Consequently, the variety $\mathbf{X}_{(25.4)}$ satisfies the identity $\mathbf{w} \approx \mathbf{w}'$. Since the identity $\mathbf{w} \approx \mathbf{w}'$ is arbitrary in Σ , it follows that $\mathbf{X}_{(25.4)} = \mathbf{X}_{(25.4)}\Sigma = \mathbf{B}_0 \vee \mathbf{W}$. □

26. $\mathcal{F}_6, \mathcal{F}_7, \mathcal{F}_8$

This section establishes the finite basis property of the following semigroups.

\mathcal{F}_6	1	2	3	4	5	6	\mathcal{F}_7	1	2	3	4	5	6	\mathcal{F}_8	1	2	3	4	5	6
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	1	1	1	2	2	2	1	1	1	1	2	2	2	1	1	1	1	2	2
3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
5	1	1	1	4	5	6	5	1	2	1	4	5	6	5	3	3	3	4	5	6
6	1	1	4	4	5	6	6	1	2	4	4	5	6	6	4	4	3	4	5	6

26.1. Finite basis property of \mathcal{F}_7

PROPOSITION 26.1. *The variety generated by \mathcal{F}_7 is defined by the identities*

$$x^2Hx \approx xHx, \tag{26.1a}$$

$$xHx^2 \approx xHx, \tag{26.1b}$$

$$xHyKxy \approx x^2HyKy, \tag{26.1c}$$

$$xHyzyx \approx xHy^2zx. \tag{26.1d}$$

In this section, a word \mathbf{w} with $\text{ini}(\mathbf{w}) = x_0 \dots x_m$ is said to be in α -canonical form if

$$\mathbf{w} = x_0^{e_0} \prod_{i=1}^m (x_i^{e_i} \mathbf{w}_i), \tag{26.2}$$

where all of the following are satisfied:

- (I) $\mathbf{w}_i \in \{\emptyset, x_0, \dots, x_{i-1}\}$;
- (II) $e_0, \dots, e_m \in \{1, 2\}$;
- (III) \mathbf{w} does not contain a factor of the form $x^2\mathbf{a}x$ where $\mathbf{a} \in \mathcal{X}^+$.

Note that (I) and (II) imply that the word \mathbf{w} in (26.2) does not contain a factor of the form \mathbf{ax}^2 where $\mathbf{a} \in \mathcal{X}^*$.

LEMMA 26.2. *Let \mathbf{w} be any word. Then there exists some word $\bar{\mathbf{w}}$ in α -canonical form such that the identities (26.1) imply the identity $\mathbf{w} \approx \bar{\mathbf{w}}$.*

Proof. It suffices to convert \mathbf{w} , using the identities (26.1), into a word in α -canonical form. Suppose $\text{ini}(\mathbf{w}) = x_0 \dots x_m$. Then \mathbf{w} can be written as a word of the form $\mathbf{w} = \prod_{i=0}^m (x_i \mathbf{w}_i)$ with $\mathbf{w}_i \in \{x_0, \dots, x_i\}^*$. Suppose $|\mathbf{w}_r| \geq 2$ for some $r \in \{0, \dots, m\}$. Then $\mathbf{w}_r = \mathbf{w}'_r x_k x_\ell$ for some $\mathbf{w}'_r \in \mathcal{X}^*$ with $\text{con}(\mathbf{w}'_r x_k x_\ell) \subseteq \{x_0, \dots, x_r\}$. Hence

$$\mathbf{w} = \mathbf{p}x_r \mathbf{w}_r \mathbf{q}$$

where $\mathbf{p} = \prod_{i=0}^{r-1} (x_i \mathbf{w}_i)$ and $\mathbf{q} = \prod_{i=r+1}^m (x_i \mathbf{w}_i)$. Since the letters x_k and x_ℓ in \mathbf{w}_r are non-first occurrences, the first x_k and x_ℓ of \mathbf{w} occur somewhere in the prefix $\mathbf{p}x_r$. If $k = \ell$, then

$$\mathbf{w} = \underbrace{\dots x_\ell \dots}_{\mathbf{p}x_r} \underbrace{\mathbf{w}'_r x_\ell^2}_{\mathbf{w}_r} \mathbf{q} \stackrel{(26.1b)}{\approx} \dots x_\ell \dots \mathbf{w}'_r x_\ell \mathbf{q}.$$

If $k < \ell$, then

$$\mathbf{w} = \underbrace{\dots x_k \dots x_\ell \dots}_{\mathbf{p}x_r} \underbrace{\mathbf{w}'_r x_k x_\ell}_{\mathbf{w}_r} \mathbf{q} \stackrel{(26.1c)}{\approx} \dots x_k^2 \dots x_\ell \dots \mathbf{w}'_r x_\ell \mathbf{q}.$$

If $\ell < k$, then

$$\mathbf{w} = \underbrace{\dots x_\ell \dots x_k \dots}_{\mathbf{p}x_r} \underbrace{\mathbf{w}'_r x_k x_\ell}_{\mathbf{w}_r} \mathbf{q} \stackrel{(26.1d)}{\approx} \dots x_\ell \dots x_k^2 \dots \mathbf{w}'_r x_\ell \mathbf{q}.$$

In each of the three cases just considered, an identity from (26.1) has been used to convert the factor $\mathbf{w}_r = \mathbf{w}'_r x_k x_\ell$ of \mathbf{w} into $\mathbf{w}'_r x_\ell$, that is, the length of \mathbf{w}_r is reduced by one. This argument can be repeated until the length of \mathbf{w}_r is reduced to one. Hence it is easily shown that \mathbf{w} can be converted by the identities (26.1) into the form (26.2) with (I) satisfied. The identities (26.1a) can then be used to convert this word into one that satisfies (II) and (III). \square

LEMMA 26.3. *Suppose that $\mathbf{w} \approx \mathbf{w}'$ is any identity satisfied by \mathcal{F}_7 . Then $\text{ini}(\mathbf{w}) = \text{ini}(\mathbf{w}')$, $\text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$, and $\text{t}(\mathbf{w}) = \text{t}(\mathbf{w}')$.*

Proof. This follows from Lemma 2.1 since the subsemigroups $\{1, 4, 5\}$, $\{1, 2, 5\}$, and $\{5, 6\}$ of \mathcal{F}_7 are isomorphic to L_2^1 , N_2^1 , and R_2 , respectively. \square

LEMMA 26.4. *Suppose that \mathcal{F}_7 satisfies an identity $\mathbf{w} \approx \mathbf{w}'$, where*

$$\mathbf{w} = x_0^{e_0} \prod_{i=1}^m (x_i^{e_i} \mathbf{w}_i) \quad \text{and} \quad \mathbf{w}' = x_0^{e'_0} \prod_{i=1}^m (x_i^{e'_i} \mathbf{w}'_i)$$

are words in α -canonical form. Then $e_i = e'_i$ for all i .

Proof. Seeking a contradiction, suppose $e_r \neq e'_r$ for some r , say $(e_r, e'_r) = (1, 2)$. Then the letter x_r is non-simple in \mathbf{w}' . By Lemma 26.3, the letter x_r is also non-simple in \mathbf{w} . Since \mathbf{w}

is in α -canonical form, the second x_r in \mathbf{w} coincides with one of $\mathbf{w}_{r+1}, \dots, \mathbf{w}_m$, say $x_r = \mathbf{w}_\ell$. Then $x_r \neq x_\ell$ by (I), and the word \mathbf{w} can be written as

$$\mathbf{w} = \mathbf{p}x_r \mathbf{w}_r \mathbf{q}x_\ell^{e_\ell} \underbrace{x_r}_{\mathbf{w}_\ell} \mathbf{u},$$

where $\mathbf{p} = x_0^{e_0} \prod_{i=1}^{r-1} (x_i^{e_i} \mathbf{w}_i)$, $\mathbf{q} = \prod_{i=r+1}^{\ell-1} (x_i^{e_i} \mathbf{w}_i)$, and $\mathbf{u} = \prod_{i=\ell+1}^m (x_i^{e_i} \mathbf{w}_i)$ with

(a) $x_r \notin \text{con}(\mathbf{p}\mathbf{w}_r\mathbf{q}x_\ell)$.

As for the word \mathbf{w}' , since $e'_r = 2$, it can be written as

$$\mathbf{w}' = \mathbf{p}'x_r^2 \mathbf{w}'_r \mathbf{q}'x_\ell^{e'_\ell} \mathbf{w}'_\ell \mathbf{u}'$$

where $\mathbf{p}' = x_0^{e'_0} \prod_{i=1}^{r-1} (x_i^{e'_i} \mathbf{w}'_i)$, $\mathbf{q}' = \prod_{i=r+1}^{\ell-1} (x_i^{e'_i} \mathbf{w}'_i)$, and $\mathbf{u}' = \prod_{i=\ell+1}^m (x_i^{e'_i} \mathbf{w}'_i)$ with $x_r \notin \text{con}(\mathbf{p}')$ and $x_\ell \notin \text{con}(\mathbf{p}'x_r\mathbf{w}'_r\mathbf{q}')$. In fact, it follows from (I) and (III) that

(b) $x_r \notin \text{con}(\mathbf{p}'\mathbf{w}'_r\mathbf{q}'x_\ell\mathbf{w}'_\ell\mathbf{u}')$.

