DIFFERENTIAL SUBORDINATIONS FOR CLASSES OF MEROMORPHIC $p$-VALENT FUNCTIONS DEFINED BY MULTIPLIER TRANSFORMATIONS

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Abstract

We investigate several inclusion relationships and other interesting properties of certain subclasses of $p$-valent meromorphic functions, which are defined by using a certain linear operator, involving the generalized multiplier transformations.


Keywords and phrases: multiplier transformations, meromorphic functions, differential subordination.

1. Introduction

For $n > -p$, let $\sum_{p,n}$ denote the class of meromorphic functions of the form

$$f(z) = z^{-p} + \sum_{k=n}^{\infty} a_k z^k, \quad p \in \mathbb{N} = \{1, 2, 3, \ldots\},$$

which are analytic and $p$-valent in the punctured unit disc $\hat{U} = U \setminus \{0\}$, where $U = \{z \in \mathbb{C} : |z| < 1\}$. For convenience, we write $\sum_p = \sum_{-p+1}^{p}$.

If $f$ and $g$ are two analytic functions in $U$, we say that $f$ is subordinate to $g$, written symbolically as $f(z) \prec g(z)$, if there exists a Schwarz function $w$, which (by definition) is analytic in $U$ with $w(0) = 0$, and $|w(z)| < 1$, $z \in U$, such that $f(z) = g(w(z))$ for all $z \in U$.

It is well known that, if $f(z) \prec g(z)$, then $f(0) = g(0)$ and $f(U) \subset g(U)$. Further, if the function $g$ is univalent in $U$, then we have the following equivalence (see [9]; see also [10, p. 4]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(U) \prec g(U).$$

For the functions $f_j \in \sum_{p,n}$, $j = 1, 2$, given by

$$f_j(z) = z^{-p} + \sum_{k=n}^{\infty} a_{k,j} z^k,$$
we define the Hadamard (or convolution) product of $f_1$ and $f_2$ by
\[(f_1 \ast f_2)(z) = z^{-p} + \sum_{k=n}^{\infty} a_{k,1}a_{k,2}z^n.\]

Define the linear operator $I^m_p(n; \lambda, l) : \sum_{p,n} \rightarrow \sum_{p,n}$, where $\lambda \geq 0$, $l > 0$, and $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, by
\[I^m_p(n; \lambda, l)f(z) = z^{-p} + \sum_{k=n}^{\infty} \left[ \frac{\lambda(k + p) + l}{l} \right]^m a_k z^k. \tag{1.1}\]

Then, we can write (1.1) as
\[I^m_p(n; \lambda, l)f(z) = (\Phi_{p,m}^n \ast f)(z), \]
where
\[\Phi_{p,m}^n(z) = z^{-p} + \sum_{k=n}^{\infty} \left[ \frac{\lambda(k + p) + l}{l} \right]^m z^k. \]

Using definition (1.1), it is easy to verify that the next formula holds for $\lambda > 0$:
\[\lambda z(I^m_p(n; \lambda, l)f(z))' = II^{m+1}_p(n; \lambda, l)f(z) - (\lambda p + l)I^m_p(n; \lambda, l)f(z). \tag{1.2}\]

**Remark 1.1.** (1) We note that $I^0_p(n; \lambda, l)f = f$ and
\[I^1_p(n; 1, 1)f(z) = \frac{(z^{p+1}f(z))'}{z^p} = (p + 1)f(z) + zf'(z). \]

(2) For some special values of the parameters $\lambda, l, m$ and $p$, we obtain the following operators studied by various authors:
(i) $I^m_p(n; 1, l) = I^m_p(n, l)$ (see Cho et al. [2]);
(ii) $I^m_p(n; 1, 1) = D^m_p$ (see Aouf and Hossen [1], and Liu and Srivastava [6]);
(iii) $I^m_1(0; 1, l) = D^m_l$ (see Cho et al. [3, 4]);
(iv) $I^m_1(0; 1, 1) = D^m_l$ (see Uralbegadd and Somanatha [18]).

Using differential subordinations as well as the linear operator $I^m_p(n; \lambda, l)$, we will introduce a subclass of $\sum_{p,n}$, as follows.

**Definition 1.2.** (1) For the fixed parameters $A$ and $B$, with $-1 \leq B < A \leq 1$, we say that a function $f \in \sum_{p,n}$ is in the class $\sum_{p,n}(\lambda, l; A, B)$, if it satisfies the subordination condition
\[-\frac{z^{p+1}(I^m_p(n; \lambda, l)f(z))'}{p} < \frac{1 + Az}{1 + Bz}, \quad l > 0, m \in \mathbb{N}_0, n > -p. \tag{1.3}\]

(2) For convenience, we write
\[\sum_{p,n}(\lambda, l; \alpha) \equiv \sum_{p,n}(\lambda, l; 1 - \frac{2\alpha}{p}, -1), \quad 0 \leq \alpha < p. \]
that is, \( \sum_{p,n}^m (\lambda, l; \alpha) \) denotes the class of functions \( f \in \sum_{p,n} \) satisfying
\[
\Re\{-z^{p+1} (I_p^m(n; \lambda, l) f(z))' \} > \alpha, \quad z \in U.
\]

