DIFFERENTIAL SUBORDINATIONS FOR CLASSES OF MEROMORPHIC $p$-VALENT FUNCTIONS DEFINED BY MULTIPLIER TRANSFORMATIONS

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(Received 28 September 2009)

Abstract

We investigate several inclusion relationships and other interesting properties of certain subclasses of $p$-valent meromorphic functions, which are defined by using a certain linear operator, involving the generalized multiplier transformations.

Keywords and phrases: multiplier transformations, meromorphic functions, differential subordination.

1. Introduction

For $n > -p$, let $\sum_{p,n}$ denote the class of meromorphic functions of the form

$$f(z) = z^{-p} + \sum_{k=n}^{\infty} a_k z^k, \quad p \in \mathbb{N} = \{1, 2, 3, \ldots\},$$

which are analytic and $p$-valent in the punctured unit disc $\hat{U} = U \setminus \{0\}$, where $U = \{z \in \mathbb{C} : |z| < 1\}$. For convenience, we write $\sum_p \equiv \sum_{p,-p+1}$.

If $f$ and $g$ are two analytic functions in $U$, we say that $f$ is subordinate to $g$, written symbolically as $f(z) \prec g(z)$, if there exists a Schwarz function $w$, which (by definition) is analytic in $U$ with $w(0) = 0$, and $|w(z)| < 1$, $z \in U$, such that $f(z) = g(w(z))$ for all $z \in U$.

It is well known that, if $f(z) \prec g(z)$, then $f(0) = g(0)$ and $f(U) \subset g(U)$. Further, if the function $g$ is univalent in $U$, then we have the following equivalence (see [9]; see also [10, p. 4]):

$$f(z) \prec g(z) \iff f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U).$$

For the functions $f_j \in \sum_{p,n}$, $j = 1, 2$, given by

$$f_j(z) = z^{-p} + \sum_{k=n}^{\infty} a_{k,j} z^k,$$

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we define the Hadamard (or convolution) product of \( f_1 \) and \( f_2 \) by
\[
(f_1 \ast f_2)(z) = z^{-p} + \sum_{k=n}^{\infty} a_k, 2a_k z^m.
\]

Define the linear operator \( I_p^m(n; \lambda, l) : \sum_{p,n} \to \sum_{p,n} \), where \( \lambda \geq 0, \ l > 0 \), and \( m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \), by
\[
I_p^m(n; \lambda, l) f(z) = z^{-p} + \sum_{k=n}^{\infty} \left[ \frac{\lambda(k + p) + l}{l} \right]^m a_k z^k.
\] (1.1)

Then, we can write (1.1) as
\[
I_p^m(n; \lambda, l) f(z) = (\Phi_{n; \lambda, l}^{p,m} \ast f)(z),
\]
where
\[
\Phi_{n; \lambda, l}^{p,m}(z) = z^{-p} + \sum_{k=n}^{\infty} \left[ \frac{\lambda(k + p) + l}{l} \right]^m z^k.
\]

Using definition (1.1), it is easy to verify that the next formula holds for \( \lambda > 0 \):
\[
\lambda z(I_p^m(n; \lambda, l) f(z))' = lI_p^{m+1}(n; \lambda, l) f(z) - (\lambda p + l)I_p^m(n; \lambda, l) f(z).
\] (1.2)

**Remark 1.1.** (1) We note that \( I_p^0(n; \lambda, l) f = f \) and
\[
I_p^1(n; 1, 1) f(z) = \frac{(z^{p+1} f(z))'}{z^p} = (p + 1) f(z) + zf'(z).
\]

(2) For some special values of the parameters \( \lambda, l, m \) and \( p \), we obtain the following operators studied by various authors:
(i) \( I_p^m(n; 1, l) = I_p^m(n, l) \) (see Cho et al. [2]);
(ii) \( I_p^m(n; 1, 1) = D_p^m \) (see Aouf and Hossen [1], and Liu and Srivastava [6]);
(iii) \( I_1^m(0; 1, l) = D_l^m \) (see Cho et al. [3, 4]);
(iv) \( I_1^m(0; 1, 1) = I_m \) (see Urallegaddi and Somanatha [18]).

Using differential subordinations as well as the linear operator \( I_p^m(n; \lambda, l) \), we will introduce a subclass of \( \sum_{p,n} \), as follows.

**Definition 1.2.** (1) For the fixed parameters \( A \) and \( B \), with \(-1 \leq B < A \leq 1\), we say that a function \( f \in \sum_{p,n} \) is in the class \( \sum_{p,n}^m (\lambda, l; A, B) \), if it satisfies the subordination condition
\[
-\frac{z^{p+1} (I_p^m(n; \lambda, l) f(z))'}{p} < \frac{1 + Az}{1 + Bz}, \quad l, \lambda > 0, \ m \in \mathbb{N}_0, \ n > -p.
\] (1.3)