In particular, $\mathbf{t}(\mathbf{w}') \neq x_r$. Since $\mathbf{t}(\mathbf{w}) = \mathbf{t}(\mathbf{w}')$ by Lemma 26.3, it follows that $\mathbf{u} \neq \emptyset$, whence

(c) $\mathbf{h}(\mathbf{u}) = x_{\ell+1} \notin \text{con}(\mathbf{p}x_r\mathbf{w}_r\mathbf{q}x_\ell)$

and $\text{ini}(\mathbf{w}) = \dots x_\ell x_{\ell+1} \dots$. Further, $\text{ini}(\mathbf{w}) = \text{ini}(\mathbf{w}')$ by Lemma 26.3, so that $\mathbf{u}' \neq \emptyset$ and

(d) $\mathbf{h}(\mathbf{u}') = x_{\ell+1} \notin \text{con}(\mathbf{p}'x_r\mathbf{w}'_r\mathbf{q}'x_\ell\mathbf{w}'_\ell)$.

Let $\varphi : \mathcal{X} \rightarrow \mathcal{F}_7$ denote the substitution

$$z \mapsto \begin{cases} 6 & \text{if } z = x_r, \\ 3 & \text{if } z = x_{\ell+1}, \\ 5 & \text{otherwise.} \end{cases}$$

Since $x_r, x_{\ell+1} \notin \text{con}(\mathbf{p}\mathbf{w}_r\mathbf{q}x_\ell)$ by (a) and (c), it follows that $\mathbf{p}\varphi \in \{\emptyset, 5\}$ and $(\mathbf{w}_r\mathbf{q}x_\ell^{e_\ell})\varphi = 5$. Thus

$$\mathbf{w}\varphi = \mathbf{p}\varphi \cdot x_r\varphi \cdot (\mathbf{w}_r\mathbf{q}x_\ell^{e_\ell})\varphi \cdot x_r\varphi \cdot \mathbf{u}\varphi = \mathbf{p}\varphi \cdot 6 \cdot 5 \cdot 6 \cdot 3 \dots = 4.$$

Similarly, since $x_r, x_{\ell+1} \notin \text{con}(\mathbf{p}'\mathbf{w}'_r\mathbf{q}'x_\ell\mathbf{w}'_\ell)$ by (b) and (d), it follows that $\mathbf{p}'\varphi \in \{\emptyset, 5\}$ and $(\mathbf{w}'_r\mathbf{q}'x_\ell^{e'_\ell}\mathbf{w}'_\ell)\varphi = 5$. Hence

$$\mathbf{w}'\varphi = \mathbf{p}'\varphi \cdot x_r^2\varphi \cdot (\mathbf{w}'_r\mathbf{q}'x_\ell^{e'_\ell}\mathbf{w}'_\ell)\varphi \cdot \mathbf{u}'\varphi = \mathbf{p}'\varphi \cdot 6 \cdot 5 \cdot 3 \dots = 1.$$

Consequently, the contradiction $\mathbf{w}\varphi \neq \mathbf{w}'\varphi$ is established. □

Proof of Proposition 26.1. It is routinely checked that \mathcal{F}_7 satisfies the identities (26.1). Hence it suffices to show that any identity $\mathbf{w} \approx \mathbf{w}'$ satisfied by \mathcal{F}_7 is implied by the identities (26.1). By Lemma 26.2, the words \mathbf{w} and \mathbf{w}' can be chosen to be in α -canonical form. By Lemmas 26.3 and 26.4,

$$\mathbf{w} = x_0^{e_0} \prod_{i=1}^m (x_i^{e_i} \mathbf{w}_i) \quad \text{and} \quad \mathbf{w}' = x_0^{e'_0} \prod_{i=1}^m (x_i^{e'_i} \mathbf{w}'_i)$$

with $\mathbf{w}_m = \mathbf{w}'_m$. Let $\ell < m$ be the least integer such that $\mathbf{w}_\ell \neq \mathbf{w}'_\ell$. Then it is convenient to write

(a) $\mathbf{p} = x_0^{e_0} \prod_{i=1}^{\ell-1} (x_i^{e_i} \mathbf{w}_i) = x_0^{e'_0} \prod_{i=1}^{\ell-1} (x_i^{e'_i} \mathbf{w}'_i)$.

Since $(\mathbf{w}_\ell, \mathbf{w}'_\ell) \neq (\emptyset, \emptyset)$, generality is not lost by assuming $\mathbf{w}_\ell \neq \emptyset$. Then (I) implies $\mathbf{w}_\ell = x_k$ for some $k \in \{0, \dots, \ell - 1\}$ and

(b) $\mathbf{w}'_\ell \in \{\emptyset, x_0, \dots, x_{\ell-1}\} \setminus \{x_k\}$,

whence

$$\mathbf{w} = \mathbf{p}x_\ell^{e_\ell} \underbrace{x_k}_{\mathbf{w}_\ell} x_{\ell+1}^{e_{\ell+1}} \mathbf{q} \quad \text{and} \quad \mathbf{w}' = \mathbf{p}x_\ell^{e_\ell} \mathbf{w}'_\ell x_{\ell+1}^{e_{\ell+1}} \mathbf{q}',$$

where $\mathbf{q} = \mathbf{w}_{\ell+1} \prod_{i=\ell+2}^m (x_i^{e_i} \mathbf{w}_i)$ and $\mathbf{q}' = \mathbf{w}'_{\ell+1} \prod_{i=\ell+2}^m (x_i^{e_i} \mathbf{w}'_i)$. Let $\varphi : \mathcal{X} \rightarrow \mathcal{F}_7$ denote the substitution

$$z \mapsto \begin{cases} 6 & \text{if } z = x_k, \\ 3 & \text{if } z = x_{\ell+1}, \\ 5 & \text{otherwise.} \end{cases}$$

Since the prefix \mathbf{p} contains the first x_k of \mathbf{w} and $x_{\ell+1} \notin \text{con}(\mathbf{p}x_\ell)$ by (I) and (a), it follows that $(\mathbf{p}x_\ell^{e_\ell})\varphi = 5$. Hence

$$\mathbf{w}\varphi = (\mathbf{p}x_\ell^{e_\ell})\varphi \cdot x_k\varphi \cdot x_{\ell+1}^{e_{\ell+1}}\varphi \cdot \mathbf{q}\varphi = 5 \cdot 6 \cdot 3 \cdot \mathbf{q}\varphi = 4.$$

On the other hand, $\mathbf{w}'_\ell\varphi \in \{\emptyset, 5\}$ by (b), so that

$$\mathbf{w}'\varphi = (\mathbf{p}x_\ell^{e_\ell})\varphi \cdot \mathbf{w}'_\ell\varphi \cdot x_{\ell+1}^{e_{\ell+1}}\varphi \cdot \mathbf{q}'\varphi = 5 \cdot \mathbf{w}'_\ell\varphi \cdot 3 \cdot \mathbf{q}'\varphi = 1.$$

The contradiction $\mathbf{w}\varphi \neq \mathbf{w}'\varphi$ implies that the integer ℓ does not exist. Consequently, $\mathbf{w}_i = \mathbf{w}'_i$ for all i , whence the identity $\mathbf{w} \approx \mathbf{w}'$ is trivial and so is implied by the identities (26.1). \square

26.2. Finite basis property of \mathcal{F}_6 and \mathcal{F}_8

The semigroup $\mathcal{F}_8^{\text{iso}}$ given by the following multiplication table is isomorphic to the semigroup \mathcal{F}_8 via the relabeling $1 \leftrightarrow 3$.

$\mathcal{F}_8^{\text{iso}}$	1	2	3	4	5	6
1	1	1	1	1	1	1
2	3	3	3	3	2	2
3	3	3	3	3	3	3
4	4	4	4	4	4	4
5	1	1	1	4	5	6
6	1	4	4	4	5	6

It turns out that arguments in the present subsection will become simpler if $\mathcal{F}_8^{\text{iso}}$ is considered instead of \mathcal{F}_8 .

PROPOSITION 26.5. *The variety generated by $S \in \{\mathcal{F}_6, \mathcal{F}_8^{\text{iso}}\}$ is defined by the identities*

$$x^2\mathbf{H}x \approx x\mathbf{H}x, \tag{26.3a}$$

$$x\mathbf{H}x^2 \approx x\mathbf{H}x, \tag{26.3b}$$

$$x\mathbf{H}y\mathbf{K}xy \approx x^2\mathbf{H}y\mathbf{K}y, \tag{26.3c}$$

$$x\mathbf{H}yzyx \approx x\mathbf{H}yzx, \tag{26.3d}$$

$$xy^2 \approx xy. \tag{26.3e}$$

In this section, a word \mathbf{w} with $\text{ini}(\mathbf{w}) = x_0 \dots x_m$ is said to be in β -canonical form if

$$\mathbf{w} = x_0^e \prod_{i=1}^m (x_i \mathbf{w}_i),$$

where all of the following are satisfied:

- (I) $\mathbf{w}_i \in \{\emptyset, x_0, \dots, x_{i-1}\}$;
- (II) $e \in \{1, 2\}$;
- (III) if $\mathbf{w}_i = x_0$ for some $i \geq 1$, then $e = 1$.