**Remark 1.3.** We have the next special cases of \( \sum_{p,n}^m (\lambda, l; A, B) \), studied previously by different authors:

(i) \( \sum_{p,n}^m (1, 1; A, B) = R_{m,p}(A, B) \) (see Liu and Srivastava [6]);

(ii) \( \sum_{p,n}^m (1, 1; A, B) = \sum_{p,n}^m (A, B) \) (see Srivastava and Patel [16]);

(iii) \( \sum_{p,n}^m (0, 1; A, B) = H(p; A, B) \) (see Mogra [11, 12]);

(iv) \( \sum_{p,n}^m (1, l; A, B) = \sum_{p,n}^m (A, B) \), where \( \sum_{p,n}^m (A, B) \) is the class of functions \( f \in \sum_{p,n} \), satisfying
\[
- \frac{z^{p+1} (I_p^m(n, l) f(z))'}{p} \leq \frac{1 + Az}{1 + Bz}, \quad l > 0, m \in \mathbb{N}_0, n > -p,
\]
and \( I_p^m(n, l) \equiv I_p^m(n; 1, l) \).

In the present paper we obtain several inclusion relationships for the function class \( \sum_{p,n}^m (\lambda, l; A, B) \), and we investigate various other properties of functions belonging to the class \( \sum_{p,n}^m (\lambda, l; A, B) \). Relevant connections of the results presented in this paper with those obtained in earlier works are also pointed out.

### 2. Preliminaries

To establish our main results, we will need the following lemmas and definition.

**Lemma 2.1** [5]. Let the function \( h \) be convex (univalent) in \( U \), with \( h(0) = 1 \). Suppose also that the function \( \varphi \) given by
\[
\varphi(z) = 1 + c_{p+n}z^{p+n} + c_{p+n+1}z^{p+n+1} + \cdots
\]
(2.1)
is analytic in \( U \). Then
\[
\varphi(z) + \frac{z\varphi'(z)}{\delta} < h(z), \quad \Re\delta \geq 0, \delta \neq 0,
\]
implies that
\[
\varphi(z) < \psi(z) = \frac{\delta}{p+n} z^{-\delta/(p+n)} \int_0^z t^{\delta/(p+n) - 1} h(t) \, dt < h(z),
\]
(2.2)
and \( \psi \) is the best dominant of (2.2).

**Definition 2.2.** We denote by \( \mathcal{P}(\gamma) \) the class of functions \( \varphi \) given by
\[
\varphi(z) = 1 + b_1z + b_2z^2 + \cdots
\]
(2.3)
which are analytic in \( U \) and satisfy the inequality
\[
\Re\varphi(z) > \gamma, \quad z \in U \ (0 \leq \gamma < 1).
\]
Lemma 2.3 [14]. Let the function \( \varphi \) given by (2.3) be in the class \( \mathcal{P}(\gamma) \). Then
\[
\text{Re} \, \varphi(z) \geq 2\gamma - 1 + \frac{2(1 - \gamma)}{1 + |z|}, \quad z \in U \quad (0 \leq \gamma < 1).
\]

Lemma 2.4 [17]. For \( 0 \leq \gamma_1 < \gamma_2 < 1 \), the inclusion
\[
\mathcal{P}(\gamma_1) \ast \mathcal{P}(\gamma_2) \subset \mathcal{P}(\gamma_3)
\]
holds and the result is the best possible. The symbol ‘\( \ast \)’ stands for the previous mentioned Hadamard product of the power series.

Lemma 2.5 [15]. Let \( \Phi \) be an analytic function in \( U \), with \( \Phi(0) = 1 \) and \( \text{Re} \, \Phi(z) > 1/2, \quad z \in U \). Then, for any function \( F \) analytic in \( U \), the set \( (\Phi \ast F)(U) \) is contained in the convex hull of \( F(U) \), that is, \( (\Phi \ast F)(U) \subset \text{co} \, F(U) \).

Lemma 2.6 [19]. For all real or complex numbers \( \alpha_1, \alpha_2, \beta_1 \), where \( \beta_1 \notin \mathbb{Z}^- = \{0, -1, -2, \ldots\} \),
\[
\int_0^1 t^{\alpha_2 - 1}(1 - t)^{\beta_1 - \alpha_2 - 1}(1 - zt)^{-\alpha_1} \, dt
= \frac{\Gamma(\alpha_2)\Gamma(\beta_1 - \alpha_2)}{\Gamma(\beta_1)} \, {}_2F_1(\alpha_1, \alpha_2, \beta_1; z) \quad \text{for} \quad \text{Re} \, \beta_1 > \text{Re} \, \alpha_2 > 0,
\]
\[
= \frac{\Gamma(\alpha_2)\Gamma(\beta_1 - \alpha_2)}{\Gamma(\beta_1)} \, {}_2F_1(\alpha_1, \alpha_2, \beta_1; z) = {}_2F_1(\alpha_2, \alpha_1, \beta_1; z),
\]
\[
{}_2F_1(\alpha_1, \alpha_2, \beta_1; z) = (1 - z)^{-\alpha_1} \, {}_2F_1\left(\alpha_1, \beta_1 - \alpha_2, \beta_1; \frac{z}{z - 1}\right),
\]
and
\[
{}_2F_1\left(\alpha_1, \alpha_2, \frac{\alpha_1 + \alpha_2 + 1}{2}; \frac{1}{2}\right) = \frac{\sqrt{\pi} \Gamma\left(\frac{\alpha_1 + \alpha_2 + 1}{2}\right)}{\Gamma\left(\frac{\alpha_1 + 1}{2}\right)\Gamma\left(\frac{\alpha_2 + 1}{2}\right)},
\]
where \( {}_2F_1 \) represents the Gauss hypergeometric function.

