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COMPLEMENTARITY PROBLEM AND DUALITY OVER CONVEX CONES

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ABSTRACT—The complementarity problem is defined and studied for cases where the constraints involve convex cones, thus extending the real and complex complementarity problems.

Special cases of the problem are equivalent to dual, linear or quadratic, programs over polyhedral cones.

1. **Introduction.** A nonempty set $S \subset C^n$ is

(a) a *cone* if $x \in S$, $\lambda \geq 0 \Rightarrow \lambda x \in S$

(b) a *convex cone* if it satisfies (a) and $x \in S$, $y \in S \Rightarrow x+y \in S$

(c) a *polyhedral* (*convex*) *cone* if there is a positive integer k and a matrix $A \in C^{n \times k}$ such that $S = AR_+^k$

The *polar*, S^*, of a nonempty subset $S \subset C^n$, is defined by

$$S^* = \{y \in C^n;\, x \in S \Rightarrow \mathrm{Re}(y, x) \geq 0\}$$

The polar S^* is a closed convex cone. A set S is a closed convex cone, if and only if $S = (S^*)^*$. The polar of a polyhedral cone is a polyhedral cone.

In this paper we study the following

Complementarity Problem over Cones: Let P and Q be convex cones in C^n, $q \in C^n$ and $M \in C^{n \times k}$. Find a vector z such that

$$z \in P, \qquad q+Mz \in Q \tag{1}$$

and

$$\mathrm{Re}(z, q+Mz) = 0.$$

For a real vector q and a real matrix M and $P=P^*=Q=R_+^n$ the problem reduces to the Real Complementarity Problem of Cottle and Dantzig [9]. For complex q and M,

$$P = \{z \mid |\arg z| \leq \gamma\}, \qquad \gamma \in R^n, \qquad 0 \leq \gamma \leq \frac{\pi}{2} e \tag{2}$$

(where e is the real vector whose elements are ones) and

$$Q = P^* = \left\{z \mid |\arg z| \leq \frac{\pi}{2} e - \gamma\right\},$$

the problem reduces to the Complex Complementarity Problem introduced by McCallum [16].

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Notice that in these special cases, P and Q are polyhedral cones. There are two cases when the Complementarity Problem (over cones) has a trivial solution. The first is when $q \in Q$ and then $z=0$ solves the problem. The second is when there exists $z \in P$ such that $Mz=-q$, where such a z is the solution of the complementarity problem. The existence of such a z is guaranteed in case MP is closed, if $\operatorname{Re}(q, y) \leq 0$ whenever y satisfies $M^H y \in P^*$. See e.g. [3, Theorem 2.4].

The purpose of this paper is to study the complementarity problem in the non-trivial cases.

In Section 2, the consistency of (1) is characterized and unboundedness of its solutions set is related to cone-copositive matrices defined by Haynsworth and Hoffman.

In Section 3, pairs of dual, linear and quadratic, programs over polyhedral cones are shown to be equivalent to special cases of the complementarity problem over cones where P and Q are polyhedral and $Q=P^*$.

For such a pair of polyhedral cones it is shown in Section 4 that if (1) is consistent and M is positive semi definite then the complementarity problem has a solution.

2. **Non-emptiness and unboundedness of the constraints set.** We start by characterizing the consistency of (1).

THEOREM 1. *Let $M \in C^{n\times n}$, P and Q be convex cones in C^n such that $Q-MP$ is closed, $q \in C^n$ and $Z_q=\{z \mid z \in P, q+Mz \in Q\}$. Then the set Z_q is non-empty if and only if*

$$y \in Q^*, \qquad -M^H y \in P^* \Rightarrow \operatorname{Re}(q, y) \geq 0. \tag{3}$$

Proof. Follows from the main theorem in [3] or from Theorem 1 in [4], by replacing T by Q, S by P, b by q and A by $-M$. □

For P given by (2) and $Q=P^*$, Theorem 1 reduces to Theorem 4.2.2 of [2].