(2) For convenience, we write
\[
\sum_{p,n}^m (\lambda, l; \alpha) \equiv \sum_{p,n}^m \left( \lambda, l; 1 - \frac{2\alpha}{p}, -1 \right), \quad 0 \leq \alpha < p,
\]
that is, \( \sum_{p,n}^m (\lambda, l; \alpha) \) denotes the class of functions \( f \in \sum_{p,n} \) satisfying
\[
\text{Re}\{-z^{p+1}(I_p^m(n; \lambda, l)f(z))'\} > \alpha, \quad z \in U.
\]

**Remark 1.3.** We have the next special cases of \( \sum_{p,n}^m (\lambda, l; A, B) \), studied previously by different authors:

(i) \( \sum_{p,n}^m (1, 1; A, B) = R_{m,p}(A, B) \) (see Liu and Srivastava [6]);
(ii) \( \sum_{p,n}^m (1, 1; A, B) = \sum_{p,n}^m (A, B) \) (see Srivastava and Patel [16]);
(iii) \( \sum_{p,n}^m (1, 1; A, B) = H(p; A, B) \) (see Mogra [11, 12]);
(iv) \( \sum_{p,n}^m (1, 1; A, B) = \sum_{p,n}^m (A, B) \), where \( \sum_{p,n}^m (A, B) \) is the class of functions \( f \in \sum_{p,n} \) satisfying
\[
\frac{-z^{p+1}(I_p^m(n, l)f(z))'}{p} < \frac{1 + Az}{1 + Bz}, \quad l > 0, m \in \mathbb{N}_0, n > -p,
\]
and \( I_p^m(n, l) \equiv I_p^m(n; 1, l) \).

In the present paper we obtain several inclusion relationships for the function class \( \sum_{p,n}^m (\lambda, l; A, B) \), and we investigate various other properties of functions belonging to the class \( \sum_{p,n}^m (\lambda, l; A, B) \). Relevant connections of the results presented in this paper with those obtained in earlier works are also pointed out.

**2. Preliminaries**

To establish our main results, we will need the following lemmas and definition.

**Lemma 2.1 [5].** Let the function \( h \) be convex (univalent) in \( U \), with \( h(0) = 1 \). Suppose also that the function \( \varphi \) given by
\[
\varphi(z) = 1 + c_{p+n}z^{p+n} + c_{p+n+1}z^{p+n+1} + \cdots
\]
(2.1)
is analytic in \( U \). Then
\[
\varphi(z) > \frac{z\varphi'(z)}{\delta} < h(z), \quad \text{Re} \delta \geq 0, \delta \neq 0,
\]
implies that
\[
\varphi(z) < \psi(z) = \frac{\delta}{p+n}z^{-\delta/(p+n)} \int_0^z t^{\delta/(p+n) - 1}h(t) \, dt < h(z),
\]
(2.2)
and \( \psi \) is the best dominant of (2.2).

**Definition 2.2.** We denote by \( P(\gamma) \) the class of functions \( \varphi \) given by
\[
\varphi(z) = 1 + b_1z + b_2z^2 + \cdots,
\]
(2.3)
which are analytic in \( U \) and satisfy the inequality
\[
\text{Re} \varphi(z) > \gamma, \quad z \in U \ (0 \leq \gamma < 1).
\]
Lemma 2.3 [14]. Let the function $\varphi$ given by (2.3) be in the class $\mathcal{P}(\gamma)$. Then

$$\text{Re } \varphi(z) \geq 2\gamma - 1 + \frac{2(1 - \gamma)}{1 + |z|}, \quad z \in U \ (0 \leq \gamma < 1).$$

Lemma 2.4 [17]. For $0 \leq \gamma_1 < \gamma_2 < 1$, the inclusion

$$\mathcal{P}(\gamma_1) * \mathcal{P}(\gamma_2) \subset \mathcal{P}(\gamma_3) \quad \text{where } \gamma_3 = 1 - 2(1 - \gamma_1)(1 - \gamma_2),$$

holds and the result is the best possible. The symbol ‘*’ stands for the previous mentioned Hadamard product of the power series.

Lemma 2.5 [15]. Let $\Phi$ be an analytic function in $U$, with $\Phi(0) = 1$ and $\text{Re } \Phi(z) > 1/2$, $z \in U$. Then, for any function $F$ analytic in $U$, the set $(\Phi * F)(U)$ is contained in the convex hull of $F(U)$, that is, $(\Phi * F)(U) \subset \text{co } F(U)$.

Lemma 2.6 [19]. For all real or complex numbers $\alpha_1$, $\alpha_2$, $\beta_1$, where $\beta_1 \notin \mathbb{Z}^- = \{0, -1, -2, \ldots\}$,

$$\int_0^1 t^{\alpha_2 - 1}(1 - t)^{\beta_1 - \alpha_2 - 1}(1 - zt)^{-\alpha_1} \, dt = \frac{\Gamma(\alpha_2)\Gamma(\beta_1 - \alpha_2)}{\Gamma(\beta_1)} \, 2F_1(\alpha_1, \alpha_2, \beta_1; z) \quad \text{for } \text{Re } \beta_1 > \text{Re } \alpha_2 > 0,$$