It is easily shown that a word in β -canonical form is also in α -canonical form.

LEMMA 26.6. *Let \mathbf{w} be any word. Then there exists some word $\bar{\mathbf{w}}$ in β -canonical form such that the identities (26.3) imply the identity $\mathbf{w} \approx \bar{\mathbf{w}}$.*

Proof. It suffices to convert \mathbf{w} , using the identities (26.3), into a word in β -canonical form. Let $\text{ini}(\mathbf{w}) = x_0 \dots x_m$. Note that the identities (26.3) imply the identities (26.1). Hence by Lemma 26.2, the identities (26.3) can be used to convert \mathbf{w} into a word in α -canonical form, that is, a word of the form $x_0^{e_0} \prod_{i=1}^m (x_i^{e_i} \mathbf{w}_i)$ that satisfies (I). The identity (26.3e) can be used to reduce the exponents e_1, \dots, e_m to 1. It is easily seen that (II) and (III) are satisfied by applying the identities (26.3a). □

LEMMA 26.7. *Let $S \in \{\mathcal{F}_6, \mathcal{F}_8^{\text{iso}}\}$. Suppose that $\mathbf{w} \approx \mathbf{w}'$ is any identity satisfied by S . Then $\text{ini}(\mathbf{w}) = \text{ini}(\mathbf{w}')$ and $\text{t}(\mathbf{w}) = \text{t}(\mathbf{w}')$.*

Proof. This follows from Lemma 2.1 since the subsemigroups $\{1, 4, 5\}$ and $\{5, 6\}$ of S are isomorphic to L_2^1 and R_2 , respectively. □

LEMMA 26.8. *Let $S \in \{\mathcal{F}_6, \mathcal{F}_8^{\text{iso}}\}$. Suppose that S satisfies an identity $\mathbf{w} \approx \mathbf{w}'$, where*

$$\mathbf{w} = x_0^e \prod_{i=1}^m (x_i \mathbf{w}_i) \quad \text{and} \quad \mathbf{w}' = x_0^{e'} \prod_{i=1}^m (x_i \mathbf{w}'_i)$$

are words in β -canonical form. Then $e = e'$.

Proof. Let $\mathbf{u} = \prod_{i=1}^m (x_i \mathbf{w}_i)$ and $\mathbf{u}' = \prod_{i=1}^m (x_i \mathbf{w}'_i)$. Seeking a contradiction, suppose $e \neq e'$, say $(e, e') = (1, 2)$. Then $x_0 \notin \text{con}(\mathbf{u}')$ by (I) and (III). There are two cases to consider.

Case 1: $x_0 \notin \text{con}(\mathbf{u})$. Let $\varphi_1 : \mathcal{X} \rightarrow S$ denote the substitution

$$z \mapsto \begin{cases} 2 & \text{if } z = x_0, \\ 5 & \text{otherwise.} \end{cases}$$

Then $\mathbf{w}\varphi_1 = 2 \cdot \mathbf{u}\varphi_1 = 2$ and $\mathbf{w}'\varphi_1 = x_0^2 \varphi_1 \cdot \mathbf{u}'\varphi_1 \in \{1, 3\}$, whence $\mathbf{w}\varphi_1 \neq \mathbf{w}'\varphi_1$ is impossible.

Case 2: $x_0 \in \text{con}(\mathbf{u})$. Then x_0 is non-simple in \mathbf{w} . Since the word \mathbf{w} is in β -canonical form, the second x_0 in \mathbf{w} coincides with one of $\mathbf{w}_1, \dots, \mathbf{w}_m$, say $x_0 = \mathbf{w}_\ell$ for some least possible $\ell \in \{1, \dots, m\}$. Then $x_0 \neq x_\ell$ by (I), and \mathbf{w} can be written as

$$\mathbf{w} = x_0 \mathbf{p} x_\ell \underbrace{x_0}_{\mathbf{w}_\ell} \mathbf{q},$$

where $\mathbf{p} = \prod_{i=1}^{\ell-1} (x_i \mathbf{w}_i)$ and $\mathbf{q} = \prod_{i=\ell+1}^m (x_i \mathbf{w}_i)$ with $x_\ell \notin \text{con}(x_0 \mathbf{p})$ and

- (a) $x_0 \notin \text{con}(\mathbf{p} x_\ell)$.

As for the word \mathbf{w}' , since $e' = 2$, it can be written as

$$\mathbf{w}' = x_0^2 \mathbf{p}' x_\ell \mathbf{w}'_\ell \mathbf{q}'$$

where $\mathbf{p}' = \prod_{i=1}^{\ell-1} (x_i \mathbf{w}'_i)$ and $\mathbf{q}' = \prod_{i=\ell+1}^m (x_i \mathbf{w}'_i)$ with $x_\ell \notin \text{con}(x_0 \mathbf{p}')$. By (I) and (III),

(b) $x_0 \notin \text{con}(\mathbf{p}' x_\ell \mathbf{w}'_\ell \mathbf{q}')$.

In particular, $\mathbf{t}(\mathbf{w}') \neq x_0$. Since $\mathbf{t}(\mathbf{w}) = \mathbf{t}(\mathbf{w}')$ by Lemma 26.7, it follows that $\mathbf{t}(\mathbf{w}) \neq x_0$. Therefore $\mathbf{q} \neq \emptyset$, whence

(c) $\mathbf{h}(\mathbf{q}) = x_{\ell+1} \notin \text{con}(x_0 \mathbf{p} x_\ell)$

and $\text{ini}(\mathbf{w}) = \dots x_\ell x_{\ell+1} \dots$. Further, $\text{ini}(\mathbf{w}) = \text{ini}(\mathbf{w}')$ by Lemma 26.7, and this implies $\mathbf{q}' \neq \emptyset$ and

(d) $\mathbf{h}(\mathbf{q}') = x_{\ell+1} \notin \text{con}(x_0 \mathbf{p}' x_\ell \mathbf{w}'_\ell)$.

Let $\varphi_2 : \mathcal{X} \rightarrow S$ denote the substitution

$$z \mapsto \begin{cases} 6 & \text{if } z = x_0, \\ 3 & \text{if } z = x_{\ell+1}, \\ 5 & \text{otherwise.} \end{cases}$$

Since $x_0, x_{\ell+1} \notin \text{con}(\mathbf{p} x_\ell)$ by (a) and (c), it follows that $(\mathbf{p} x_\ell) \varphi_2 = 5$. Hence

$$\mathbf{w} \varphi_2 = x_0 \varphi_2 \cdot (\mathbf{p} x_\ell) \varphi_2 \cdot x_0 \varphi_2 \cdot \mathbf{q} \varphi_2 = 6 \cdot 5 \cdot 6 \cdot 3 \dots = 4.$$

On the other hand, $x_0, x_{\ell+1} \notin \text{con}(\mathbf{p}' x_\ell \mathbf{w}'_\ell)$ by (b) and (d), so that $(\mathbf{p}' x_\ell \mathbf{w}'_\ell) \varphi_2 = 5$. Therefore

$$\mathbf{w}' \varphi_2 = x_0^2 \varphi_2 \cdot (\mathbf{p}' x_\ell \mathbf{w}'_\ell) \varphi_2 \cdot \mathbf{q}' \varphi_2 = 6 \cdot 5 \cdot 3 \dots = 1.$$

Consequently, the contradiction $\mathbf{w} \varphi_2 \neq \mathbf{w}' \varphi_2$ is deduced. □

Proof of Proposition 26.5. Let $S \in \{\mathcal{F}_6, \mathcal{F}_8^{\text{iso}}\}$. It is routinely checked that S satisfies the identities (26.3). Hence it suffices to show that any identity $\mathbf{w} \approx \mathbf{w}'$ satisfied by S is implied by the identities (26.3). By Lemma 26.6, the words \mathbf{w} and \mathbf{w}' can be assumed to be in β -canonical form. By Lemmas 26.7 and 26.8,

$$\mathbf{w} = x_0^e \prod_{i=1}^m (x_i \mathbf{w}_i) \quad \text{and} \quad \mathbf{w}' = x_0^e \prod_{i=1}^m (x_i \mathbf{w}'_i)$$

with $\mathbf{w}_m = \mathbf{w}'_m$. Let $\ell < m$ be the least integer such that $\mathbf{w}_\ell \neq \mathbf{w}'_\ell$. For convenience, write

(a) $\mathbf{p} = x_0^e \prod_{i=1}^{\ell-1} (x_i \mathbf{w}_i) = x_0^e \prod_{i=1}^{\ell-1} (x_i \mathbf{w}'_i)$.