The following conditions are necessary and sufficient for a nonempty constraints set Z_q to be unbounded.

THEOREM 2. *Let $Z_q \neq \varnothing$. Then the following are equivalent:*

(a) *The set Z_q is unbounded.*

(b) $Z_0=\{z \mid z \in P, Mz \in Q\} \neq 0$.

Proof. (a)⇒(b). By (a) there exist $z_0 \neq 0$ and z_1 such that (a′) $z_1+z_0 \in P$, and (a″) $q+Mz_1+\lambda Mz_0 \in Q$ for every $\lambda \geq 0$. (a′)⇒$z_0 \in P$, for suppose $z_0 \notin P$ then there exists $x \in P^*$ such that $\operatorname{Re}(x, z_0)=-1$ and $\operatorname{Re}(x, z_1+\lambda z_0)=\operatorname{Re}(x, z_1)-\lambda$ which for λ big enough contradicts (a′). Similarly (a″)⇒$Mz_0 \in Q$ so that $0 \neq z_0 \in Z_0$ which proves (b).

(b)⇒(a). Let $0 \neq z_0 \in Z_0$ and $z_1 \in Z_q$. Then $z_1+\lambda z_0 \in P$, $q+M(z_1+\lambda z_0) \in Q$, so that $z_1+\lambda z_0 \in Z_q$ for every $\lambda \geq 0$, proving (a). □

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Notice that Theorem 2 is a (non polyhedral) generalization of the well known fact that a polyhedral set is bounded if and only if its corresponding polyhedral cone consists only of the origin.

The interior of P^* is given algebraically by

$$\text{int } P^* = \{y \in P^* \mid 0 \neq p \in P \Rightarrow \text{Re}(p, y) > 0\}$$

A cone S is *pointed* if $S \cap (-S) = 0$. A cone P is pointed if and only if $\text{int } P^* \neq \varnothing$

THEOREM 3. *Let P and Q be pointed cones. Then the set Z_q is unbounded if and only if it is non-empty and the system*

$$(4) \qquad y \in Q^*, \qquad -M^H y \in \text{int } P$$

has no solution.

Proof. By Theorem 2, the non-empty Z_q is unbounded if and only if Z_0 contains a nontrivial vector, that is, there exists $0 \neq z_0 \in P$ such that $Mz_0 \in Q$. This is equivalent by Corollary 1.4 of [5] to the inconsistency of

$$(4') \qquad y \in \text{int } Q^*, \qquad -M^H y \in \text{int } P^*$$

which, in turn, is equivalent to the inconsistency of (4), since Q is pointed.

The definition of a copositive matrix was introduced by Motzkin in [17], namely, $M \in R^{n \times n}$ is *copositive* if $x \geq 0 \Rightarrow X^T Mx \geq 0$.

This was generalized by Haynsworth and Hoffman [13]. Following them, a matrix $M \in C^{n \times n}$ will be called copositive with respect to a cone $P \subset C^n$, if $z \in P \Rightarrow \text{Re } z^H Mz \geq 0$.

THEOREM 4. *Let M be copositive with respect to a pointed cone P and $Q = P^*$. Then the system* (4) *has no solution.*

Proof. Suppose y is a solution of (4). Then $y \neq 0$ and $\text{Re } y^H M^H y = \text{Re } y^H My < 0$ contradicting the copositivity of M with respect to P. □

Choosing P as in (2), or $P = R^n_+$ and $Q = P^*$, one gets Theorem 4.5.5 and Lemma 4.5.6 in [16] and their real counterparts in [8], respectively. Notice that in both cases P and Q are pointed.

3. **Duality in linear and quadratic programming over polyhedral cones.** In this section and in the next one the discussion will be confined to the case where $Q = P^*$ and the cones are polyhedral. In this section it will be shown that for appropriate choice of M, q and P the complementarity problem over cones is equivalent to dual programming problems over cones as is the case for the real and complex complementarity problems e.g. [19] and [16].