3. Subordination theorems and the associated functional inequalities

Unless otherwise mentioned, we shall assume throughout the paper that \( n \) is an integer with \( n > -p \), that \(-1 \leq B < A \leq 1, \lambda, \ l > 0, \ m \in \mathbb{N}_0, \beta > 0, \) and \( p \in \mathbb{N} \).

Theorem 3.1. If the function \( f \in \sum_{p, n} \) satisfies the subordination condition
\[
- \frac{(1 - \beta)z^{p + 1}(I_p^m(n; \lambda, l) \, f(z))' + \beta z^{p + 1}(I_p^{m + 1}(n; \lambda, l) \, f(z))'}{p} < \frac{1 + Az}{1 + Bz},
\]
then
\[
- \frac{z^{p + 1}(I_p^m(n; \lambda, l) \, f(z))'}{p} < Q(z) < \frac{1 + Az}{1 + Bz},
\]
where the function $Q$ is given by

$$
Q(z) = \begin{cases}
\frac{A}{B} + \left(1 - \frac{A}{B}\right)(1 + Bz)^{-1} \left(1, 1, \frac{l}{\lambda \beta(p + n)} + 1; \frac{Bz}{1 + Bz}\right), & B \neq 0, \\
1 + \frac{l}{\lambda \beta(p + n) + l} Az, & B = 0,
\end{cases}
$$

and it is the best dominant of (3.1).

Furthermore, for all $k \in \mathbb{N}$, we have

$$
\text{Re} \left[ - \frac{z^{p+1}(I_p^m(n; \lambda, l)f(z))'}{p} \right]^{1/k} > \rho^{1/k}, \quad z \in U,
$$

where $\rho = Q(-1)$, and the inequality (3.2) is the best possible.

**Proof.** If we consider the function $\varphi$ defined by

$$
\varphi(z) = - \frac{z^{p+1}(I_p^m(n; \lambda, l)f(z))'}{p},
$$

then $\varphi$ has the form (2.1) and is analytic in $U$. Applying the identity (1.2) in (3.3), and differentiating the resulting equation with respect to $z$, we get

$$
\left(1 - \beta\right)z^{p+1}(I_p^m(n; \lambda, l)f(z))' + \beta z^{p+1}(I_p^{m+1}(n; \lambda, l)f(z))' = \varphi(z) + \frac{\beta\lambda}{l} z\varphi'(z) < \frac{1 + Az}{1 + Bz}.
$$

Now by using Lemma 2.1 for $\gamma = l/(\lambda \beta)$, we deduce that

$$
- \frac{z^{p+1}(I_p^m(n; \lambda, l)f(z))'}{p} < Q(z)
$$

$$
= \frac{l}{\lambda \beta(p + n)} \int_0^z t^{(l/\lambda \beta(p+n))-1} \frac{1 + At}{1 + Bt} dt,
$$

$$
= \begin{cases}
\frac{A}{B} + \left(1 - \frac{A}{B}\right)(1 + Bz)^{-1} \left(1, 1, \frac{l}{\lambda \beta(p + n)} + 1; \frac{Bz}{1 + Bz}\right), & B \neq 0, \\
1 + \frac{l}{\lambda \beta(p + n) + l} Az, & B = 0,
\end{cases}
$$

where we made a changes of variables, followed by the use of the identities (2.4), (2.5), and (2.6) (with $b = 1$ and $c = a + 1$). Hence, assertion (3.1) is proved.

In order to prove assertion (3.2), it is sufficient to show that

$$
\inf\{\text{Re} \ Q(z) : |z| < 1\} = Q(-1).
$$
Indeed, for $|z| \leq r < 1$,

$$\text{Re} \left( \frac{1 + Az}{1 + Bz} \right) \geq \frac{1 - Ar}{1 - Br}, \quad |z| \leq r < 1.$$ 

Setting

$$G(s, z) = \frac{1 + Az}{1 + Bs}$$

and

$$d\nu(s) = \frac{l}{\lambda \beta (p + n)} s^{\lambda \beta (p + n)} ds, \quad 0 \leq s \leq 1,$$

which is a positive measure on $[0, 1]$, we get

$$Q(z) = \int_0^1 G(s, z) d\nu(s),$$

so that

$$\text{Re} \ Q(z) \geq \int_0^1 \frac{1 - Asr}{1 - Bsr} \ d\nu(s) = Q(-r), \quad |z| \leq r < 1.$$ 

Letting $r \to 1^-$ in the above inequality, and using the elementary inequality

$$\text{Re} \ w^{1/k} \geq (\text{Re} \ w)^{1/k}, \quad \text{Re} \ w > 0, \ k \in \mathbb{N},$$

we obtain (3.2). Finally, inequality (3.2) is the best possible, as the function $Q$ is the best dominant of (3.1). \hfill \Box

**Remark 3.2.** Putting $\lambda = l = 1$ in Theorem 3.1, we obtain the result of Srivastava and Patel [16, Theorem 1].

For $\lambda = l = 1$, $n = 0$, and $\beta = 1$, Theorem 3.1 yields the following result, which improves the corresponding one of Liu and Srivastava [7, Theorem 1].