Since $(\mathbf{w}_\ell, \mathbf{w}'_\ell) \neq (\emptyset, \emptyset)$, generality is not lost by assuming $\mathbf{w}_\ell \neq \emptyset$. Then (I) implies $\mathbf{w}_\ell = x_k$ for some $k \in \{0, \dots, \ell - 1\}$ and

(b) $\mathbf{w}'_\ell \in \{\emptyset, x_0, \dots, x_{\ell-1}\} \setminus \{x_k\}$,

whence

$$\mathbf{w} = \mathbf{p} x_\ell \underbrace{x_k}_{\mathbf{w}_\ell} x_{\ell+1} \mathbf{q} \quad \text{and} \quad \mathbf{w}' = \mathbf{p} x_\ell \mathbf{w}'_\ell x_{\ell+1} \mathbf{q}'$$

where $\mathbf{q} = \mathbf{w}_{\ell+1} \prod_{i=\ell+2}^m (x_i \mathbf{w}_i)$ and $\mathbf{q}' = \mathbf{w}'_{\ell+1} \prod_{i=\ell+2}^m (x_i \mathbf{w}'_i)$. Let $\varphi : \mathcal{X} \rightarrow S$ denote the substitution

$$z \mapsto \begin{cases} 6 & \text{if } z = x_k, \\ 3 & \text{if } z = x_{\ell+1}, \\ 5 & \text{otherwise.} \end{cases}$$

Since \mathbf{p} contains the first x_k of \mathbf{w} and $x_{\ell+1} \notin \text{con}(\mathbf{p} x_\ell)$ by (I) and (a), it follows that $(\mathbf{p} x_\ell) \varphi = 5$. Hence

$$\mathbf{w} \varphi = (\mathbf{p} x_\ell) \varphi \cdot x_k \varphi \cdot x_{\ell+1} \varphi \cdot \mathbf{q} \varphi = 5 \cdot 6 \cdot 3 \cdot \mathbf{q} \varphi = 4.$$

On the other hand, $\mathbf{w}'_\ell \varphi \in \{\emptyset, 5\}$ by (b), so that

$$\mathbf{w}' \varphi = (\mathbf{p} x_\ell) \varphi \cdot \mathbf{w}'_\ell \varphi \cdot x_{\ell+1} \varphi \cdot \mathbf{q}' \varphi = 5 \cdot \mathbf{w}'_\ell \varphi \cdot 3 \cdot \mathbf{q}' \varphi = 1.$$

The contradiction $\mathbf{w} \varphi \neq \mathbf{w}' \varphi$ implies that the integer ℓ does not exist. Consequently, $\mathbf{w}_i = \mathbf{w}'_i$ for all i , whence the identity $\mathbf{w} \approx \mathbf{w}'$ is trivial and so is implied by the identities (26.3). □

27. $\mathcal{F}_9, \mathcal{F}_{10}, \mathcal{G}_1, \mathcal{G}_2$

This section establishes the finite basis property of the following semigroups.

\mathcal{F}_9	1 2 3 4 5 6	\mathcal{F}_{10}	1 2 3 4 5 6	\mathcal{G}_1	1 2 3 4 5 6	\mathcal{G}_2	1 2 3 4 5 6
1	1 1 1 1 1 1	1	1 1 1 1 1 1	1	1 1 1 1 1 1	1	1 1 1 1 1 1
2	1 1 1 1 2 3	2	1 1 1 1 2 3	2	1 1 1 1 2 3	2	1 1 1 2 2 3
3	3 3 3 3 3 3	3	3 3 3 3 3 3	3	3 3 3 3 3 3	3	3 3 3 3 3 3
4	1 2 3 4 1 1	4	1 2 3 4 2 3	4	1 2 3 4 3 3	4	1 2 3 4 5 6
5	1 1 1 1 5 6	5	1 1 1 1 5 6	5	3 3 3 3 5 6	5	1 2 3 4 5 6
6	6 6 6 6 6 6	6	6 6 6 6 6 6	6	6 6 6 6 6 6	6	6 6 6 6 6 6

27.1. Finite basis property of \mathcal{F}_{10}

PROPOSITION 27.1. The variety generated by \mathcal{F}_{10} is defined by the identities

$$xHx^2 \approx xHx, \tag{27.1a}$$

$$xHyKxy \approx xHyKyx, \tag{27.1b}$$

$$xHyKyx \approx xHyKx, \tag{27.1c}$$

$$xHyKxTx \approx xHyKxTx. \tag{27.1d}$$

In this section, a word \mathbf{w} with $\text{ini}(\mathbf{w}) = x_1 \dots x_m$ is said to be in α -canonical form if

$$\mathbf{w} = \prod_{i=1}^m (x_i \mathbf{w}_i),$$

where:

- (I) $\mathbf{w}_i \in \{\emptyset, x_1, \dots, x_i\}$;
- (II) \mathbf{w} does not contain any factor of the form $x_j \mathbf{a} x_k \mathbf{b} x_j \mathbf{c} x_k$ where $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathcal{X}^*$ and $j < k$.

LEMMA 27.2. Let \mathbf{w} be any word. Then there exists some word $\bar{\mathbf{w}}$ in α -canonical form such that the identities (27.1) imply the identity $\mathbf{w} \approx \bar{\mathbf{w}}$.

Proof. It suffices to convert \mathbf{w} , using the identities (27.1), into a word in α -canonical form. Let $\text{ini}(\mathbf{w}) = x_1 \dots x_m$, so that $\mathbf{w} = \prod_{i=1}^m (x_i \mathbf{w}_i)$ for some $\mathbf{w}_i \in \{x_1, \dots, x_i\}^*$. Suppose $\mathbf{w}_i \neq \emptyset$. Then there exists a least integer ℓ such that $x_\ell \in \text{con}(\mathbf{w}_i)$, whence $\text{con}(\mathbf{w}_i) \subseteq \{x_\ell, \dots, x_i\}$. Since the letters of \mathbf{w}_i are non-first occurrences in \mathbf{w} , the identities (27.1b) can be used to arrange them, within \mathbf{w}_i , in any order. In particular, the identities (27.1b) can be used to move all occurrences of x_ℓ in \mathbf{w}_i to the right, so that

$$\mathbf{w} = \dots x_\ell \mathbf{w}_\ell \dots x_i \mathbf{w}_i \dots \stackrel{(27.1b)}{\approx} \dots x_\ell \mathbf{w}_\ell \dots x_i \mathbf{w}'_i x_\ell^e \dots,$$

where $e = \text{occ}(x_\ell, \mathbf{w}_i)$ and \mathbf{w}'_i is obtained from \mathbf{w}_i by removing all occurrences of x_ℓ . Note that $\text{con}(\mathbf{w}'_i) \subseteq \{x_{\ell+1}, \dots, x_i\}$ and any letter in \mathbf{w}'_i is a non-first occurrence in \mathbf{w} that is sandwiched between two occurrences of x_ℓ . The identities (27.1c) can be used to remove the letters of \mathbf{w}'_i one by one starting from the right, and the identities (27.1a) can be used to reduce the exponent e to 1:

$$\dots x_\ell \mathbf{w}_\ell \dots x_i \mathbf{w}'_i x_\ell^e \dots \stackrel{(27.1c)}{\approx} \dots x_\ell \mathbf{w}_\ell \dots x_i x_\ell^e \dots \stackrel{(27.1a)}{\approx} \dots x_\ell \mathbf{w}_\ell \dots x_i x_\ell \dots$$

Therefore the factor \mathbf{w}_i is converted into x_ℓ , whence (I) is satisfied. Since

$$\mathbf{w} = \dots x_j \mathbf{a} x_k \mathbf{b} x_j \mathbf{c} x_k \dots \stackrel{(27.1d)}{\approx} \dots x_j \mathbf{a} x_k \mathbf{b} x_j \mathbf{c} x_j \dots,$$

the identities (27.1d) can be used to convert \mathbf{w} into a word that satisfies (II). □

Proof of Proposition 27.1. It is routinely checked that \mathcal{F}_{10} satisfies the identities (27.1). Hence it suffices to show that any identity $\mathbf{w} \approx \mathbf{w}'$ satisfied by \mathcal{F}_{10} is implied by the identities (27.1). By Lemma 27.2, the words \mathbf{w} and \mathbf{w}' can be assumed to be in α -canonical form. Since the subsemigroup $\{1, 3, 4\}$ of \mathcal{F}_{10} is isomorphic to L_2^1 , it follows from Lemma 2.1(iii) that

$$\mathbf{w} = \prod_{i=1}^m (x_i \mathbf{w}_i) \quad \text{and} \quad \mathbf{w}' = \prod_{i=1}^m (x_i \mathbf{w}'_i).$$

Let ℓ be the least integer such that $\mathbf{w}_\ell \neq \mathbf{w}'_\ell$. Multiplying both sides of the identity $\mathbf{w} \approx \mathbf{w}'$ on the right by x_{m+1}^2 if necessary, generality is not lost by further assuming $\ell < m$. Hence

$$\mathbf{w} = \mathbf{p} x_\ell \mathbf{w}_\ell x_{\ell+1} \mathbf{q} \quad \text{and} \quad \mathbf{w}' = \mathbf{p} x_\ell \mathbf{w}'_\ell x_{\ell+1} \mathbf{q}'$$

where $\mathbf{p} = \prod_{i=1}^{\ell-1} (x_i \mathbf{w}_i)$, $\mathbf{q} = \mathbf{w}_{\ell+1} \prod_{i=\ell+2}^m (x_i \mathbf{w}_i)$, and $\mathbf{q}' = \mathbf{w}'_{\ell+1} \prod_{i=\ell+2}^m (x_i \mathbf{w}'_i)$. By symmetry, there are four cases to consider.