Duality in quadratic programming is described in the following theorem.

THEOREM 5. (Abrams and Ben-Israel [2].) *Let $B \in C^{n \times n}$ be a positive semi-definite matrix, $A \in C^{m \times n}$, $b \in C^m$, $c \in C^n$ and let $S \subset C^n$, $T \subset C^m$ be polyhedral cones.*

Consider the pair of problems:

(Q.P) $\quad$ *minimize* $P(x) \equiv \mathrm{Re}(\frac{1}{2}x^H Bx + c^H x)$

$$\text{s.t.} \quad Ax - b \in T$$
$$x \in S$$

(Q.D) $\quad$ *maximize* $g(u, y) \equiv \mathrm{Re}(-\frac{1}{2}u^H Bu + b^H y)$

$$\text{s.t.} \quad Bu - A^H y + c \in S^*$$
$$y \in T^*$$

If x^0 is an optimal solution to (Q.P.), then there exists a vector y^0 such that (x^0, y^0) is an optimal solution to (Q.D.). Moreover, $f(x^0) = g(x^0, y^0)$.

If (u^0, y^0) is an optimal solution to (Q.D.) then there exists a vector x^0, for which $Bx^0 = Bu^0$, such that x^0 is an optimal solution to (Q.P.). Moreover, $f(x^0) = g(u^0, y^0)$. □

Special cases of Theorem 5, include the real version of Dorn [10] and the complex version of Hanson and Mond [12].

When $B=0$, the problems become linear:

(L.P) $\quad$ minimize $\mathrm{Re}\, c^H x$

$$\text{s.t.} \quad Ax - b \in T$$
$$x \in S$$

(L.D) $\quad$ maximize $\mathrm{Re}\, b^H y$

$$\text{s.t.} \quad c - A^H y \in S^*$$

The duality theorem becomes Theorem 2 of [6], where special cases are the classical duality theorem of real linear programming and its complex version due to Levinson [15].

THEOREM 6. *Consider the complementarity problem with*

$$M = \begin{bmatrix} B & -A^H \\ A & 0 \end{bmatrix}, \quad q = \begin{bmatrix} c \\ -b \end{bmatrix}, \quad z = \begin{bmatrix} x \\ y \end{bmatrix}$$
$$P = S \times T^*, \quad Q = P^* = S^* \times T.$$

A solution of this problem solves the pair of problems (Q.P) and (Q.D) and vice-versa.

Proof. With the choice of M, q, z and P as above, the complementarity problem becomes:

Find $x \in S$, $y \in T^*$ such that $c + Bx - A^H y \in S^*$, $-b + Ax \in T$, and

$$\mathrm{Re}(c^H x + x^H Bx - x^H A^H y - b^H y + x^H A^H y) = \mathrm{Re}(c^H x - b^H y + x^H Bx) = 0. \tag{5}$$

Notice that the constraints that x and y have to satisfy, are those that x and $(u = x, y)$ have to satisfy in the pair of the quadratic problems.

Let x^0 and (x^0, y^0) be optimal solutions of (Q.P) and (Q.D). Then by Theorem 5, $\mathrm{Re}(c^H x^0 + x^0 B x^0/2) = \mathrm{Re}(b^H y^0 - x^0 B x^0/2)$, which imply (5), so that x^0 and y^0 solve the complementarity problem.

On the other hand let (x, y) be a solution of the complementarity problem. Then x and (x, y) are feasible solutions of (Q.P) and (Q.D) respectively and satisfy (5), and $f(x) = g(x, y)$.

Let S be a feasible solution of (Q.P). We now show that $f(s) \geq f(x)$ and thus x is a feasible solution of (Q.P). First, notice that

$$\mathrm{Re}(c^H s + s^H B x) \geq \mathrm{Re}\, y^H A s \geq \mathrm{Re}\, b^H y \tag{6}$$

The left side of (6) follows from $c + Bx - Ay \in S^*$ (x and y being solutions of the complementarity problem) and $s \in S$ (being a feasible solution). The right side follows from $As - b \in T$ and $y \in T^*$.