**Corollary 3.3.** The inclusions

$$R_{m+1,p}(A, B) \subset R_{m,p}(A, B) \subset R_{m,p}(1 - 2\rho, -1)$$

hold, where

$$\rho = \begin{cases} 
\frac{A}{B} + \left( 1 - \frac{A}{B} \right) (1 - B)^{-1} F_1 \left( \frac{1}{p + 1}; \frac{B}{B - 1} \right), & B \neq 0, \\
1 - \frac{A}{p + 1}, & B = 0,
\end{cases}$$

and the result is the best possible.

Putting $A = 1 - 2\alpha/p$, $B = -1$, $\beta = \lambda = l = 1$, $m = 0$ and $n = -p + 2$ in Theorem 3.1, and using (2.7), we get the following result.
Corollary 3.4. If the function \( f \in \sum_{p,-p+2} \) satisfies the inequality
\[
\Re[-z^{p+1}((p + 2)f'(z) + zf''(z))] > \alpha, \quad z \in U \quad (0 \leq \alpha < p),
\]
then
\[
\Re[-z^{p+1}f'(z)] > \alpha + (p - \alpha)\left(\frac{\pi}{2} - 1\right), \quad z \in U,
\]
and the result is the best possible.

Remark 3.5. Taking \( \alpha = -p(\pi - 2)/(4 - \pi) \) in the above corollary, we obtain that if the function \( f \in \sum_{p,-p+2} \) satisfies
\[
\Re[-z^{p+1}((p + 2)f'(z) + zf''(z))] > -\frac{p(\pi - 2)}{4 - \pi}, \quad z \in U,
\]
then \( \Re[-z^{p+1}f'(z)] > 0, z \in U \) (see Pap [13]).

Theorem 3.6. If the function \( f \in \sum_{p,n}(\lambda, l; \alpha), 0 \leq \alpha < p, \) then
\[
\Re[-z^{p+1}((1 - \beta)(I_p^m(n; \lambda, l)f(z))' + \beta(I_p^{m+1}(n; \lambda, l)f(z))') + \alpha] > \alpha,
\]
for \( |z| < R \), where
\[
R = \left[1 + \left(\frac{\beta\lambda}{l}\right)^2(p + n)^2 - \frac{\beta\lambda}{l}(p + n)\right]^{1/(p+n)}.
\]

The result is the best possible.

Proof. If we let
\[
-z^{p+1}(I_p^m(n; \lambda, l)f(z))' = \alpha + (p - \alpha)\varphi(z),
\]
then \( \varphi \) has the form (2.1), and is analytic with positive real part in \( U \). Using the identity (1.2) in (3.5), and differentiating the resulting equation with respect to \( z \),
\[
\frac{z^{p+1}[(1 - \beta)(I_p^m(n; \lambda, l)f(z))' + \beta(I_p^{m+1}(n; \lambda, l)f(z))'] + \alpha}{p - \alpha} = \varphi(z) + \frac{\beta\lambda}{l}z\varphi'(z).
\]

Applying in (3.6) the estimate (see [8])
\[
\frac{|z\varphi'(z)|}{\Re \varphi(z)} \leq \frac{2(p + n)r^{p+n}}{1 - r^{2(p+n)}}, \quad |z| = r < 1,
\]
we get
\[
\Re\left\{\frac{z^{p+1}[(1 - \beta)(I_p^m(n; \lambda, l)f(z))' + \beta(I_p^{m+1}(n; \lambda, l)f(z))'] + \alpha}{p - \alpha}\right\} \geq \left[1 - \frac{2\beta\lambda(p + n)r^{p+n}}{l(1 - r^{2(p+n)})}\right]\Re \varphi(z).
\]
and it is easy to see that the right-hand side of (3.7) is positive, provided that \( r < R \), where \( R \) is given by (3.4).

In order to show that the bound \( R \) is the best possible, we consider the function \( f \in \sum p, n \) defined by

\[
-z^{p+1}(I_p^m(n; \lambda, l)f(z))' = \alpha + (p - \alpha) \frac{1 + z^{p+n}}{1 - z^{p+n}}.
\]

Then

\[
-z^{p+1}[(1 - \beta)(I_p^m(n; \lambda, l)f(z))' + \beta(I_p^{m+1}(n; \lambda, l)f(z))'] + \alpha = \frac{p - \alpha}{1 - z^{2(p+n)} + \frac{2\beta}{l}(p + n)z^{p+n}} = 0,
\]

for \( z = R \exp(i\pi/(p + n)) \), which completes the proof of the theorem.