Case 1: $(\mathbf{w}_\ell, \mathbf{w}'_\ell) = (x_\ell, \emptyset)$. Then

$$\mathbf{w} = \mathbf{p} x_\ell^2 x_{\ell+1} \mathbf{q} \quad \text{and} \quad \mathbf{w}' = \mathbf{p} x_\ell x_{\ell+1} \mathbf{q}'$$

with $x_\ell, x_{\ell+1} \notin \text{con}(\mathbf{p})$. Let $\varphi_1 : \mathcal{X} \rightarrow \mathcal{F}_{10}$ denote the substitution

$$z \mapsto \begin{cases} 4 & \text{if } z \in \text{con}(\mathbf{p}), \\ 2 & \text{if } z = x_\ell, \\ 6 & \text{otherwise.} \end{cases}$$

Then $\mathbf{w}\varphi_1 = \mathbf{p}\varphi_1 \cdot 2^2 \cdot 6 \cdot \mathbf{q}\varphi_1 = 1$ and $\mathbf{w}'\varphi_1 = \mathbf{p}\varphi_1 \cdot 2 \cdot 6 \cdot \mathbf{q}'\varphi_1 = 3$, whence the contradiction $\mathbf{w}\varphi_1 \neq \mathbf{w}'\varphi_1$ is deduced.

Case 2: $(\mathbf{w}_\ell, \mathbf{w}'_\ell) = (x_i, \emptyset)$ for some $i < \ell$. Then

$$\mathbf{w} = \underbrace{\mathbf{a} x_i \mathbf{b}}_{\mathbf{p}} x_\ell x_i x_{\ell+1} \mathbf{q} \quad \text{and} \quad \mathbf{w}' = \underbrace{\mathbf{a} x_i \mathbf{b}}_{\mathbf{p}} x_\ell x_{\ell+1} \mathbf{q}' \quad \text{for some } \mathbf{a}, \mathbf{b} \in \mathcal{X}^*$$

such that $x_i \notin \text{con}(\mathbf{a})$, $x_\ell \notin \text{con}(\mathbf{a} x_i \mathbf{b})$, and $x_{\ell+1} \notin \text{con}(\mathbf{a} x_i \mathbf{b} x_\ell)$. Let $\varphi_2 : \mathcal{X} \rightarrow \mathcal{F}_{10}$ denote the substitution

$$z \mapsto \begin{cases} 4 & \text{if } z \in \text{con}(\mathbf{a} x_i \mathbf{b}), \\ 2 & \text{if } z = x_\ell, \\ 6 & \text{otherwise.} \end{cases}$$

Then $\mathbf{w}\varphi_2 = 4 \cdot 2 \cdot 4 \cdot 6 \cdot \mathbf{q}\varphi_2 = 1$ and $\mathbf{w}'\varphi_2 = 4 \cdot 2 \cdot 6 \cdot \mathbf{q}'\varphi_2 = 3$, whence the contradiction $\mathbf{w}\varphi_2 \neq \mathbf{w}'\varphi_2$ is deduced.

Case 3: $(\mathbf{w}_\ell, \mathbf{w}'_\ell) = (x_i, x_\ell)$ for some $i < \ell$. Then

$$\mathbf{w} = \underbrace{\mathbf{c}x_i\mathbf{d}}_{\mathbf{p}} x_\ell x_i x_{\ell+1} \mathbf{q} \quad \text{and} \quad \mathbf{w}' = \underbrace{\mathbf{c}x_i\mathbf{d}}_{\mathbf{p}} x_\ell^2 x_{\ell+1} \mathbf{q}' \quad \text{for some } \mathbf{c}, \mathbf{d} \in \mathcal{X}^*$$

such that $x_i \notin \text{con}(\mathbf{c})$, $x_\ell \notin \text{con}(\mathbf{c}x_i\mathbf{d})$, and $x_{\ell+1} \notin \text{con}(\mathbf{c}x_i\mathbf{d}x_\ell)$. Let $\varphi_3 : \mathcal{X} \rightarrow \mathcal{F}_{10}$ denote the substitution

$$z \mapsto \begin{cases} 4 & \text{if } z \in \text{con}(\mathbf{c}x_i\mathbf{d}), \\ 5 & \text{if } z = x_\ell, \\ 6 & \text{otherwise.} \end{cases}$$

Then $\mathbf{w}\varphi_3 = 4 \cdot 5 \cdot 4 \cdot 6 \cdot \mathbf{q}\varphi_3 = 1$ and $\mathbf{w}'\varphi_3 = 4 \cdot 5^2 \cdot 6 \cdot \mathbf{q}'\varphi_3 = 3$, whence the contradiction $\mathbf{w}\varphi_3 \neq \mathbf{w}'\varphi_3$ is deduced.

Case 4: $(\mathbf{w}_\ell, \mathbf{w}'_\ell) = (x_i, x_j)$ for some i and j with $i < j < \ell$. Then

$$\mathbf{w} = \underbrace{\mathbf{e}x_i\mathbf{f}x_j\mathbf{g}}_{\mathbf{p}} x_\ell x_i x_{\ell+1} \mathbf{q} \quad \text{and} \quad \mathbf{w}' = \underbrace{\mathbf{e}x_i\mathbf{f}x_j\mathbf{g}}_{\mathbf{p}} x_\ell x_j x_{\ell+1} \mathbf{q}' \quad \text{for some } \mathbf{e}, \mathbf{f}, \mathbf{g} \in \mathcal{X}^*$$

such that $x_i \notin \text{con}(\mathbf{e})$, $x_j \notin \text{con}(\mathbf{e}x_i\mathbf{f})$, $x_\ell \notin \text{con}(\mathbf{e}x_i\mathbf{f}x_j\mathbf{g})$, and $x_{\ell+1} \notin \text{con}(\mathbf{e}x_i\mathbf{f}x_j\mathbf{g}x_\ell)$. If the words $\mathbf{e}x_i\mathbf{f}$ and $\mathbf{g}x_\ell$ share a common letter, then (II) is violated by the word \mathbf{w}' . It follows that $\text{con}(\mathbf{e}x_i\mathbf{f}) \cap \text{con}(x_j\mathbf{g}x_\ell) = \emptyset$. Let $\varphi_4 : \mathcal{X} \rightarrow \mathcal{F}_{10}$ denote the substitution

$$z \mapsto \begin{cases} 4 & \text{if } z \in \text{con}(\mathbf{e}x_i\mathbf{f}), \\ 5 & \text{if } z \in \text{con}(x_j\mathbf{g}x_\ell), \\ 6 & \text{otherwise.} \end{cases}$$

Then the contradiction $\mathbf{w}\varphi_4 \neq \mathbf{w}'\varphi_4$ is deduced because

$$\mathbf{w}\varphi_4 = (\mathbf{e}x_i\mathbf{f})\varphi_4 \cdot (x_j\mathbf{g}x_\ell)\varphi_4 \cdot x_i\varphi_4 \cdot x_{\ell+1}\varphi_4 \cdot \mathbf{q}\varphi_4 = 4 \cdot 5 \cdot 4 \cdot 6 \cdot \mathbf{q}\varphi_4 = 1$$

and

$$\mathbf{w}'\varphi_4 = (\mathbf{e}x_i\mathbf{f})\varphi_4 \cdot (x_j\mathbf{g}x_\ell x_j)\varphi_4 \cdot x_{\ell+1}\varphi_4 \cdot \mathbf{q}'\varphi_4 = 4 \cdot 5 \cdot 6 \cdot \mathbf{q}'\varphi_4 = 3.$$

Since none of the four cases just considered is possible, the integer ℓ does not exist. Consequently, $\mathbf{w}_i = \mathbf{w}'_i$ for all i , whence the identity $\mathbf{w} \approx \mathbf{w}'$ is trivial and so is implied by the identities (27.1). □