The positive semi definiteness of B implies

$$\tfrac{1}{2} s^H B s + \tfrac{1}{2} x^H B x \geq \mathrm{Re}\, s^H B x \tag{7}$$

(7) and (6) imply

$$\tfrac{1}{2} s^H B s + \tfrac{1}{2} x^H B x \geq \mathrm{Re}(b^H y - c^H x) + \mathrm{Re}(c^H x - c^H s) \tag{8}$$

Substituting (5) into (8) one gets

$$\mathrm{Re}(\tfrac{1}{2} s^H B s + c^H s) \geq \mathrm{Re}(\tfrac{1}{2} x^H B x + c^H x)$$

which was to be shown.

By Theorem 5, the value of the dual problem is equal to that of the primal. By (5) $g(x, y) = f(x)$ and thus (x, t) is an optimal solution of (Q.D). □

The complementarity problem equivalent to (L.P) and (L.D) is the same as the one in Theorem 6, with $B = 0$.

4. **Solvability of the complementarity problem.** The real and the complex complementarity problems were shown to have a solution (if $Z_q \neq \varnothing$) for given families of matrices M, e.g. [7], [14]. This may be extended to the problem over cones using the Kuhn Tucker conditions for nonlinear programming over cones, e.g. [1]. Here we shall apply Theorem 5 to prove the existence of solutions for positive semi definite matrices M. Thus extending Theorem 4.5.1 of [16]. Notice that for positive semi definite M, Z_q is unbounded if it is not empty, since copositive matrices (with respect to a cone) are positive semi definite.

THEOREM 7. *Let M be positive semi definite (not necessarily hermitian) matrix, P a polyhedral cone and $Q = P^*$ in the complementarity problem for which the constraints set Z_q is non empty. Then the problem has a solution.*

Proof. Consider the related quadratic program.

$$\text{(Q)} \qquad \begin{aligned} &\text{Minimize } f(z) = \mathrm{Re}(z, q + \tfrac{1}{2}(M + M^H) z) \\ &\text{s.t. } z \in P,\ q + Mz \in P^*. \end{aligned}$$

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To prove the theorem it suffices to show that (Q) has a solution $\hat{z}$ and $f(\hat{z})=0$. The fact that (Q) has an optimal solution $\hat{z}$ is guaranteed by the complex version (Theorem 4.3.6 in [16]) of the Frank Wolfe Theorem [17]. To show that $f(\hat{z})=0$, consider the dual of (Q):

$$\text{Maximize } g(u, y) = \mathrm{Re}(-\tfrac{1}{2}u^H(M+M^H)u-q^Hy)$$
s.t.

$$(M+M^H)u-M^Hy+q \in P^* \tag{9}$$

$$y \in P \tag{10}$$

From (9) and (10) it follows that

$$g \equiv \mathrm{Re}(2y^HMu-y^HMy+q^Hy) \geq 0. \tag{11}$$

Also the positive semi definiteness of M implies:

$$-g(u, y) \geq g \tag{12}$$

since

$$\begin{aligned} -g(u, y)-g &= \mathrm{Re}(q^Hy+u^HMu-2y^HMu+y^HMy-q^Hy) \\ &= \mathrm{Re}(u^HMu+y^HMy-2y^HMy) \\ &= \mathrm{Re}(u-y)^HM(u-y) \geq 0. \end{aligned}$$

By (11) and (12) the maximum of $g(u, y)$ is nonpositive and by Theorem 5, $f(\hat{z})=\max g(u, y)$. The constraints of the quadratic program imply that $f(\hat{z})\geq 0$ and so $f(\hat{z})=0$ which completes the proof. □

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