(2.4)

$$2F_1(\alpha_1, \alpha_2, \beta_1; z) = 2F_1(\alpha_2, \alpha_1, \beta_1; z),$$

(2.5)

$$2F_1(\alpha_1, \alpha_2, \beta_1; z) = (1 - z)^{-\alpha_1} \, 2F_1\left(\alpha_1, \beta_1 - \alpha_2, \beta_1; \frac{z}{z - 1}\right),$$

(2.6)

and

$$2F_1\left(\alpha_1, \alpha_2, \frac{\alpha_1 + \alpha_2 + 1}{2}; \frac{1}{2}\right) = \frac{\sqrt{\pi}\Gamma\left(\frac{\alpha_1 + \alpha_2 + 1}{2}\right)}{\Gamma\left(\frac{\alpha_1 + 1}{2}\right)\Gamma\left(\frac{\alpha_2 + 1}{2}\right)},$$

(2.7)

where $2F_1$ represents the Gauss hypergeometric function.

3. Subordination theorems and the associated functional inequalities

Unless otherwise mentioned, we shall assume throughout the paper that $n$ is an integer with $n > -p$, that $-1 \leq B < A \leq 1$, $\lambda, l > 0$, $m \in \mathbb{N}_0$, $\beta > 0$, and $p \in \mathbb{N}$.

Theorem 3.1. If the function $f \in \sum_{p,n}$ satisfies the subordination condition

$$\left(1 - \beta\right)z^{p+1}(I_p^m(n; \lambda, l) f(z))' + \beta z^{p+1}(I_p^{m+1}(n; \lambda, l) f(z))' < \frac{1 + Az}{1 + Bz},$$

then

$$\frac{z^{p+1}(I_p^m(n; \lambda, l) f(z))'}{p} < Q(z) < \frac{1 + Az}{1 + Bz},$$

(3.1)
where the function $Q$ is given by

$$Q(z) = \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1 + Bz)^{-1} {}_2F_1\left(1, 1, \frac{l}{\lambda \beta(p + n)} + 1; \frac{Bz}{1 + Bz}\right), & B \neq 0, \\ 1 + \frac{l}{\lambda \beta(p + n)} + lAz, & B = 0, \end{cases}$$

and it is the best dominant of (3.1).

Furthermore, for all $k \in \mathbb{N}$, we have

$$\Re \left[ -\frac{z^{p+1}(I_p^m(n; \lambda, l) f(z))'}{p} \right]^{1/k} > \rho^{1/k}, \quad z \in U,$$

where $\rho = Q(-1)$, and the inequality (3.2) is the best possible.

**Proof.** If we consider the function $\varphi$ defined by

$$\varphi(z) = -\frac{z^{p+1}(I_p^m(n; \lambda, l) f(z))'}{p},$$

then $\varphi$ has the form (2.1) and is analytic in $U$. Applying the identity (1.2) in (3.3), and differentiating the resulting equation with respect to $z$, we get

$$-\frac{(1 - \beta)z^{p+1}(I_p^m(n; \lambda, l) f(z))' + \beta z^{p+1}(I_p^{m+1}(n; \lambda, l) f(z))'}{p} = \varphi(z) + \frac{\beta \lambda}{l} z \varphi'(z) < \frac{1 + Az}{1 + Bz}.$$

Now by using Lemma 2.1 for $\gamma = l/(\lambda \beta)$, we deduce that

$$-\frac{z^{p+1}(I_p^m(n; \lambda, l) f(z))'}{p} < Q(z)$$

$$= \frac{l}{\lambda \beta(p + n)} z^{-l/\lambda \beta(p + n)} \int_0^z t^{(l/\lambda \beta(p + n)) - 1} \frac{1 + At}{1 + Bt} dt$$

$$= \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1 + Bz)^{-1} {}_2F_1\left(1, 1, \frac{l}{\lambda \beta(p + n)} + 1; \frac{Bz}{1 + Bz}\right), & B \neq 0, \\ 1 + \frac{l}{\lambda \beta(p + n)} + lAz, & B = 0, \end{cases}$$

where we made a changes of variables, followed by the use of the identities (2.4), (2.5), and (2.6) (with $b = 1$ and $c = a + 1$). Hence, assertion (3.1) is proved.

In order to prove assertion (3.2), it is sufficient to show that

$$\inf \{\Re Q(z) : |z| < 1\} = Q(-1).$$
Indeed, for $|z| \leq r < 1$,

$$\text{Re} \left( \frac{1 + Az}{1 + Bz} \right) \geq \frac{1 - Ar}{1 - Br}, \quad |z| \leq r < 1.$$ 

Setting

$$G(s, z) = \frac{1 + A z}{1 + B s z},$$

and

$$d \nu(s) = \frac{l}{\lambda \beta (p + n)} s^{l / \lambda (p + n)} ds, \quad 0 \leq s \leq 1,$$

which is a positive measure on $[0, 1]$, we get

$$Q(z) = \int_{0}^{1} G(s, z) \, d \nu(s),$$

so that

$$\text{Re} \, Q(z) \geq \int_{0}^{1} \frac{1 - A sr}{1 - B sr} d \nu(s) = Q(-r), \quad |z| \leq r < 1.$$ 

Letting $r \to 1^{-}$ in the above inequality, and using the elementary inequality

$$\text{Re} \, w^{1/k} \geq (\text{Re} \, w)^{1/k}, \quad \text{Re} \, w > 0, \, k \in \mathbb{N},$$

we obtain (3.2). Finally, inequality (3.2) is the best possible, as the function $Q$ is the best dominant of (3.1).