**Remark 3.7.** Putting \( \lambda = l = 1 \) in Theorem 3.6, we obtain the result of Srivastava and Patel [16, Theorem 2].

For \( \beta = 1 \), Theorem 3.6 reduces to the following result.

**Corollary 3.8.** If the function \( f \in \sum m, (\lambda, l; \alpha) \), \( 0 \leq \alpha < p \), then

\[
f \in \sum m+1 (\lambda, l; \alpha)
\]

for \(|z| < \tilde{R}\), where

\[
\tilde{R} = \left[ \alpha + \left( \frac{\lambda}{l} \right)^2 (p + n)^2 - \frac{\lambda}{l}(p + n) \right]^{1/(p+n)}\]

The result is the best possible.

**Theorem 3.9.** Let \( f \in \sum m, (\lambda, l; A, B) \), and let

\[
F_{p,c}(f)(z) = \frac{c}{z^{c+p}} \int_0^z t^{c+p-1} f(t) \, dt, \quad c > 0.
\]

Then

\[
-\frac{z^{p+1}(I_p^m(n; \lambda, l)F_{p,c}(f)(z))'}{p} < \Theta(z) < \frac{1 + Az}{1 + Bz},
\]

where \( \Theta \) is defined by

\[
\Theta(z) = \begin{cases} 
\frac{A}{B} + \left(1 - \frac{A}{B}\right)(1 + Bz)^{-1} F_1\left(1, 1, \frac{c}{p + n} + 1; \frac{Bz}{1 + Bz}\right), & B \neq 0, \\
1 + \frac{Ac}{c + p + n}z, & B = 0,
\end{cases}
\]

and it is the best dominant of (3.9).
Furthermore,
\[
\text{Re} \left[ -\frac{z^{p+1}(I_p^m(n; \lambda, l)F_{p,c}(f)(z))'}{p} \right] > k, \quad z \in U,
\]
where \( k = \Theta(-1) \), and this inequality is the best possible.

**Proof.** Setting
\[
\varphi(z) = -\frac{z^{p+1}(I_p^m(n; \lambda, l)F_{p,c}(f)(z))'}{p},
\]
then \( \varphi \) has the form (2.1), and is analytic in \( U \). Using in (3.10) the operator identity
\[
z(I_p^m(n; \lambda, l)F_{p,c}(f)(z))' = cI_p^m(n; \lambda, l)f(z) - (c + p)(I_p^m(n; \lambda, l)F_{p,c}(f)(z)),
\]
and differentiating the resulting equation with respect to \( z \), we find that
\[
-\frac{z^{p+1}(I_p^m(n; \lambda, l)f(z))'}{p} = \varphi(z) + \frac{z\varphi'(z)}{c} < \frac{1 + Az}{1 + Bz}.
\]

Now, the remaining part of the proof follows by employing the same techniques that we used in the proof of Theorem 3.1. \( \square \)

**Remark 3.10.** (1) Setting \( n = 0 \) and \( l = \lambda = 1 \) in Theorem 3.9, we obtain the following result which improves the corresponding work of Liu and Srivastava [7, Theorem 2]. If \( c > 0 \) and \( f \in R_{m,p}(A, B) \), then
\[
F_{p,c}(R_{m,p}(A, B)) \subset R_{m,p}(1 - 2\zeta, -1) \subset R_{m,p}(A, B),
\]
where
\[
\zeta = \begin{cases} 
\frac{A}{B} + \left(1 - \frac{A}{B}\right)(1 - B)^{-1}F_1 \left(1, 1, \frac{c}{p} + 1; \frac{B}{B - 1}\right), & B \neq 0, \\
1 - \frac{Ac}{c + p}, & B = 0.
\end{cases}
\]
The result is the best possible.

(2) Observing that
\[
z^{p+1}(I_p^m(n; \lambda, l)F_{p,c}(f)(z))' = \frac{c}{z^{c+1}} \int_0^z t^{c+p}(I_p^m(n; \lambda, l)f(t))' \, dt,
\]
whenever \( f \in \sum_{p,n} \) and \( c > 0 \), the above remark can be restated as follows. If \( c > 0 \) and \( f \in R_{m,p}(A, B) \), then
\[
\text{Re} \left[ -\frac{c}{pz^{c+1}} \int_0^z t^{c+p}(I_p^m(n; \lambda, l)f(t))' \, dt \right] > \zeta, \quad z \in U,
\]
where \( \zeta \) is given by (3.11).
According to (3.12), and taking in the above theorem $A = 1 - 2\alpha/p$, $B = -1$, and $m = 0$, we obtain the following special case.

**Corollary 3.11.** If $c > 0$ and if $f \in \sum_{p,n}$ satisfies the inequality

$$\text{Re}[-z^{p+1} f'(z)] > \alpha, \quad z \in U \ (0 \leq \alpha < p),$$

then

$$\text{Re} \left[ -\frac{c}{z^c} \int_0^z t^{c+p} f'(t) dt \right] > \alpha + (p - \alpha) \left[ \text{Re} \left( \frac{c}{p} + 1 : \frac{1}{2} \right) - 1 \right], \quad z \in U,$$

and the inequality is the best possible.