27.2. Finite basis property of \mathcal{F}_9 and \mathcal{G}_1

PROPOSITION 27.3. *The variety generated by $S \in \{\mathcal{F}_9, \mathcal{G}_1\}$ is defined by the identities*

$$x\mathbf{H}x^2 \approx x\mathbf{H}x, \tag{27.2a}$$

$$x\mathbf{H}y\mathbf{K}xy \approx x\mathbf{H}y\mathbf{K}yx, \tag{27.2b}$$

$$x\mathbf{H}y\mathbf{K}yx \approx x\mathbf{H}y\mathbf{K}x, \tag{27.2c}$$

$$x\mathbf{H}y\mathbf{K}x\mathbf{T}y \approx x\mathbf{H}y\mathbf{K}x\mathbf{T}x, \tag{27.2d}$$

$$x\mathbf{H}xy\mathbf{K}y \approx x\mathbf{H}xy\mathbf{K}x. \tag{27.2e}$$

In this section, a word \mathbf{w} with $\text{ini}(\mathbf{w}) = x_1 \dots x_m$ is said to be in β -canonical form if

$$\mathbf{w} = \prod_{i=1}^m (x_i \mathbf{w}_i),$$

where all of the following are satisfied:

- (I) $\mathbf{w}_i \in \{\emptyset, x_1, \dots, x_i\}$;
 - (II) \mathbf{w} does not contain any factor of the form $x_j \mathbf{a} x_k \mathbf{b} x_j \mathbf{c} x_k$ where $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathcal{X}^*$ and $j < k$;
 - (III) \mathbf{w} does not contain any factor of the form $x_j \mathbf{d} x_j x_k \mathbf{e} x_k$ where $\mathbf{d}, \mathbf{e} \in \mathcal{X}^*$ and $j < k$.
- Observe that a word in β -canonical form is also in α -canonical form.

LEMMA 27.4. *Let \mathbf{w} be any word. Then there exists some word $\bar{\mathbf{w}}$ in β -canonical form such that the identities (27.2) imply the identity $\mathbf{w} \approx \bar{\mathbf{w}}$.*

Proof. It suffices to convert \mathbf{w} , using the identities (27.2), into a word in β -canonical form. Let $\text{ini}(\mathbf{w}) = x_1 \dots x_m$. Then \mathbf{w} can be written as $\mathbf{w} = \prod_{i=1}^m (x_i \mathbf{w}_i)$ where $\mathbf{w}_i \in \{x_1, \dots, x_i\}^*$. Note that the identities (27.2) contain the identities (27.1). Hence by Lemma 27.2, the identities (27.2) can be used to convert \mathbf{w} into a word in α -canonical form, that is, a word that satisfies (I) and (II). Since

$$\mathbf{w} = \dots x_j \mathbf{d} x_j x_k \mathbf{e}_1 x_k \mathbf{e}_2 x_k \dots \mathbf{e}_r x_k \dots \stackrel{(27.2e)}{\approx} \dots x_j \mathbf{d} x_j x_k \mathbf{e}_1 x_j \mathbf{e}_2 x_j \dots \mathbf{e}_r x_j \dots,$$

the identities (27.2e) can then be used to convert \mathbf{w} into a word that satisfies (III). □

Proof of Proposition 27.3. Let $S \in \{\mathcal{F}_9, \mathcal{G}_1\}$. It is routinely checked that S satisfies the identities (27.2). Hence it suffices to show that any identity $\mathbf{w} \approx \mathbf{w}'$ satisfied by S is implied by the identities (27.2). By Lemma 27.4, the words \mathbf{w} and \mathbf{w}' can be assumed to be in β -canonical form. Since the subsemigroup $\{1, 3, 4\}$ of S is isomorphic to L_2^1 , it follows from Lemma 2.1(iii) that

$$\mathbf{w} = \prod_{i=1}^m (x_i \mathbf{w}_i) \quad \text{and} \quad \mathbf{w}' = \prod_{i=1}^m (x_i \mathbf{w}'_i).$$

Let ℓ be the least integer such that $\mathbf{w}_\ell \neq \mathbf{w}'_\ell$. Multiplying both sides of the identity $\mathbf{w} \approx \mathbf{w}'$ on the right by x_{m+1}^2 if necessary, generality is not lost by further assuming $\ell < m$. Hence

$$\mathbf{w} = \mathbf{p} x_\ell \mathbf{w}_\ell x_{\ell+1} \mathbf{q} \quad \text{and} \quad \mathbf{w}' = \mathbf{p} x_\ell \mathbf{w}'_\ell x_{\ell+1} \mathbf{q}',$$

where $\mathbf{p} = \prod_{i=1}^{\ell-1} (x_i \mathbf{w}_i)$, $\mathbf{q} = \mathbf{w}_{\ell+1} \prod_{i=\ell+2}^m (x_i \mathbf{w}_i)$, and $\mathbf{q}' = \mathbf{w}'_{\ell+1} \prod_{i=\ell+2}^m (x_i \mathbf{w}'_i)$. By symmetry, there are three cases to consider.

Case 1: $(\mathbf{w}_\ell, \mathbf{w}'_\ell) = (x_i, \emptyset)$ for some $i \leq \ell$. Then a contradiction is obtained by the same arguments as in Cases 1 and 2 of the proof of Proposition 27.1.

Case 2: $(\mathbf{w}_\ell, \mathbf{w}'_\ell) = (x_i, x_\ell)$ for some $i < \ell$. Then

$$\mathbf{w} = \underbrace{\mathbf{c} x_i \mathbf{d}}_{\mathbf{p}} x_\ell x_i x_{\ell+1} \mathbf{q} \quad \text{and} \quad \mathbf{w}' = \underbrace{\mathbf{c} x_i \mathbf{d}}_{\mathbf{p}} x_\ell^2 x_{\ell+1} \mathbf{q}' \quad \text{for some } \mathbf{c}, \mathbf{d} \in \mathcal{X}^*$$

such that $x_i \notin \text{con}(\mathbf{c})$, $x_\ell \notin \text{con}(\mathbf{c} x_i \mathbf{d})$, and $x_{\ell+1} \notin \text{con}(\mathbf{c} x_i \mathbf{d} x_\ell)$. There are two subcases.

2.1. $\mathbf{d} \neq \emptyset$. Then $\mathbf{d} = \mathbf{u} t$ for some $\mathbf{u} \in \mathcal{X}^*$ with $t = t(\mathbf{d})$, whence

$$\mathbf{w} = \underbrace{\mathbf{c} x_i \mathbf{u} t}_{\mathbf{p}} x_\ell x_i x_{\ell+1} \mathbf{q} \quad \text{and} \quad \mathbf{w}' = \underbrace{\mathbf{c} x_i \mathbf{u} t}_{\mathbf{p}} x_\ell^2 x_{\ell+1} \mathbf{q}'.$$

If $t \in \text{con}(\mathbf{c} x_i \mathbf{u})$, then (III) is violated by the word \mathbf{w}' . Hence $t \notin \text{con}(\mathbf{c} x_i \mathbf{u})$. Let $\varphi_1 : \mathcal{X} \rightarrow S$ denote the substitution

$$z \mapsto \begin{cases} 4 & \text{if } z \in \text{con}(\mathbf{c} x_i \mathbf{u}), \\ 2 & \text{if } z = t, \\ 5 & \text{if } z = x_\ell, \\ 6 & \text{otherwise.} \end{cases}$$

Then $\mathbf{w}\varphi_1 = 4 \cdot 2 \cdot 5 \cdot 4 \cdot 6 \cdot \mathbf{q}\varphi_1 = 1$ and $\mathbf{w}'\varphi_1 = 4 \cdot 2 \cdot 5^2 \cdot 6 \cdot \mathbf{q}'\varphi_1 = 3$, whence the contradiction $\mathbf{w}\varphi_1 \neq \mathbf{w}'\varphi_1$ is deduced.

2.2. $\mathbf{d} = \emptyset$. Then

$$\mathbf{w} = \underbrace{\mathbf{c}x_i}_{\mathbf{p}} x_\ell x_i x_{\ell+1} \mathbf{q} \quad \text{and} \quad \mathbf{w}' = \underbrace{\mathbf{c}x_i}_{\mathbf{p}} x_\ell^2 x_{\ell+1} \mathbf{q}'.$$

Let $\varphi_2 : \mathcal{X} \rightarrow S$ denote the substitution

$$z \mapsto \begin{cases} 4 & \text{if } z \in \text{con}(\mathbf{c}), \\ 2 & \text{if } z = x_i, \\ 5 & \text{if } z = x_\ell, \\ 6 & \text{otherwise.} \end{cases}$$

Then $\mathbf{w}\varphi_2 = \mathbf{c}\varphi_2 \cdot 2 \cdot 5 \cdot 2 \cdot 6 \cdot \mathbf{q}\varphi_2 = 1$ and $\mathbf{w}'\varphi_2 = \mathbf{c}\varphi_2 \cdot 2 \cdot 5^2 \cdot 6 \cdot \mathbf{q}'\varphi_2 = 3$, whence the contradiction $\mathbf{w}\varphi_2 \neq \mathbf{w}'\varphi_2$ is deduced.