**Remark 3.2.** Putting $\lambda = l = 1$ in Theorem 3.1, we obtain the result of Srivastava and Patel [16, Theorem 1].

For $\lambda = l = 1$, $n = 0$, and $\beta = 1$, Theorem 3.1 yields the following result, which improves the corresponding one of Liu and Srivastava [7, Theorem 1].

**Corollary 3.3.** The inclusions

$$R_{m+1,p}(A, B) \subset R_{m,p}(A, B) \subset R_{m,p}(1 - 2 \rho, -1)$$

hold, where

$$\rho = \begin{cases} \frac{A}{B} + \left( 1 - \frac{A}{B} \right) (1 - B)^{-1} F_1 \left( 1, 1, \frac{1}{p} + 1; \frac{B}{B - 1} \right), & B \neq 0, \\ \frac{1 - A}{p + 1}, & B = 0, \end{cases}$$

and the result is the best possible.

Putting $A = 1 - 2 \alpha / p$, $B = -1$, $\beta = \lambda = l = 1$, $m = 0$ and $n = -p + 2$ in Theorem 3.1, and using (2.7), we get the following result.
**Corollary 3.4.** If the function \( f \in \sum_{p,-p+2} \) satisfies the inequality

\[
\Re \{-z^{p+1}[(p + 2)f'(z) + zf''(z)]\} > \alpha, \quad z \in U \quad (0 \leq \alpha < p),
\]

then

\[
\Re \{-z^{p+1}f'(z)\} > \alpha + (p - \alpha)\left(\frac{\pi}{2} - 1\right), \quad z \in U,
\]

and the result is the best possible.

**Remark 3.5.** Taking \( \alpha = -p(\pi - 2)/(4 - \pi) \) in the above corollary, we obtain that if the function \( f \in \sum_{p,-p+2} \) satisfies

\[
\Re \{-z^{p+1}[(p + 2)f'(z) + zf''(z)]\} > -\frac{p(\pi - 2)}{4 - \pi}, \quad z \in U,
\]

then \( \Re \{-z^{p+1}f'(z)\} > 0, z \in U \) (see Pap [13]).

**Theorem 3.6.** If the function \( f \in \sum_{p,n}^m(\lambda, l; \alpha), 0 \leq \alpha < p, \) then

\[
\Re \{-z^{p+1}[(1 - \beta)(I_p^m(n; \lambda, l)f(z))' + \beta(I_p^{m+1}(n; \lambda, l)f(z))']\} > \alpha,
\]

for \( |z| < R, \) where

\[
R = \left[\sqrt{1 + \left(\frac{\beta\lambda}{l}\right)^2(p + n)^2 - \frac{\beta\lambda}{l}(p + n)}\right]^{1/(p + n)}.
\]

The result is the best possible.

**Proof.** If we let

\[
-z^{p+1}(I_p^m(n; \lambda, l)f(z))' = \alpha + (p - \alpha)\varphi(z), \quad \text{(3.5)}
\]

then \( \varphi \) has the form (2.1), and is analytic with positive real part in \( U \). Using the identity (1.2) in (3.5), and differentiating the resulting equation with respect to \( z \),

\[
\frac{z^{p+1}[(1 - \beta)(I_p^m(n; \lambda, l)f(z))' + \beta(I_p^{m+1}(n; \lambda, l)f(z))'] + \alpha}{p - \alpha} = \varphi(z) + \frac{\beta\lambda}{l}z\varphi'(z).
\]

Applying in (3.6) the estimate (see [8])

\[
\frac{|\varphi'(z)|}{\Re \varphi(z)} \leq \frac{2(p + n)r^{p+n}}{1 - r^{2(p+n)}}, \quad |z| = r < 1,
\]

we get

\[
\Re\left\{-\frac{z^{p+1}[(1 - \beta)(I_p^m(n; \lambda, l)f(z))' + \beta(I_p^{m+1}(n; \lambda, l)f(z))'] + \alpha}{p - \alpha}\right\} \geq \left[1 - \frac{2\beta\lambda(p + n)r^{p+n}}{l(1 - r^{2(p+n)})}\right]\Re \varphi(z).
\]

(3.7)
and it is easy to see that the right-hand side of (3.7) is positive, provided that \( r < R \), where \( R \) is given by (3.4).

In order to show that the bound \( R \) is the best possible, we consider the function \( f \in \sum_{p,n} \) defined by

\[
-z^{p+1}(I^m_p(n; \lambda, l)f(z))' = \alpha + (p - \alpha) \frac{1 + z^{p+n}}{1 - z^{p+n}}.
\]

Then

\[
-z^{p+1}[(1 - \beta)(I^m_p(n; \lambda, l)f(z))' + \beta(I^{m+1}_p(n; \lambda, l)f(z))'] + \alpha
\]

\[
= \frac{p - \alpha}{1 - z^{2(p+n)} + \frac{2\beta}{l}(p + n)z^{p+n}} = 0,
\]

for \( z = R \exp(i\pi/(p + n)) \), which completes the proof of the theorem.