Using the technique of Srivastava and Patel [16, Theorem 4], we can prove the next theorem.

**Theorem 3.12.** Let the function $f \in \sum_{p,n}$, and suppose that $g \in \sum_{p,n}$ satisfies the inequality

$$\text{Re} \left[ z^p I^m_p (n; \lambda, l) g(z) \right] > 0, \quad z \in U.$$

If

$$\left| \frac{I^m_p (n; \lambda, l) f(z)}{I^m_p (n; \lambda, l) g(z)} - 1 \right| < 1, \quad z \in U \ (m \in \mathbb{N}_0, l, \lambda > 0),$$

then

$$\text{Re} \left[ -\frac{z (I^m_p (n; \lambda, l) f(z))'}{I^m_p (n; \lambda, l) f(z)} \right] > 0,$$

for $|z| < R_0$, where

$$R_0 = \frac{\sqrt{g(p+n)^2 + 4p(2p+n) - 3(p+n)}}{2(2p+n)}. \quad (3.13)$$

**Proof.** Letting

$$w(z) = \frac{I^m_p (n; \lambda, l) f(z)}{I^m_p (n; \lambda, l) g(z)} - 1 = k_{p+n} z^{p+n} + k_{p+n+1} z^{p+n+1} + \cdots, \quad (3.14)$$

then $w$ is analytic in $U$, with $w(0) = 0$, $|w(z)| < 1$ for all $z \in U$, and $w(z) = k_{p+m} z^{p+m} + k_{p+m+1} z^{p+m+1} + \cdots$. Defining the function $\psi$ by

$$\psi(z) = \begin{cases} w(z), & z \in \hat{U}, \\ z^{p+m}, & z = 0, \end{cases}$$

we have $\psi(0) = 0$. Applying the above inequality to $\psi(z)/z^{p+m}$ results in

$$\text{Re} \left[ -\frac{\psi'(z)}{\psi(z)} \right] > 0,$$

for $|z| < R_0$, where

$$R_0 = \frac{\sqrt{g(p+n)^2 + 4p(2p+n) - 3(p+n)}}{2(2p+n)}. \quad (3.13)$$
then \( \psi \) is analytic in \( \hat{U} \) and continuous in \( U \), hence it is analytic in the whole unit disc \( U \). If \( r \in (0, 1) \) is an arbitrary number, since \(|w(z)| < 1\) for all \( z \in U \), we deduce that

\[
|\psi(z)| \leq \max_{|z|=r} \frac{|w(z)|}{z^{p+m}} \leq \max_{|z|=r} \frac{|w(z)|}{|z|^p} < \frac{1}{r^{p+m}}, \quad |z| \leq r < 1.
\]

By letting \( r \to 1^- \) in the above inequality, we get \(|\psi(z)| < 1\) for all \( z \in U \), that is, \( w(z) = z^{p+n} \psi(z) \), where the function \( \psi \) is analytic in \( U \), and \(|\psi(z)| < 1\), \( z \in U \).

Therefore, \((3.14)\) leads us to

\[
I_p^m (n; \lambda, l) \psi(z) = I_p^m (n; \lambda, l) g(z)(1 + z^{p+n} \psi(z)), \quad z \in U,
\]

and differentiating logarithmically the above relation, we obtain

\[
\frac{z(I_p^m (n; \lambda, l) f(z))'}{I_p^m (n; \lambda, l) f(z)} = \frac{z(I_p^m (n; \lambda, l) g(z))'}{I_p^m (n; \lambda, l) g(z)} + \frac{z^{p+n}[((p + n) \psi(z) + z \psi'(z)]}{1 + z^{p+n} \psi(z)}.
\]

Setting \( \varphi(z) = z^n I_p^m (n; \lambda, l) g(z) \), we see that the function \( \varphi \) has the form \((2.1)\), is analytic in \( U \) with \( \Re \varphi(z) > 0 \), for all \( z \in U \), and

\[
\frac{z(I_p^m (n; \lambda, l) g(z))'}{I_p^m (n; \lambda, l) g(z)} = \frac{z \varphi'(z)}{\varphi(z)} = p.
\]

Hence, from \((3.15)\) we find that

\[
\Re \left[ -\frac{z(I_p^m (n; \lambda, l) f(z))'}{I_p^m (n; \lambda, l) f(z)} \right] \geq p - \left| \frac{z \varphi'(z)}{\varphi(z)} - \frac{z^{p+n}[(p + n) \psi(z) + z \psi'(z)]}{1 + z^{p+n} \psi(z)} \right|.
\]

Now, by using in \((3.16)\) the known estimates (see \[8\])

\[
\left| \frac{\varphi'(z)}{\varphi(z)} \right| \leq \frac{2(p + n) r^{p+n-1}}{1 - r^{2(p+n)}}, \quad |z| = r < 1,
\]

\[
\left| \frac{(p + n) \psi(z) + z \psi'(z)}{1 + z^{p+n} \psi(z)} \right| \leq \frac{p + n}{1 - r^{p+n}}, \quad |z| = r < 1,
\]

we conclude that

\[
\Re \left[ -\frac{z(I_p^m (n; \lambda, l) f(z))'}{I_p^m (n; \lambda, l) f(z)} \right] \geq \frac{p - 3(p + n) r^{p+n} - (2p + n) r^{2(p+n)}}{1 - r^{2(p+n)}},
\]

for \(|z| = r < 1\), which is positive provided that \( r < R_0 \), where \( R_0 \) is given by \((3.13)\).