Case 3: $(\mathbf{w}_\ell, \mathbf{w}'_\ell) = (x_i, x_j)$ for some i and j with $i < j < \ell$. Then

$$\mathbf{w} = \underbrace{\mathbf{e}x_i \mathbf{f}x_j \mathbf{g}}_{\mathbf{p}} x_\ell x_i x_{\ell+1} \mathbf{q} \quad \text{and} \quad \mathbf{w}' = \underbrace{\mathbf{e}x_i \mathbf{f}x_j \mathbf{g}}_{\mathbf{p}} x_\ell x_j x_{\ell+1} \mathbf{q}' \quad \text{for some } \mathbf{e}, \mathbf{f}, \mathbf{g} \in \mathcal{X}^*$$

such that $x_i \notin \text{con}(\mathbf{e})$, $x_j \notin \text{con}(\mathbf{e}x_i \mathbf{f})$, $x_\ell \notin \text{con}(\mathbf{e}x_i \mathbf{f}x_j \mathbf{g})$, and $x_{\ell+1} \notin \text{con}(\mathbf{e}x_i \mathbf{f}x_j \mathbf{g}x_\ell)$. If the words $\mathbf{e}x_i \mathbf{f}$ and $\mathbf{g}x_\ell$ share a common letter, then (II) is violated by the word \mathbf{w}' . It follows that $\text{con}(\mathbf{e}x_i \mathbf{f}) \cap \text{con}(x_j \mathbf{g}x_\ell) = \emptyset$. There are two subcases.

3.1. $\mathbf{f} \neq \emptyset$. Then $\mathbf{f} = \mathbf{v}t$ for some $\mathbf{v} \in \mathcal{X}^*$ with $t = t(\mathbf{f})$, whence

$$\mathbf{w} = \underbrace{\mathbf{e}x_i \mathbf{v}t x_j \mathbf{g}}_{\mathbf{p}} x_\ell x_i x_{\ell+1} \mathbf{q} \quad \text{and} \quad \mathbf{w}' = \underbrace{\mathbf{e}x_i \mathbf{v}t x_j \mathbf{g}}_{\mathbf{p}} x_\ell x_j x_{\ell+1} \mathbf{q}'.$$

If $t \in \text{con}(\mathbf{e}x_i \mathbf{v})$, then (III) is violated by \mathbf{w}' . Hence the observation $\text{con}(\mathbf{e}x_i \mathbf{f}) \cap \text{con}(x_j \mathbf{g}x_\ell) = \emptyset$ from above implies $t \notin \text{con}(\mathbf{e}x_i \mathbf{v}x_j \mathbf{g}x_\ell)$. Let $\varphi_3 : \mathcal{X} \rightarrow S$ denote the substitution

$$z \mapsto \begin{cases} 4 & \text{if } z \in \text{con}(\mathbf{e}x_i \mathbf{v}), \\ 2 & \text{if } z = t, \\ 5 & \text{if } z \in \text{con}(x_j \mathbf{g}x_\ell), \\ 6 & \text{otherwise.} \end{cases}$$

Then the contradiction $\mathbf{w}\varphi_3 \neq \mathbf{w}'\varphi_3$ is deduced because

$$\mathbf{w}\varphi_3 = (\mathbf{e}x_i \mathbf{v})\varphi_3 \cdot t\varphi_3 \cdot (x_j \mathbf{g}x_\ell)\varphi_3 \cdot x_i\varphi_3 \cdot x_{\ell+1}\varphi_3 \cdot \mathbf{q}\varphi_3 = 4 \cdot 2 \cdot 5 \cdot 4 \cdot 6 \cdot \mathbf{q}\varphi_3 = 1$$

and

$$\mathbf{w}'\varphi_3 = (\mathbf{e}x_i \mathbf{v})\varphi_3 \cdot t\varphi_3 \cdot (x_j \mathbf{g}x_\ell x_j)\varphi_3 \cdot x_{\ell+1}\varphi_3 \cdot \mathbf{q}'\varphi_3 = 4 \cdot 2 \cdot 5 \cdot 6 \cdot \mathbf{q}'\varphi_3 = 3.$$

3.2. $\mathbf{f} = \emptyset$. Then

$$\mathbf{w} = \underbrace{\mathbf{e}x_i x_j \mathbf{g}}_{\mathbf{p}} x_\ell x_i x_{\ell+1} \mathbf{q} \quad \text{and} \quad \mathbf{w}' = \underbrace{\mathbf{e}x_i x_j \mathbf{g}}_{\mathbf{p}} x_\ell x_j x_{\ell+1} \mathbf{q}'.$$

Let $\varphi_4 : \mathcal{X} \rightarrow S$ denote the substitution

$$z \mapsto \begin{cases} 4 & \text{if } z \in \text{con}(\mathbf{e}), \\ 2 & \text{if } z = x_i, \\ 5 & \text{if } z \in \text{con}(x_j \mathbf{g}x_\ell), \\ 6 & \text{otherwise.} \end{cases}$$

Then $\mathbf{w}\varphi_4 = \mathbf{e}\varphi_4 \cdot 2 \cdot 5 \cdot 2 \cdot 6 \cdot \mathbf{q}\varphi_4 = 1$ and $\mathbf{w}'\varphi_4 = \mathbf{e}\varphi_4 \cdot 2 \cdot 5 \cdot 6 \cdot \mathbf{q}'\varphi_4 = 3$, whence the contradiction $\mathbf{w}\varphi_4 \neq \mathbf{w}'\varphi_4$ is deduced.

Since none of the three cases just considered is possible, the integer ℓ does not exist. Consequently, $\mathbf{w}_i = \mathbf{w}'_i$ for all i , whence the identity $\mathbf{w} \approx \mathbf{w}'$ is trivial and so is implied by the identities (27.2). □

27.3. Finite basis property of \mathcal{G}_2

PROPOSITION 27.5. The variety generated by \mathcal{G}_2 is defined by the identities

$$xHx^2 \approx xHx, \tag{27.3a}$$

$$xHxKxt \approx xHxKt, \tag{27.3b}$$

$$xHyKxyt \approx xHyKyt. \tag{27.3c}$$

LEMMA 27.6. Suppose that $\mathbf{w} \approx \mathbf{w}'$ is any identity satisfied by \mathcal{G}_2 . Then:

- (i) $\text{ini}(\mathbf{w}) = \text{ini}(\mathbf{w}')$;
- (ii) $\text{con}(\mathbf{w}) = \text{con}(\mathbf{w}')$ and $\text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$;
- (iii) $\text{t}(\mathbf{w}) = \text{t}(\mathbf{w}')$;
- (iv) $\text{t}(\mathbf{w}) \in \text{sim}(\mathbf{w})$ if and only if $\text{t}(\mathbf{w}') \in \text{sim}(\mathbf{w}')$.

Further, if $\text{occ}(x, \mathbf{w}), \text{occ}(x, \mathbf{w}') \leq 2$ for all $x \in \mathcal{X}$, then

- (v) $\text{occ}(y, z, \mathbf{w}) = \text{occ}(y, z, \mathbf{w}')$ for all $y, z \in \mathcal{X}$.

Proof. The subsemigroups $\{1, 3, 4\}$, $\{1, 2, 4\}$, and $\{4, 5\}$ of \mathcal{G}_2 are isomorphic to L_2^1 , N_2^1 , and R_2 , respectively. Therefore parts (i)–(iii) follow from Lemma 2.1, and part (iv) is a consequence of parts (ii) and (iii).

(v) Suppose $\text{occ}(x, \mathbf{w}), \text{occ}(x, \mathbf{w}') \leq 2$ for all $x \in \mathcal{X}$. Seeking a contradiction, suppose $\text{occ}(y, z, \mathbf{w}) < \text{occ}(y, z, \mathbf{w}')$ for some $y, z \in \mathcal{X}$. Let $p = \text{occ}(y, z, \mathbf{w})$ and $p' = \text{occ}(y, z, \mathbf{w}')$. Then $0 \leq p < p' \leq 2$ so that $(p, p') \in \{(0, 1), (0, 2), (1, 2)\}$. If $p = 0$, then $\text{ini}(\mathbf{w}) \neq \text{ini}(\mathbf{w}')$ by (ii), whence (i) is violated. Therefore $(p, p') = (1, 2)$, so that

$$\mathbf{w} = \mathbf{w}_1y\mathbf{w}_2z\mathbf{w}_3 \quad \text{and} \quad \mathbf{w}' = \mathbf{w}'_1y\mathbf{w}'_2y\mathbf{w}'_3z\mathbf{w}'_4$$

for some $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}'_1, \mathbf{w}'_2, \mathbf{w}'_3, \mathbf{w}'_4 \in \mathcal{X}^*$ with $y, z \notin \text{con}(\mathbf{w}_1\mathbf{w}_2\mathbf{w}'_1\mathbf{w}'_2\mathbf{w}'_3)$. Let $\varphi : \mathcal{X} \rightarrow \mathcal{G}_2$ denote the substitution

$$x \mapsto \begin{cases} 2 & \text{if } x = y, \\ 6 & \text{if } x = z, \\ 4 & \text{otherwise.} \end{cases}$$

Then $\mathbf{w}\varphi \in \{4\}^* \cdot 2 \cdot \{4\}^* \cdot 6 \cdot \mathbf{w}_3\varphi = \{3\}$ and $\mathbf{w}'\varphi \in \{4\}^* \cdot 2 \cdot \{4\}^* \cdot 2 \cdot \{4\}^* \cdot 6 \cdot \mathbf{w}'_4\varphi = \{1\}$, so the contradiction $\mathbf{w}\varphi \neq \mathbf{w}'\varphi$ is deduced. □

A word \mathbf{w} with $\text{t}(\mathbf{w}) \in \text{sim}(\mathbf{w})$ is said to be *simple-tailed*. An identity $\mathbf{w} \approx \mathbf{w}'$ is *simple-tailed* if \mathbf{w} and \mathbf{w}' are simple-tailed words.