**Remark 3.7.** Putting \( \lambda = l = 1 \) in Theorem 3.6, we obtain the result of Srivastava and Patel [16, Theorem 2].

For \( \beta = 1 \), Theorem 3.6 reduces to the following result.

**Corollary 3.8.** If the function \( f \in \sum_{p,n}^m(\lambda, l; \alpha) \), \( 0 \leq \alpha < p \), then \( f \in \sum_{p,n}^{m+1}(\lambda, l; \alpha) \) for \( |z| < \tilde{R} \), where

\[
\tilde{R} = \left[ \sqrt{1 + \left( \frac{\lambda}{l} \right)^2 (p + n)^2 - \frac{\lambda}{l}(p + n)} \right]^{1/(p+n)}.
\]

The result is the best possible.

**Theorem 3.9.** Let \( f \in \sum_{p,n}(\lambda, l; A, B) \), and let

\[
F_{p,c}(f)(z) = \frac{c}{z^{c+p}} \int_0^z t^{c+p-1} f(t) \, dt, \quad c > 0.
\]

Then

\[
-\frac{z^{p+1}(I^m_p(n; \lambda, l)F_{p,c}(f)(z))'}{p} < \Theta(z) < \frac{1 + Az}{1 + Bz},
\]

where \( \Theta \) is defined by

\[
\Theta(z) = \begin{cases} 
\frac{A}{B} + \left(1 - \frac{A}{B}\right)(1 + Bz)^{-1}F_1\left(1, 1, \frac{c}{p+n} + 1; \frac{Bz}{1 + Bz}\right), & B \neq 0, \\
1 + \frac{Ac}{c + p + n}z, & B = 0,
\end{cases}
\]

and it is the best dominant of (3.9).
Furthermore,

$$\text{Re}\left[ -\frac{z^{p+1}(I_p^m(n; \lambda, l)F_{p,c}(f)(z))'}{p} \right] > k, \quad z \in U,$$

where \(k = \Theta(-1)\), and this inequality is the best possible.

**Proof.** Setting

$$\varphi(z) = -\frac{z^{p+1}(I_p^m(n; \lambda, l)F_{p,c}(f)(z))'}{p},$$

then \(\varphi\) has the form (2.1), and is analytic in \(U\). Using in (3.10) the operator identity

$$z(I_p^m(n; \lambda, l)F_{p,c}(f)(z))' = cI_p^m(n; \lambda, l)f(z) - (c + p)(I_p^m(n; \lambda, l)F_{p,c}(f)(z)),$$

and differentiating the resulting equation with respect to \(z\), we find that

$$-\frac{z^{p+1}(I_p^m(n; \lambda, l)f(z))'}{p} = \varphi(z) + \frac{z\varphi'(z)}{c} < \frac{1 + Az}{1 + Bz}.$$

Now, the remaining part of the proof follows by employing the same techniques that we used in the proof of Theorem 3.1. \(\square\)

**Remark 3.10.** (1) Setting \(n = 0\) and \(l = \lambda = 1\) in Theorem 3.9, we obtain the following result which improves the corresponding work of Liu and Srivastava [7, Theorem 2]. If \(c > 0\) and \(f \in R_{m,p}(A, B)\), then

$$F_{p,c}(R_{m,p}(A, B)) \subset R_{m,p}(1 - 2\zeta, -1) \subset R_{m,p}(A, B),$$

where

$$\zeta = \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1 - B)^{-1} F_1\left(1, 1, \frac{c}{p} + 1; \frac{B}{B - 1}\right), & B \neq 0, \\ 1 - \frac{Ac}{c + p}, & B = 0. \end{cases}$$

(3.11)

The result is the best possible.

(2) Observing that

$$z^{p+1}(I_p^m(n; \lambda, l)F_{p,c}(f)(z))' = \frac{c}{z^c} \int_0^z t^{c+p}(I_p^m(n; \lambda, l)f(t))' \, dt,$$

whenever \(f \in \sum_{p,n}\) and \(c > 0\), the above remark can be restated as follows. If \(c > 0\) and \(f \in R_{m,p}(A, B)\), then

$$\text{Re}\left[ -\frac{c}{pz^c} \int_0^z t^{c+p}(I_p^m(n; \lambda, l)f(t))' \, dt \right] > \zeta, \quad z \in U,$$

where \(\zeta\) is given by (3.11).
According to (3.12), and taking in the above theorem $A = 1 - 2\alpha/p$, $B = -1$, and $m = 0$, we obtain the following special case.

**Corollary 3.11.** If $c > 0$ and if $f \in \sum_{p,n}$ satisfies the inequality
$$\text{Re}[−z^{p+1} f'(z)] > \alpha, \quad z \in U \ (0 \leq \alpha < p),$$

then
$$\text{Re}\left[−\frac{c}{z^c} \int_0^z t^{c+p} f'(t) \, dt\right] > \alpha + (p - \alpha) \left[2 F_1\left(1, 1, \frac{c}{p+n} + 1; \frac{1}{2}\right) - 1\right], \quad z \in U,$$

and the inequality is the best possible.