**Theorem 3.13.** Let \(-1 \leq B_i < A_i \leq 1, \quad i = 1, 2\), and suppose that each of the functions \( f_i \in \sum_p \) satisfies the subordination condition

\[
(1 - \beta)z^p I_p^m (\lambda, l) f_i(z) + \beta z^p I_p^{m+1} (\lambda, l) f_i(z) < \frac{1 + A_i z}{1 + B_i z}, \quad i = 1, 2.
\]
where $I_p^m(\lambda, l) \equiv I_p^m(-p + 1; \lambda, l)$. Then

$$\left(1 - \beta\right)z^pI_p^m(\lambda, l)G(z) + \beta z^pI_p^{m+1}(\lambda, l)G(z) < \frac{1 + (1 - 2\eta)z}{1 - z},$$

where

$$G(z) = I_p^m(\lambda, p)(f_1 * f_2)(z)$$

and

$$\eta = 1 - \frac{4(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)} \left[ 1 - \frac{1}{2}F_1 \left(1, 1, \frac{l}{\beta\lambda} + 1; \frac{1}{2} \right) \right].$$

The result is the best possible when $B_1 = B_2 = -1$.

**Proof.** Since each of the functions $f_i \in \sum_p, i = 1, 2$, satisfies condition (3.17), then by letting

$$\varphi_i(z) = (1 - \beta)z^pI_p^m(\lambda, l)f_i(z) + \beta z^pI_p^{m+1}(\lambda, l)f_i(z), \quad i = 1, 2,$$

(3.18)

we have

$$\varphi_i \in \mathcal{P}(\gamma_i) \quad \text{where} \quad \gamma_i = \frac{1 - A_i}{1 - B_i} (i = 1, 2).$$

Using identity (1.2) in (3.18),

$$I_p^m(\lambda, l)f_i(z) = \frac{1}{\beta\lambda}z^{-p-l/\beta\lambda}\int_0^z t^{(l/\beta\lambda) - 1}\varphi_i(t)\,dt, \quad i = 1, 2,$$

which, according to the definition of $G$, yields

$$I_p^m(\lambda, l)G(z) = \frac{1}{\beta\lambda}z^{-p-l/\beta\lambda}\int_0^z t^{(l/\beta\lambda) - 1}\varphi_0(t)\,dt,$$

where

$$\varphi_0(z) = (1 - \beta)z^pI_p^m(\lambda, l)G(z) + \beta z^pI_p^{m+1}(\lambda, l)G(z)
\quad = \frac{1}{\beta\lambda}z^{-l/\beta\lambda}\int_0^z t^{(l/\beta\lambda) - 1} (\varphi_1 * \varphi_2)(t)\,dt.$$  \hspace{1cm} (3.19)

Since $\varphi_i \in \mathcal{P}(\gamma_i), i = 1, 2$, it follows from Lemma 2.4 that

$$\varphi_1 * \varphi_2 \in \mathcal{P}(\gamma_3) \quad \text{where} \quad \gamma_3 = 1 - 2(1 - \gamma_1)(1 - \gamma_2).$$  \hspace{1cm} (3.20)

By using (3.20) and (3.19), from Lemmas 2.3 and 2.6, we get

\[
\text{Re } \varphi_0(z) = \frac{1}{\beta\lambda}z^{-l/\beta\lambda}\int_0^1 u^{(l/\beta\lambda) - 1} \text{Re}(\varphi_1 * \varphi_2)(uz)\,du \geq \frac{l}{\beta\lambda}\int_0^1 u^{(l/\beta\lambda) - 1} \left[ 2\gamma_3 - 1 + \frac{2(1 - \gamma_3)}{1 + u|z|} \right] du \\
> \frac{l}{\beta\lambda}\int_0^1 u^{(l/\beta\lambda) - 1} \left[ 2\gamma_3 - 1 + \frac{2(1 - \gamma_3)}{1 + u} \right] du
\]
assumptions (3.17) and are defined by

\[
I^m_p(\alpha, l) f_i(z) = \frac{l}{\beta \lambda} z^{-l/\beta \lambda} \int_0^z \left( \frac{1 + A_i t}{1 - t} \right) dt, \quad i = 1, 2.
\]

