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On the Vanishing Orders of Vector Fields on Fano Varieties of Picard Number 1*

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Abstract. We show that the vanishing order of a non-zero vector field at a generic point of a smooth Fano variety of Picard number 1 cannot exceed the dimension of the Fano variety. Furthermore, if there exist only finitely many rational curves of minimal degree through a generic point of the Fano variety, we show that a non-zero vector field cannot vanish at a generic point of the Fano variety.

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1. Introduction

The dimensions of the automorphism groups of projective varieties of dimension n cannot be bounded in terms of n. For example, the dimension of the automorphism group of the Hirzebruch surface $\mathbf{P}(\mathcal{O}(m) \oplus \mathcal{O}), m > 0,$ is $m + 5$.

In this paper, we will give a bound on the dimension of the automorphism group of a smooth Fano variety X of Picard number 1 in terms of $n = \dim(X)$ by giving a bound on the vanishing orders of vector fields at a generic point of X. Here the vanishing order of a vector field is defined as follows. A non-zero vector field V on a smooth variety X has *vanishing order* $k \geqslant 0$ at $x \in X$ if $V \in H^0(X, T(X) \otimes \mathbf{m}^k)$ but $V \notin H^0(X, T(X) \otimes \mathbf{m}^{k+1})$, where $T(X)$ is the tangent bundle of X and $\mathbf{m}$ is the maximal ideal at x. Throughout the paper, we will work over the complex numbers.

To state our results, we need the concept of standard rational curves. Let X be a smooth uniruled projective variety of dimension n. By Mori's bend-and-break trick ([Ko] Ch.II), there exists a rational curve $C \subset X$, such that under the normalization $\nu: \mathbf{P}_1 \to C \subset X, \nu^*T(X) = \mathcal{O}(2) \oplus [\mathcal{O}(1)]^p \oplus \mathcal{O}^q, p + q + 1 = n$. Such a rational curve C will be called a *standard rational curve*. For example, choose a generic point x and consider rational curves passing through x which has minimal degree with respect to a fixed ample divisor. Then a generic choice of such a curve is a standard rational

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curve. A standard rational curve C needs not be smooth. But its normalization $\nu: \mathbf{P}_1 \to C \subset X$ is an immersion. For convenience, we will call the bundle $\nu^* T(X)/T(\mathbf{P}_1) \cong [\mathcal{O}(1)]^p \oplus \mathcal{O}^q$ on the normalization of C as the *normal bundle* of C.

Note that many Fano varieties have standard rational curves with $p = 0$. For example, any Fano threefold of Picard number 1, except the projective space and the hyperquadric, has standard rational curves with $p = 0$. In general, if there exist only finitely many rational curves of minimal degree through a generic point of a smooth Fano variety, it is easy to see from the basic deformation theory (e.g. [Ko]), that these rational curves are standard rational curves with $p = 0$. In this case, we will prove the following.

THEOREM 1. *Let X be a smooth Fano variety of Picard number 1 of dimension* $\geqslant 3$ *having standard rational curves with $p = 0$ and $x \in X$ be a generic point. Then there exists no non-zero vector field on X which vanishes at x.*

An immediate consequence is

COROLLARY 1. *Let X be a smooth Fano variety of Picard number 1 of dimension n having standard rational curves with $p = 0$. Then the dimension of the automorphism group of X is $\leqslant n$.*

Theorem 1 implies that if the dimension is n in Corollary 1, the variety must be almost homogeneous. This is the case for Mukai–Umemura threefolds ([MU]), which are SL(2, $\mathbf{C}$)-almost homogeneous Fano threefolds satisfying the assumption of Theorem 1. In this sense, Corollary 1 seems optimal.

For $p > 0$, we have the following result.