Using the technique of Srivastava and Patel [16, Theorem 4], we can prove the next theorem.

**Theorem 3.12.** Let the function $f \in \sum_{p,n}$, and suppose that $g \in \sum_{p,n}$ satisfies the inequality
$$\text{Re}[z^{p} I^m_p(n; \lambda, l) g(z)] > 0, \quad z \in U.$$

If
$$\left|\frac{I^m_p(n; \lambda, l) f(z)}{I^m_p(n; \lambda, l) g(z)} - 1\right| < 1, \quad z \in U \ (m \in \mathbb{N}_0, l, \lambda > 0),$$

then
$$\text{Re}\left[-\frac{z(I^m_p(n; \lambda, l) f(z))'}{I^m_p(n; \lambda, l) f(z)}\right] > 0,$$

for $|z| < R_0$, where
$$R_0 = \frac{\sqrt{g(p+n)^2 + 4p(2p+n)} - 3(p+n)}{2(2p+n)}. \quad (3.13)$$

**Proof.** Letting
$$w(z) = \frac{I^m_p(n; \lambda, l) f(z)}{I^m_p(n; \lambda, l) g(z)} - 1 = k_{p+n} z^{p+n} + k_{p+n+1} z^{p+n+1} + \cdots, \quad (3.14)$$

then $w$ is analytic in $U$, with $w(0) = 0$, $|w(z)| < 1$ for all $z \in U$, and $w(z) = k_{p+m} z^{p+m} + k_{p+m+1} z^{p+m+1} + \cdots$. Defining the function $\psi$ by
$$\psi(z) = \begin{cases} w(z), & z \in \hat{U}, \\ \frac{w(p+m)(0)}{(p+m)!}, & z = 0, \end{cases}$$

we have
$$\text{Re}\left[-\frac{z(I^m_p(n; \lambda, l) f(z))'}{I^m_p(n; \lambda, l) f(z)}\right] > 0,$$

for $|z| < R_0$. 

https://doi.org/10.1017/S0004972711002103 Published online by Cambridge University Press
then $\psi$ is analytic in $\hat{U}$ and continuous in $U$, hence it is analytic in the whole unit disc $U$. If $r \in (0, 1)$ is an arbitrary number, since $|w(z)| < 1$ for all $z \in U$, we deduce that

$$|\psi(z)| \leq \max_{|z|=r} \left| \frac{w(z)}{z^{p+m}} \right| \leq \max_{|z|=r} \frac{|w(z)|}{|z|^{p+m}} < \frac{1}{r^{p+m}}, \quad |z| \leq r < 1.$$ 

By letting $r \to 1^-$ in the above inequality, we get $|\psi(z)| < 1$ for all $z \in U$, that is, $w(z) = z^{p+n}\psi(z)$, where the function $\psi$ is analytic in $U$, and $|\psi(z)| < 1$, $z \in U$.

Therefore, (3.14) leads us to

$$I^m_p(n; \lambda, l) f(z) = I^m_p(n; \lambda, l) g(z)(1 + z^{p+n}\psi(z)), \quad z \in U,$$

and differentiating logarithmically the above relation, we obtain

$$\frac{z(I^m_p(n; \lambda, l) f(z))'}{I^m_p(n; \lambda, l) f(z)} = \frac{z(I^m_p(n; \lambda, l) g(z))'}{I^m_p(n; \lambda, l) g(z)} + \frac{z^{p+n}[(p + n)\psi(z) + z\psi'(z)]}{1 + z^{p+n}\psi(z)}. \tag{3.15}$$

Setting $\varphi(z) = z^n(I^m_p(n; \lambda, l) g(z))$, we see that the function $\varphi$ has the form (2.1), is analytic in $U$ with $\text{Re} \varphi(z) > 0$, for all $z \in U$, and

$$\frac{z(I^m_p(n; \lambda, l) g(z))'}{I^m_p(n; \lambda, l) g(z)} = \frac{z\varphi'(z)}{\varphi(z)} - p.$$ 

Hence, from (3.15) we find that

$$\text{Re} \left[ -\frac{z(I^m_p(n; \lambda, l) f(z))'}{I^m_p(n; \lambda, l) f(z)} \right] \geq p - \left| \frac{z\varphi'(z)}{\varphi(z)} \right| - \left| \frac{z^{p+n}[(p + n)\psi(z) + z\psi'(z)]}{1 + z^{p+n}\psi(z)} \right|. \tag{3.16}$$

