INFINITE DOUBLY STOCHASTIC MATRICES

J. R. Isbell*

(received August 28, 1961)

This note proves two propositions on infinite doubly stochastic matrices, both of which already appear in the literature: one with an unnecessarily sophisticated proof (Kendall [2]) and the other with the incorrect assertion that the proof is trivial (Isbell [1]). Both are purely algebraic; so we are, if you like, in the linear space of all real doubly infinite matrices $A = (a_{ij})$.

Proposition 1. Every extreme point of the convex set of all doubly stochastic matrices is a permutation matrix.

Kendall's proof of this depends on an ingenious choice of a topology and the Krein-Milman theorem for general locally convex spaces [2]. The following proof depends on practically nothing: for example, not on the axiom of choice.

Proof. Let A be a doubly stochastic matrix which is not a permutation matrix; we may assume $0 < a_{11} < 1$. We must find a non-zero matrix E such that both A + E and A - E are doubly stochastic, with $A = \frac{1}{2}(A+E) + \frac{1}{2}(A-E)$ non-extreme.

I shall define certain finite sets R(n) of row indices and C(n) of column indices for $n = 0, \pm 1, \pm 2, \ldots$, beginning with $R(0) = \{1\}$, $C(0) = \{1\}$. Each j in C(n) will be associated with at least one i in R(n-1) and with at least one i in R(n), so that among other facts we have $0 < a_{ij} < 1$ when i and j are associated. More fully, for $n > 0$, each j in C(n) is

*This work was supported by the Rand Corporation Combinatorial Symposium in the summer of 1961.

Canad. Math. Bull. vol. 5, no. 1, January 1962.

https://doi.org/10.4153/CMB-1962-001-4 Published online by Cambridge University Press

associated with exactly one i in $R(n-1)$; each i in $R(n)$ is associated with exactly one j in $C(n)$. In the other direction the association is one-to-many. For $n < 0$ the direction is reversed.

Note now that should $C(m)$ and $C(n)$, $m < n$, ever have a common element, we should be done. Consider the case $m < 0 < n$. Then j_1 in $C(m) \cap C(n)$ is associated with a unique i_1 in $R(m)$ and a unique i_{n-m} in $R(n-1)$. In turn i_1 is associated with a unique j_2 in $C(m+1)$, and so on. Working toward zero, we obtain a closed loop of $2(n-m)$ places, cyclically ordered, in which some non-zero ε can be alternately added to or subtracted from the entries of A to yield two doubly stochastic matrices $A + E$, $A - E$. Moreover, however m and n lie with respect to zero, the same result can be achieved by working toward zero. (The closed loop may have more than $2(n-m)$ places; one may have to go to $R(0)$ or $C(0)$ to close it.) Further, should we ever find two distinct column indices j, k in $C(n)$ $(n > 0)$ such that for some row index i not in $R(n-1)$, both a_{ij} and a_{ik} are non-zero, we could again find a closed loop. Similar remarks hold for row indices and for $n \leq 0$.

Then select $\varepsilon_o > 0$, strictly less than $\min(a_{11}, 1-a_{11})$. For some finite set $C(1)$ of column indices, disjoint from $C(0)$, the sum of a_{1j} as j runs over $C(1)$ exceeds ε_o; and it is certainly less than $1 - \varepsilon_o$. Select numbers $\delta_{1j} \geq 0$ for j in $C(1)$, with sum ε_o, such that each a_{1j} is strictly between δ_{1j} and $1 - \delta_{1j}$. (Clearly the apparent free choice here can be replaced by rigid formulas.) Generally, having $C(n)$ and δ_{nj}, $n > 0$, select finite sets $R(n, j)$ of row indices i not in $R(n-1)$ over which a_{ij} sums to more than δ_{nj}. For fixed n and different j, these are disjoint sets, or we have a closed loop. Partition δ_{nj} into numbers ε_{ni} as before; put $R(n) = \cup R(n, j)$; and define $C(n+1)$ in the same manner as $C(1)$. The recursion for $n < 0$ differs only trivially from this.

Finally we define E: e_{oo} is ε_o; for $n > 0$, for $i \in R(n)$, $j \in C(n)$, e_{ij} is ε_{ni}; for $i \in R(n-1)$, $j \in C(n)$, e_{ij} is $-\delta_{nj}$; and similarly for $n < 0$. By construction, both $A + E$ and $A - E$ are doubly stochastic.

Proposition 2. A doubly stochastic matrix $A = (a_{ij})$ in which a_{ij} takes only finitely many distinct values is a convex combination of permutation matrices.

In [1] I said this followed trivially from the theorem that there exists a permutation matrix P such that $a_{ij} > 0$ whenever $p_{ij} > 0$ (for any doubly stochastic A). In using the result (for approximations), Peck and Rattray added the restriction that a_{ij} takes only rational values [3]; then it does follow trivially. To make the proof trivial without this restriction, we seem to need the

Lemma. For any finite set of positive real numbers $\lambda_1, \ldots, \lambda_n$ there exists a Hamel basis for the reals over the rationals, $\{b_\alpha\}$, such that each λ_i is $\Sigma\, r_{ij} b_{\alpha_j}$ with non-negative rational coefficients r_{ij}.

Proof. Since 0 cannot be represented as a positive rational combination of the λ_i, the convex cone which they generate in the vector space of reals over the rationals contains no line. In the finite-dimensional subspace generated by the λ_i, the polar cone has an interior point and hence generates the whole subspace. We pick a basis for the subspace from this polar cone and extend to the required Hamel basis.

Now the proof of Proposition 2 presents no difficulty, if we begin by rewriting each a_{ij} in terms of our special Hamel basis.

Let us note in conclusion that the restricted form of

Proposition 2, with a_{ij} rational, actually follows from the construction for Proposition 1 (suitably extended). It would be interesting to know whether there is a choice-free proof of the theorem that each infinite doubly stochastic matrix has a positive diagonal. In [1], that was deduced from the marriage theorem; but the constant row and column sums might impose enough order on the array to avoid this.

REFERENCES

1. J. R. Isbell, Birkhoff's Problem 111, Proc. Amer. Math. Soc. 6(1955), 217-218.

2. M. G. Kendall, On infinite doubly stochastic matrices and Birkhoff's Problem 111, J. London Math. Soc. 35(1960), 81-84.

3. B. A. Rattray and J. E. L. Peck, Infinite doubly stochastic matrices, Trans. Roy. Soc. Canada III(3), 49(1955), 55-57.

University of Washington

https://doi.org/10.4153/CMB-1962-001-4 Published online by Cambridge University Press