THEOREM 2. *Let X be a smooth Fano variety of Picard number 1 of dimension $n \geqslant 2$ having standard rational curves with $p > 0$ and $x \in X$ be a generic point. Then there exists a positive integer m and a nonnegative integer l satisfying $l + (p+1)m \leqslant n$ such that the vanishing order at x of any non-zero vector field on X cannot exceed $l + 2m$. In particular, the vanishing order at x cannot exceed n.*

The idea of the proof of Theorem 2 can be best illustrated by proving it for $p = n - 1$. Since $m = 1$ and $l = 0$ for $p = n - 1$, we have to show that the vanishing order cannot exceed 2. Suppose the vanishing order at x of a vector field V is $\geqslant 3$. Then the one-parameter group of automorphisms of X induced by V acts trivially on the tangent space $T_x(X)$. We claim that this action preserves each standard rational curve through x. Otherwise, this action sends some standard rational curve through x to a family of standard rational curves through x having the same tangent vector at x. Then the infinitesimal deformation will give a section of the normal bundle vanishing at x with multiplicity $\geqslant 2$. This is impossible from the splitting type of the normal bundle of a standard rational curve. Thus V is tangent to each standard rational curve through x. Since the vanishing order of V at x is $\geqslant 3$ while $c_1(\mathbf{P}_1) = 2$, V vanishes identically on each standard rational curve through x. But from $p = n - 1$, standard rational curves passing through x cover a Zariski dense open subset in X. This shows that V vanishes identically on X. The proof of Theorem 2 is a refinement of this argument.

Since the dimension of the vector space of polynomial vector fields in n variables with coefficients of degree $\leqslant n$ is

$$n + n \times \binom{n}{n-1} + n \times \binom{n+1}{n-1} + \cdots + n \times \binom{2n-1}{n-1} = n \times \binom{2n}{n},$$

Theorem 2 gives the following bound on the dimensions of automorphism groups of Fano varieties.

COROLLARY 2. *Let X be a smooth Fano variety of Picard number 1 of dimension n. Then the dimension of the automorphism group of X is less than or equal to $n \times \binom{2n}{n}$.*

It should be mentioned that it is possible to get a bound on the dimension of the automorphism group of a smooth Fano variety of Picard number 1 by known results. In fact, by the results on Fujita's conjecture, e.g. [Si], we have a bound on the integer m for which $|mK^{-1}|$ is very ample for all smooth Fano varieties of dimension n with Picard number 1. Then Alan Nadel's proof of the boundedness of degree of Fano varieties of Picard number 1 of a fixed dimension gives a bound N on the dimension of $|mK^{-1}|$ ([Na]). So the dimensions of automorphism groups will be bounded by the dimension of $\mathrm{PGL}(N+1)$. But this bound is quite huge because the known bounds on m and the dimension of $|mK^{-1}|$ are huge, and usually there is a big difference between the automorphism group of a Fano variety X and $\mathrm{PGL}(|mK_X^{-1}|)$. For example, even assuming K^{-1} is very ample, i.e. $m = 1$, the bound one can get by this method is the square of $\binom{n^2+2n}{n}$, which is much larger than ours. Moreover, it is unclear that such a bound on the dimensions of automorphism groups gives a bound on the vanishing orders of vector fields at generic points.

We expect that the bound in Theorem 2 is far from being optimal. In this regard, we would like to raise the following questions.

QUESTION 1. Let X be a smooth Fano variety of Picard number 1 and $x \in X$ be a generic point. Is the vanishing order at x of any non-zero vector field on X less than or equal to 2?

QUESTION 2. Is the dimension of the automorphism group of an n-dimensional smooth Fano variety of Picard number 1 bounded by that of $\mathbf{P}_n$?

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2. Proof of Theorem 1

Given a smooth uniruled projective variety X, choose an irreducible component $\mathcal{K}$ of the Chow scheme of curves on X so that a generic point of $\mathcal{K}$ corresponds to a standard rational curve. By taking normalization, we can construct universal family morphisms $\psi: \mathcal{F} \to \mathcal{K}$ and $\phi: \mathcal{F} \to X$ (e.g. [Ko] Ch.II) so that for a point $\kappa \in \mathcal{K}$ corresponding to a standard rational curve, the fiber $\psi^{-1}(\kappa)$ is $\mathbf{P}_1$ and $\phi|_{\psi^{-1}(\kappa)}$ is an immersion of $\mathbf{P}_1$. The fiber of ϕ over a point in $\phi(\psi^{-1}(\kappa))$ has dimension p, where p is the number of $\mathcal{O}(1)$-factors in the splitting of $T(X)$ over the normalization of the standard rational curve $\phi(\psi^{-1}(\kappa))$.