Now, by using in (3.16) the known estimates (see [8])

$$\left| \frac{\varphi'(z)}{\varphi(z)} \right| \leq \frac{2(p + n)r^{p+n-1}}{1 - r^{2(p+n)}}, \quad |z| = r < 1,$$

$$\left| \frac{(p + n)\psi(z) + z\psi'(z)}{1 + z^{p+n}\psi(z)} \right| \leq \frac{p + n}{1 - r^{(p+n)}}, \quad |z| = r < 1,$$

we conclude that

$$\text{Re} \left[ -\frac{z(I^m_p(n; \lambda, l) f(z))'}{I^m_p(n; \lambda, l) f(z)} \right] \geq \frac{p - 3(p + n)r^{p+n} - (2p + n)r^{2(p+n)}}{1 - r^{2(p+n)}},$$

for $|z| = r < 1$, which is positive provided that $r < R_0$, where $R_0$ is given by (3.13). □

**Theorem 3.13.** Let $-1 \leq B_i < A_i \leq 1$, $i = 1, 2$, and suppose that each of the functions $f_i \in \sum_p$ satisfies the subordination condition

$$(1 - \beta)z^p I^m_p(\lambda, l) f_i(z) + \beta z^p I^{m+1}_p(\lambda, l) f_i(z) < \frac{1 + A_i z}{1 + B_i z}, \quad i = 1, 2, \tag{3.17}$$
where $I^m_p(\lambda, l) \equiv I^m_p(-p+1; \lambda, l)$. Then
\[
(1 - \beta)z^p I^m_p(\lambda, l)G(z) + \beta z^p I^{m+1}_p(\lambda, l)G(z) < \frac{1 + (1 - 2\eta)z}{1 - z},
\]
where
\[
G(z) = I^m_p(\lambda, p)(f_1 * f_2)(z)
\]
and
\[
\eta = 1 - \frac{4(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)}\left[1 - \frac{1}{2}F_1\left(1, 1, \frac{l}{\beta \lambda} + 1; \frac{1}{2}\right)\right].
\]
The result is the best possible when $B_1 = B_2 = -1$.

**Proof.** Since each of the functions $f_i \in \sum_p$, $i = 1, 2$, satisfies condition (3.17), then by letting
\[
\varphi_i(z) = (1 - \beta)z^p I^m_p(\lambda, l)f_i(z) + \beta z^p I^{m+1}_p(\lambda, l)f_i(z), \quad i = 1, 2,
\]
we have
\[
\varphi_i \in \mathcal{P}(\gamma_i) \quad \text{where} \quad \gamma_i = \frac{1 - A_i}{1 - B_i} (i = 1, 2).
\]
Using identity (1.2) in (3.18),
\[
I^m_p(\lambda, l)f_i(z) = \frac{l}{\beta \lambda}z^{-p-l/\beta \lambda} \int_0^z t^{(l/\beta \lambda) - 1}\varphi_i(t) \, dt, \quad i = 1, 2,
\]
which, according to the definition of $G$, yields
\[
I^m_p(\lambda, l)G(z) = \frac{l}{\beta \lambda}z^{-p-l/\beta \lambda} \int_0^z t^{(l/\beta \lambda) - 1}\varphi_0(t) \, dt,
\]
where
\[
\varphi_0(z) = (1 - \beta)z^p I^m_p(\lambda, l)G(z) + \beta z^p I^{m+1}_p(\lambda, l)G(z)
\]
\[
= \frac{l}{\beta \lambda}z^{-l/\beta \lambda} \int_0^z t^{(l/\beta \lambda) - 1}(\varphi_1 * \varphi_2)(t) \, dt. \tag{3.19}
\]
Since $\varphi_i \in \mathcal{P}(\gamma_i)$, $i = 1, 2$, it follows from Lemma 2.4 that
\[
\varphi_1 * \varphi_2 \in \mathcal{P}(\gamma_3) \quad \text{where} \quad \gamma_3 = 1 - 2(1 - \gamma_1)(1 - \gamma_2). \tag{3.20}
\]
By using (3.20) and (3.19), from Lemmas 2.3 and 2.6, we get
\[
\text{Re} \varphi_0(z) = \frac{l}{\beta \lambda}z^{-l/\beta \lambda} \int_0^1 u^{(l/\beta \lambda) - 1} \text{Re}(\varphi_1 * \varphi_2)(uz) \, du
\]
\[
\geq \frac{l}{\beta \lambda} \int_0^1 u^{(l/\beta \lambda) - 1}\left[2\gamma_3 - 1 + \frac{2(1 - \gamma_3)}{1 + u|z|}\right] du
\]
\[
> \frac{l}{\beta \lambda} \int_0^1 u^{(l/\beta \lambda) - 1}\left[2\gamma_3 - 1 + \frac{2(1 - \gamma_3)}{1 + u}\right] du
\]
\[ \frac{4(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)} \left[ 1 - \frac{1}{\beta \lambda} \int_0^1 u^{(l/\beta \lambda) - 1} (1 + u)^{-1} \, du \right] \]

and are defined by

\[ I_p^m(\lambda, l) f_i(z) = \frac{l}{\beta \lambda} z^{-l/\beta \lambda} \int_0^z t^{(l/\beta \lambda) - 1} \left( \frac{1 + A_i t}{1 - t} \right) \, dt, \quad i = 1, 2. \]