Thus, from (3.19) and Lemma 2.6, it follows that

\[
\varphi_0(z) = \frac{l}{\beta \lambda} \int_0^1 u^{(l/\beta \lambda) - 1} \left[ 1 - (1 + A_1)(1 + A_2) + \frac{(1 + A_1)(1 + A_2)}{1 - uz} \right] du
\]

\[
= 1 - (1 + A_1)(1 + A_2) + (1 + A_1)(1 + A_2)(1 - z)^{-1}
\times 2 F_1 \left( 1, 1, \frac{l}{\beta \lambda} + 1; \frac{z}{1 - t} \right)
\]

\[
\rightarrow 1 - (1 + A_1)(1 + A_2) + \frac{1}{2} (1 + A_1)(1 + A_2) 2 F_1 \left( 1, 1, \frac{l}{\beta \lambda} + 1; \frac{1}{2} \right),
\]

as \( z \rightarrow -1 \), which completes the proof. \( \square \)

Taking \( A_i = 1 - 2\alpha_i, B_i = -1 \ (i = 1, 2) \), \( m = 0 \) and \( l = \lambda = 1 \) in Theorem 3.13, we obtain the following result which refines the work of Yang [20, Theorem 4].

**Corollary 3.14.** If the functions \( f_i \in \sum_p \), \( i = 1, 2 \), satisfy the inequality

\[
\text{Re}\{(1 + \beta p) z^p f_i(z) + \beta z^{p + 1} f_i'(z)\} > \alpha_i, \quad z \in U \ (0 \leq \alpha_i < 1, i = 1, 2), \quad (3.21)
\]

then

\[
\text{Re}\{(1 + \beta p) z^p (f_1 * f_2)(z) + \beta z^{p + 1} (f_1 * f_2)(z)\} > \eta_0, \quad z \in U,
\]

where

\[
\eta_0 = 1 - 4(1 - \alpha_1)(1 - \alpha_2) \left[ 1 - \frac{1}{2} 2 F_1 \left( 1, 1, \frac{l}{\beta \lambda} + 1; \frac{1}{2} \right) \right].
\]

The result is the best possible.

**Theorem 3.15.** If the function \( f \in \sum_{p,n} \) satisfies the subordination condition

\[
(1 - \beta) z^p I^m_p(n; \lambda, l) f(z) + \beta z^p I^{m+1}_p(n; \lambda, l) f(z) < \frac{1 + A z}{1 + B z},
\]

then

\[
\text{Re}\left[ z^p I^m_p(n; \lambda, l) f(z) \right]^{1/q} > \rho^{1/q}, \quad z \in U \ (q \in \mathbb{N}),
\]

where \( \rho = Q(-1) \) is given as in Theorem 3.1. The result is the best possible.
PROOF. Defining the function \( \varphi \) by

\[
\varphi(z) = z^p I_p^m(n; \lambda, l) f(z),
\]

we see that \( \varphi \) has the form (2.1) and is analytic in \( U \). Using identity (1.2) in (3.22), and differentiating the resulting equation with respect to \( z \), we obtain

\[
(1 - \beta) z^p I_p^m(n; \lambda, l) f(z) + \beta z^p I_p^{m+1}(n; \lambda, l) f(z) = \varphi(z) + \frac{\beta \lambda}{l} z \varphi'(z) \prec \frac{1 + A z}{1 + B z}.
\]

Now, by following similar steps to the proof of Theorem 3.1, and using the elementary inequality

\[
\Re \left( \frac{w^1}{q} \right) \geq \left( \Re w \right)^{1/q}, \quad \Re w > 0, \quad q \in \mathbb{N},
\]

we obtain the result asserted by Theorem 3.15. \( \square \)

From Corollary 3.14 and Theorem 3.15, for the special case \( n = -p + 1, m = 0, A = 1 - 2 \eta_0, B = -1 \) and \( q = 1 \), we deduce the next result.

COROLLARY 3.16. Let the functions \( f_i \in \sum_p \) (\( i = 1, 2 \)), satisfy inequality (3.21). Then

\[
\Re[z^p(f_1 \ast f_2)(z)] > \eta_0 + (1 - \eta_0) \left[ 2 F_1 \left( 1, 1, \frac{1}{\beta} + 1; \frac{1}{2} \right) - 1 \right], \quad z \in U,
\]

where \( \eta_0 \) is given as in Corollary 3.14. The result is the best possible.

THEOREM 3.17. If the function \( g \in \sum_{p,n} \) satisfies the inequality

\[
\Re[z^p g(z)] > \frac{1}{2}, \quad z \in U,
\]

then, for any function \( f \in \sum_{p,n}(\lambda, l; A, B) \), we have

\[
f \ast g \in \sum_{p,n}(\lambda, l; A, B).
\]

PROOF. It is easy to check that

\[
- \frac{z^{p+1}(I_p^m(n; \lambda, l)(f \ast g)(z))'}{p} = \left[ - \frac{z^{p+1}(I_p^m(n; \lambda, l)f(z))'}{p} \right] \ast [z^p g(z)].
\]

According to this relation, by applying Lemma 2.5 for the functions

\[
F(z) = - \frac{z^{p+1}(I_p^m(n; \lambda, l)f(z))'}{p}
\]

and \( \Phi(z) = z^p g(z) \), and using the fact that the function \( h(z) = (1 + A z)/(1 + B z) \) is convex (univalent) in \( U \), we deduce the conclusion of the theorem. \( \square \)
References


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