Proof of Theorem 1. Choose $\mathcal{K}$ as above with $p = 0$ and the universal family morphisms $\psi: \mathcal{F} \to \mathcal{K}$ and $\phi: \mathcal{F} \to X$. From $p = 0$, ϕ is generically finite and a standard rational curve is an immersed $\mathbf{P}_1$ with trivial normal bundle. Thus ϕ is unramified at every point on a generic fiber of ψ. Replacing $\mathcal{F}$ by its desingularization, we assume that $\mathcal{F}$ is smooth. ϕ remains to be generically finite and unramified at every point on a generic fiber of ψ.

Let $R \subset \mathcal{F}$ be the ramification loci of ϕ. A generic fiber F of ψ is disjoint from the ramification loci R and ϕ is biholomorphic in an analytic neighborhood $\mathcal{U} \subset \mathcal{F}$ of F.

We claim that ϕ is not birational. Otherwise, we may assume that $\phi^{-1}(\phi(\mathcal{U})) = \mathcal{U}$. Shrinking $\mathcal{U}$ if necessary, we can choose a general hypersurface $H \subset \mathcal{K}$ disjoint from $\psi(\mathcal{U})$. Then $\phi(\psi^{-1}(H))$ is a hypersurface on X disjoint from $C = \phi(F)$. This is a contradiction to the assumption that X has Picard number 1. Thus ϕ is not birational.

Let $B \subset X$ be the codimension 1 loci of $\phi(R)$, which is nonempty since ϕ is not birational and X is simply connected. From the triviality of the normal bundle, we may assume that the generic curve C is disjoint from the codimension 2 set $\phi(R) \setminus B$. We claim that $\phi^{-1}(C)$ contains an irreducible component C' such that $\phi: C' \to C$ is not birational. In fact, since C intersects B from the Picard number of X, some component C' intersects R. If $\phi: C' \to C$ is birational, deformations of C' induce deformations of C by the genericity of C. It follows that both C and C' have trivial normal bundles. This is a contradiction to $K_{\mathcal{F}} = \phi^* K_X + R$.

Let $\tilde{\phi}: \tilde{C}' \to \tilde{C}$ be the induced morphism on the normalizations. Then $\tilde{\phi}$ has at least two distinct branch points on $\tilde{C}$. Otherwise, we have a finite unramified covering of $\mathbf{C}$, a contradiction. We conclude that $\nu^{-1}(B)$ has at least two distinct points, where $\nu: \tilde{C} \to X$ is the normalization of C

Now let $x \in X$ be a generic point and suppose there exists a vector field V on X vanishing at x. Then the one-parameter group of automorphisms of X induced by V fixes the finitely many curves $C_1, \ldots, C_m$ through x belonging to the family $\mathcal{K}$. Thus V must be tangent to each C_i. Let $\nu_i : \tilde{C}_i \to C_i$ be the normalization. Since the divisor B is determined by $\mathcal{K}$, B is invariant under V. So V vanishes at the points $C_i \cap B$. But from the above discussion, the lifted vector field $\tilde{V}$ on $\tilde{C}_i$ vanishes at least at three distinct points $\nu_i^{-1}(B)$ and $\nu_i^{-1}(x)$. It follows that V vanishes identically on C_i.

Arguing at a generic point on C_i in place of x, we see that V vanishes on points which can be joined to x by the union of two intersecting rational curves belonging to the family $\mathcal{K}$. Repeating the same argument, V vanishes on points which can be joined to x by the connected chain of finitely many curves belonging to the family $\mathcal{K}$. Since the Picard number of X is 1, this means that V vanishes on generic points of X (e.g. [Ko] IV.4) and $V \equiv 0$. □

3. Proof of Theorem 2

We start with a discussion on how the vanishing orders of a vector field change along standard rational curves.

PROPOSITION 1. *Let X be a smooth uniruled projective variety. Let V be a vector field on X with vanishing order $k \geqslant 1$ at $x \in X$. Suppose there exists a standard rational curve C through x at a generic point of which the vanishing order of V is k. Assume that the vanishing order of V is $l \geqslant k$ at some point $y \in C$. Then*

(i) $l - k \leqslant 2$;
(ii) *if $l - k = 2$, then the k-jet of V at x regarded as an element of $T_x(X) \otimes \mathrm{Sym}^k T_x^*(X)$ lies in the subspace $T_x(C) \otimes \mathrm{Sym}^k T_x^*(X)$.*

In the statement of (ii), the standard rational curve C is an immersed $\mathbf{P}_1$ and may have several branches at x. But the proof of Proposition 1 shows that all the branches must have the same tangent direction at x, which we denote by $T_x(C)$.