Thus, from (3.19) and Lemma 2.6, it follows that

\[ \varphi_0(z) = \frac{l}{\beta \lambda} \int_0^1 u^{(l/\beta \lambda) - 1} \left[ 1 - (1 + A_1)(1 + A_2) + \frac{(1 + A_1)(1 + A_2)}{1 - uz} \right] \, du \]

\[ = 1 - (1 + A_1)(1 + A_2) + (1 + A_1)(1 + A_2)(1 - z)^{-1} \times 2F1 \left( 1, 1, \frac{l}{\beta \lambda} + 1; \frac{z}{z - 1} \right) \]

\[ \to 1 - (1 + A_1)(1 + A_2) + \frac{1}{2}(1 + A_1)(1 + A_2)2F1 \left( 1, 1, \frac{l}{\beta \lambda} + 1; \frac{1}{2} \right), \]

as \( z \to -1 \), which completes the proof. \( \square \)

Taking \( A_i = 1 - 2\alpha_i, B_i = -1 (i = 1, 2), m = 0 \) and \( l = \lambda = 1 \) in Theorem 3.13, we obtain the following result which refines the work of Yang [20, Theorem 4].

**Corollary 3.14.** If the functions \( f_i \in \sum_p, i = 1, 2, \) satisfy the inequality

\[ \text{Re} \{(1 + \beta p)z^p f_i(z) + \beta z^{p+1} f_i'(z)\} > \alpha_i, \quad z \in U (0 \leq \alpha_i < 1, i = 1, 2), \quad (3.21) \]

then

\[ \text{Re} \{(1 + \beta p)z^p (f_1 * f_2)(z) + \beta z^{p+1} (f_1 * f_2)'(z)\} > \eta_0, \quad z \in U, \]

where

\[ \eta_0 = 1 - 4(1 - \alpha_1)(1 - \alpha_2) \left[ 1 - \frac{1}{2} 2F1 \left( 1, 1, \frac{1}{\beta} + 1; \frac{1}{2} \right) \right]. \]

The result is the best possible.

**Theorem 3.15.** If the function \( f \in \sum_{p,n} \) satisfies the subordination condition

\[ (1 - \beta)z^p I_p^m(n; \lambda, l) f(z) + \beta z^{p+1} I_p^{m+1}(n; \lambda, l) f(z) < \frac{1 + Az}{1 + Bz}, \]

then

\[ \text{Re} [z^p I_p^m(n; \lambda, l) f(z)]^{1/q} > \rho^{1/q}, \quad z \in U (q \in \mathbb{N}), \]

where \( \rho = Q(-1) \) is given as in Theorem 3.1. The result is the best possible.
PROOF. Defining the function $\varphi$ by

$$
\varphi(z) = z^p I_p^m(n; \lambda, l) f(z), 
$$

(3.22)

we see that $\varphi$ has the form (2.1) and is analytic in $U$. Using identity (1.2) in (3.22), and differentiating the resulting equation with respect to $z$, we obtain

$$(1 - \beta)z^p I_p^m(n; \lambda, l) f(z) + \beta z^p I_p^{m+1}(n; \lambda, l) f(z) = \varphi(z) + \frac{\beta \lambda}{l} z \varphi'(z) < \frac{1 + Az}{1 + Bz}.$$ 

Now, by following similar steps to the proof of Theorem 3.1, and using the elementary inequality

$$
\text{Re} w^{1/q} \geq (\text{Re} w)^{1/q}, \quad \text{Re} w > 0, \quad q \in \mathbb{N},
$$

we obtain the result asserted by Theorem 3.15.

From Corollary 3.14 and Theorem 3.15, for the special case $n = -p + 1, m = 0, A = 1 - 2\eta_0, B = -1$ and $q = 1$, we deduce the next result.

**Corollary 3.16.** Let the functions $f_i \in \sum_p (i = 1, 2)$, satisfy inequality (3.21). Then

$$
\text{Re}[z^p(f_1 * f_2)(z)] > \eta_0 + (1 - \eta_0) \left[ 2 F_1 \left( 1, 1, \frac{1}{2}; 1; 1 - 1 \right) \right], \quad z \in U,
$$

where $\eta_0$ is given as in Corollary 3.14. The result is the best possible.

**Theorem 3.17.** If the function $g \in \sum_{p,n}$ satisfies the inequality

$$
\text{Re}[z^p g(z)] > \frac{1}{2}, \quad z \in U,
$$

(3.23)

then, for any function $f \in \sum_{p,n}^m(\lambda, l; A, B)$, we have

$$
f * g \in \sum_{p,n}^m(\lambda, l; A, B).
$$

**Proof.** It is easy to check that

$$
-\frac{z^{p+1}(I_p^m(n; \lambda, l) f * g(z)')}{p} = \left[ -\frac{z^{p+1}(I_p^m(n; \lambda, l) f(z)')}{p} \right] * [z^p g(z)].
$$

According to this relation, by applying Lemma 2.5 for the functions

$$
F(z) = -\frac{z^{p+1}(I_p^m(n; \lambda, l) f(z)')}{p}
$$

and $\Phi(z) = z^p g(z)$, and using the fact that the function $h(z) = (1 + Az)/(1 + Bz)$ is convex (univalent) in $U$, we deduce the conclusion of the theorem. $\square$
References


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