LEMMA 27.7. The variety generated by \mathcal{G}_2 is defined by the identities (27.3) and some set of simple-tailed identities.

Proof. Let \mathbf{V} denote the variety generated by \mathcal{G}_2 . It is routinely checked that \mathcal{G}_2 satisfies the identities (27.3), so that \mathbf{V} is defined by (27.3) and some set Σ of identities. Suppose that $\sigma : \mathbf{w} \approx \mathbf{w}'$ is any identity from Σ that is not simple-tailed. Then Lemma 27.6 implies $\text{t}(\mathbf{w}) = \text{t}(\mathbf{w}') = t$ and $t \notin \text{sim}(\mathbf{w}) = \text{sim}(\mathbf{w}')$. Hence $\mathbf{w} = \mathbf{w}_1t\mathbf{w}_2t$ and $\mathbf{w}' = \mathbf{w}'_1t\mathbf{w}'_2t$ for some

$\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}'_1, \mathbf{w}'_2 \in \mathcal{X}^*$. Choose any letter $z \notin \text{con}(\mathbf{w}) = \text{con}(\mathbf{w}')$. Then the identity σ clearly implies the simple-tailed identity $\tilde{\sigma} : \mathbf{w}z \approx \mathbf{w}'z$. Conversely, since

$$\mathbf{w} \stackrel{(27.3a)}{\approx} \mathbf{w}t \stackrel{\tilde{\sigma}}{\approx} \mathbf{w}'t \stackrel{(27.3a)}{\approx} \mathbf{w}',$$

the identities $\{(27.3), \tilde{\sigma}\}$ imply the identity σ . It follows that if σ in Σ is replaced by the simple-tailed identity $\tilde{\sigma}$, then the resulting set, together with (27.3), is still a basis for \mathbf{V} .

The same argument can be repeated to replace any non-simple-tailed identity σ in Σ with a simple-tailed identity $\tilde{\sigma}$. The resulting set $\tilde{\Sigma}$ consists of simple-tailed identities; these identities, together with (27.3), constitute a basis for \mathbf{V} . □

In this subsection, a simple-tailed word \mathbf{w} with $\text{ini}(\mathbf{w}) = x_1 \dots x_m t$ and $t \in \text{sim}(\mathbf{w})$ is said to be in *canonical form* if

$$\mathbf{w} = \left(\prod_{i=1}^m (x_i \mathbf{w}_i) \right) t, \tag{27.4}$$

where:

- (I) $\text{occ}(x_i, \mathbf{w}) \leq 2$;
- (II) $\mathbf{w}_i \in \{x_1^{e_1} \dots x_i^{e_i} \mid e_1, \dots, e_i \in \{0, 1\}\}$;
- (III) $\mathbf{w}_1, \dots, \mathbf{w}_m$ are pairwise disjoint.

LEMMA 27.8. *Let \mathbf{w} be any simple-tailed word. Then there exists some simple-tailed word $\bar{\mathbf{w}}$ in canonical form such that the identities (27.3) imply the identity $\mathbf{w} \approx \bar{\mathbf{w}}$.*

Proof. Let \mathbf{w} be any simple-tailed word with $\text{ini}(\mathbf{w}) = x_1 \dots x_m t$. Then it suffices to convert \mathbf{w} , using the identities (27.3), into a simple-tailed word in canonical form. It is easily shown that \mathbf{w} can be written in the form (27.4) with $\mathbf{w}_i \in \{x_1, \dots, x_i\}^*$ for all i . For each i , since the letters of \mathbf{w}_i are non-first occurrences, the identities (27.3c) can be used to arrange them within \mathbf{w}_i in any order, whence $\mathbf{w}_i \in \{x_1^{e_1} \dots x_i^{e_i} \mid e_1, \dots, e_i \geq 0\}$. Any x_j that is neither a first nor a second occurrence can be eliminated by the identities (27.3b), so that (I) is satisfied. It is then easily seen that (II) and (III) are also satisfied. □

Proof of Proposition 27.5. By Lemma 27.7, the variety generated by \mathcal{G}_2 is defined by the identities (27.3) and some set Σ of simple-tailed identities. By Lemma 27.8, the simple-tailed words forming the identities in Σ can further be chosen to be in canonical form. Let $\mathbf{w} \approx \mathbf{w}'$ be any such identity, so that by Lemma 27.6,

$$\mathbf{w} = \left(\prod_{i=1}^m (x_i \mathbf{w}_i) \right) t \quad \text{and} \quad \mathbf{w}' = \left(\prod_{i=1}^m (x_i \mathbf{w}'_i) \right) t.$$

Suppose that ℓ is the least integer such that $\text{con}(\mathbf{w}_\ell) \neq \text{con}(\mathbf{w}'_\ell)$. Then by (II), there exists some $k \leq \ell$ such that x_k belongs to either \mathbf{w}_ℓ or \mathbf{w}'_ℓ but not both, say $x_k \in \text{con}(\mathbf{w}_\ell) \setminus \text{con}(\mathbf{w}'_\ell)$. It follows from (II) and the minimality of ℓ that $\mathbf{w}_i = \mathbf{w}'_i$ for all $i < \ell$. Hence

$$\text{occ} \left(x_k, \prod_{i=1}^{\ell-1} (x_i \mathbf{w}_i) \right) = \text{occ} \left(x_k, \prod_{i=1}^{\ell-1} (x_i \mathbf{w}'_i) \right). \tag{27.5}$$

There are two cases to consider.

Case 1: $\ell < m$. Then

$$\text{occ}(x_k, x_{\ell+1}, \mathbf{w}) = \text{occ}\left(x_k, \prod_{i=1}^{\ell-1} (x_i \mathbf{w}_i)\right) + \text{occ}(x_k, x_\ell \mathbf{w}_\ell)$$

and

$$\text{occ}(x_k, x_{\ell+1}, \mathbf{w}') = \text{occ}\left(x_k, \prod_{i=1}^{\ell-1} (x_i \mathbf{w}'_i)\right) + \text{occ}(x_k, x_\ell \mathbf{w}'_\ell).$$

But Lemma 27.6(v) and (27.5) imply $\text{occ}(x_k, x_\ell \mathbf{w}_\ell) = \text{occ}(x_k, x_\ell \mathbf{w}'_\ell)$, and this is impossible because $x_k \in \text{con}(\mathbf{w}_\ell) \setminus \text{con}(\mathbf{w}'_\ell)$.

Case 2: $\ell = m$. Then

$$\text{occ}(x_k, \mathbf{w}) = \text{occ}\left(x_k, \prod_{i=1}^{m-1} (x_i \mathbf{w}_i)\right) + \text{occ}(x_k, x_m \mathbf{w}_m)$$

and

$$\text{occ}(x_k, \mathbf{w}') = \text{occ}\left(x_k, \prod_{i=1}^{m-1} (x_i \mathbf{w}'_i)\right) + \text{occ}(x_k, x_m \mathbf{w}'_m).$$

But Lemma 27.6(ii), (I), and (27.5) imply $\text{occ}(x_k, x_m \mathbf{w}_m) = \text{occ}(x_k, x_m \mathbf{w}'_m)$, and this is impossible because $x_k \in \text{con}(\mathbf{w}_m) \setminus \text{con}(\mathbf{w}'_m)$.

Neither of the two cases just considered is possible. Therefore the integer ℓ does not exist, whence $\text{con}(\mathbf{w}_i) = \text{con}(\mathbf{w}'_i)$ for all i . It follows from (II) that $\mathbf{w}_i = \mathbf{w}'_i$ for all i . The identity $\mathbf{w} \approx \mathbf{w}'$ is thus trivial. Since the identity $\mathbf{w} \approx \mathbf{w}'$ is arbitrary in Σ , every identity in Σ is trivial. Consequently, the variety generated by \mathcal{G}_2 is defined by the identities (27.3). \square

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