Proof. Let $J^m T(X)$ be the mth order jet bundle of $T(X)$. We may pull-back the exact sequence of vector bundles

$$0 \longrightarrow T(X) \otimes \mathrm{Sym}^k T^*(X) \longrightarrow J^k T(X) \longrightarrow J^{k-1} T(X) \longrightarrow 0$$

by the normalization of C, and regard all bundles to be defined on $\mathbf{P}_1$. Let $\mathbf{m}_y$ be the ideal sheaf on $\mathbf{P}_1$ corresponding to the point y. Since V vanishes to the order k along C and to the order l at $y \in C$, it defines a non-zero section τ of $H^0(\mathbf{P}_1, T(X) \otimes \mathrm{Sym}^k T^*(X) \otimes \mathbf{m}_y^{l-k})$. From the splitting type

$$T(X) \otimes \mathrm{Sym}^k T^*(X)|_{\mathbf{P}_1} \cong (\mathcal{O}(2) \oplus [\mathcal{O}(1)]^p \oplus \mathcal{O}^q) \otimes \mathrm{Sym}^k(\mathcal{O}(-2) \oplus [\mathcal{O}(-1)]^p \oplus \mathcal{O}^q),$$

we see (i) immediately. Furthermore if $l - k = 2$, then τ must be a section of $\mathcal{O}(2) \otimes \mathrm{Sym}^k(\mathcal{O}^q)$ vanishing to the order 2 at y. Since the $\mathcal{O}(2)$-factor of $T(X)|_{\mathbf{P}_1}$ corresponds to $T_x(C)$, (ii) follows. □

PROPOSITION 2. *Let X be a smooth uniruled projective variety and $C_t, t \in \Delta := \{|t| < 1\}$ be a family of distinct standard rational curves sharing a common point $x \in X$. Suppose there exists a vector field V on X such that the vanishing*

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order of V is $k \geqslant 0$ at x and at generic points of C_t for each $t \in \Delta$. If the vanishing order is $l \geqslant 2$ at some point $y_t \in C_t$ for each $t \in \Delta$, then $l \leqslant k + 1$.

Proof. First we show that $k > 0$, namely, V vanishes on C_t for all $t \in \Delta$. Since the one-parameter group of automorphisms of X induced by V acts trivially on the tangent space of X at y_t, this action moves C_t with its tangent vector at y_t fixed. But standard rational curves cannot be deformed with a tangent vector at a point fixed because the infinitesimal deformation gives a section of the normal bundle of the curve vanishing to order 2 at that point. It follows that the action preserves C_t for each $t \in \Delta$ and fixes the point x. In other words, V is tangent to C_t and vanishes at x. So $V|_{C_t}$ has at least three zeroes, a double zero at y_t and a single zero at x, showing that V vanishes on C_t.

Now we can apply Proposition 1 to each C_t. Suppose $l = k + 2$. From Proposition 1 (ii), the k-jet of V at x lies in $T_x(C_t) \otimes \mathrm{Sym}^k T^*_x(X) \subset T_x(X) \otimes \mathrm{Sym}^k T^*_x(X)$. Thus the the tangent direction of C_t at x is independent of $t \in \Delta$ and C_t's give a family of standard rational curves with the tangent vector at x fixed, a contradiction. □

Now we assume that X is a smooth Fano variety of Picard number 1. Fix an irreducible component $\mathcal{K}$ of the Chow scheme of curves on X so that a generic point of $\mathcal{K}$ corresponds to a standard rational curve on X. We say that an irreducible subvariety $A \subset X$ is *saturated* if for any standard rational curve C belonging to $\mathcal{K}$, either $C \subset A$ or $C \cap A = \emptyset$.

LEMMA 1. *Let X be a smooth Fano variety of Picard number 1. There exists a countable union of proper subvarieties of X, so that the only saturated subvariety of X containing a point outside this countable union is X itself.*

Proof. Otherwise the union of saturated subvarieties of dimension $< n = \dim(X)$ cover a Zariski-open subset of X. Thus there exists an irreducible subvariety $\mathcal{H}$ of the Hilbert scheme of X whose generic point corresponds to a saturated proper subvariety of X so that the members of $\mathcal{H}$ cover the whole X. By choosing a suitable subvariety of $\mathcal{H}$, we get a hypersurface $H \subset X$ which is the closure of the union of some collection of saturated proper subvarieties of X. Choose a standard rational curve C_1 belonging to $\mathcal{K}$ which is not contained in H. From the condition on the Picard number, C_1 intersects H. Thus small deformations of C_1 intersect generic points of H. This gives standard rational curves not contained in H but intersects saturated subvarieties lying in H, a contradiction to the definition of saturated subvarieties. □

If $A \subset X$ is not saturated and $A \neq X$, then we can find a standard rational curve C belonging to $\mathcal{K}$ which is not contained in A but contains a point of A. Small deformations of standard rational curves are standard rational curves, and the union of all such deformations contain an open neighborhood of C. Thus given a generic point $a \in A$, there exists a standard rational curve belonging to $\mathcal{K}$ which is not contained in A but contains a. Let $\psi: \mathcal{F} \to \mathcal{K}$ and $\phi: \mathcal{F} \to X$ be the universal family morphisms, as explained in Section 2. Given $A \subset X$ as above, $\phi \circ \psi^{-1} \circ \psi \circ \phi^{-1}(A)$ contains an irreducible component A' which contains A properly so that given a generic point $a \in A'$ there exists a standard $\mathcal{K}$-curve C containing a with $C \cap A \neq \emptyset$. There may be many possibilities for A'. We choose one such A' with maximal dimension and say that A' is obtained from A by *attaching standard rational curves*.

PROPOSITION 3. *Given an irreducible subvariety $A \subset X$ which is not saturated, let A' be an irreducible subvariety obtained from A by attaching standard rational curves. Then either* $\dim(A') \geqslant \dim(A) + p + 1$, *or for a generic point $a \in A'$, there exists a family $C_t, t \in \Delta$ of distinct standard rational curves belonging to $\mathcal{K}$ such that $a \in C_t$ and $C_t \cap A \neq \emptyset$ for all $t \in \Delta$.*

Proof. Note that ϕ has a generic fiber of dimension p. Thus a component $\hat{A}$ of $\psi^{-1} \circ \psi \circ \phi^{-1}(A)$ with $\phi(\hat{A}) = A'$ has dimension $\geqslant \dim(A) + p + 1$. If ϕ is generically finite on this component, we have $\dim(A') \geqslant \dim(A) + p + 1$. Otherwise, for each generic $a \in A'$, $\psi(\phi|_{\hat{A}}^{-1}(a))$ will give the required family of standard rational curves. □

We are ready to finish the proof of Theorem 2.

Proof of Theorem 2. If the bound on the vanishing order holds for some point on X, it will hold for generic points of X. Thus we may prove it for some $x \in X$.

Choose a point $x \in X$ so that any proper irreducible subvariety of X containing x is not saturated (Lemma 1). Choose a sequence of irreducible subvarieties $A_0 \subset A_1 \subset \cdots \subset A_{N-1} \subset A_N = X$ so that $A_0 = x$ and A_i is obtained from A_{i-1} by attaching standard rational curves. Let m be the number of inclusions $A_{i-1} \subset A_i$ with $\dim(A_i) \geqslant \dim(A_{i-1}) + p + 1$. Note that there does not exist a non-trivial family of standard rational curves sharing two distinct points, from the splitting type of their normal bundles. Thus $\dim(A_1) \geqslant \dim(A_0) + p + 1$ and $m \geqslant 1$. Let $l = N - m$. Then $(p+1)m + l \leqslant n$.

Let V be a vector field on X which has order k_i at generic points of A_i. If $k_{i-1} \geqslant 3$, then V vanishes on A_i as in the proof of Proposition 2, and applying Proposition 1 (i), we see that $k_{i-1} - k_i \leqslant 2$. If $k_{i-1} \geqslant 2$ and $\dim(A_i) < \dim(A_{i-1}) + p + 1$, we have $k_{i-1} - k_i \leqslant 1$ by Proposition 2 and Proposition 3. Combining these, if $k_0 > l + 2m$, then $k_N > 0$ and V vanishes on $A_N = X$ identically. Thus $k_0 \leqslant l + 2m$. □

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