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BANACH SPACES OF BOUNDED SOLUTIONS OF $\Delta u = Pu$ ($P \geq 0$) ON HYPERBOLIC RIEMANN SURFACES

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Consider a nonnegative Hölder continuous 2-form $P(z)dxdy$ on a hyperbolic Riemann surface R ($z = x + iy$). We denote by $PB(R)$ the Banach space of solutions of the equation $\Delta u = Pu$ on R with finite supremum norms. We are interested in the question how the Banach space structure of $PB(R)$ depends on P. Precisely we consider two such 2-forms P and Q on R and compare $PB(R)$ and $QB(R)$. If there exists a bijective linear isometry T of $PB(R)$ to $QB(R)$, then we say that $PB(R)$ and $QB(R)$ are isomorphic. If moreover $|u - Tu|$ is dominated by a potential p_u on R for every u in $PB(R)$, then we say that $PB(R)$ and $QB(R)$ are canonically isomorphic. Intuitively this means that u and Tu have the same ideal boundary values. Using these terminologies our problem is formulated as follows:

1° *When are Banach spaces $PB(R)$ and $QB(R)$ isomorphic?*

2° *When are Banach spaces $PB(R)$ and $QB(R)$ canonically isomorphic?*

For this purpose we consider the set Δ_P, which we call the nondensity point set associated with $P(z)dxdy$, of points z^* in the Wiener harmonic boundary Δ of R such that there exists a neighborhood U^* of z^* in the Wiener compactification R^* of R such that

$$\int_{U^* \cap R} G(z, \zeta)P(\zeta)d\xi d\eta < \infty \qquad (\zeta = \xi + i\eta)$$

for some and hence for every z in R where $G(z, \zeta)$ is the harmonic Green's function on R. We shall see that the set Δ_P is a compact Stonean space with the relative topology of R^*. Our main result in this paper is that the Banach space structure of $PB(R)$ is completely determined by the space Δ_P. More precisely the questions 1° and 2° can be answered in terms of Δ_P as follows:

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THE MAIN THEOREM. *Banach spaces $PB(R)$ and $QB(R)$ are isomorphic (canoncally isomorphic, resp.) if and only if spaces Δ_P and Δ_Q are homeomorphic (identical, resp.).*

Sufficient conditions to the question 2° thus far obtained by Royden [18], Nakai [14], Maeda [9], Lahtinen [7], etc. are then direct consequences of the above theorem. These will be discussed in nos. 15–18. In nos. 1–5 we shall discuss behaviors of functions in $PB(R)$ on the Wiener compactification R^* of R. The Heins canonical extension λ_P^W is one of the important tools in our study. Another important tool, the reduction operator T_P, is discussed in nos. 6–7. The structure of Δ_P will be studied in nos. 8–11. The main theorem will then be proven in nos. 12–13. An alternate definition of Δ_P is appended in no. 14.

Wiener compactification

1. Let $P(z)dxdy$ be a 2-form on a Riemann surface R such that $P(z)$ is a Hölder continuous function of each local parameter $z = x + iy$, i.e. $|P(z_1) - P(z_2)| \leq K|z_1 - z_2|^\alpha$ for every pair of points z_1 and z_2 in the parametric disk of z with a $K \in (0, \infty)$ and an $\alpha \in (0, 1]$. Then the elliptic equation $\Delta u(z) = P(z)u(z)$ can be invariently defined on R where $\Delta u(z)dxdy = d{*}du(z)$. Denote by $P(U)$ the linear space of solutions of $\Delta u = Pu$ on an open subset U of R. We also use the standard notation $H(U)$ for $P(U)$ with $P \equiv 0$.

LEMMA. *The absolute value $|u|$ of any u in $P(U)$ is subharmonic on U for every open subset U of R if and only if $P(z)dxdy$ is nonnegative, i.e. $P(z) \geq 0$ for every choice of local parameters z.*

Proof. Suppose $P \geq 0$ and $u \in P(U)$. Then $\Delta|u(z)| = P(z)|u(z)| \geq 0$ on the open set $U' = \{z \in U; u(z) \neq 0\}$, i.e. $|u|$ is subharmonic on U'. The submean value property is clearly valid for $|u|$ at each point of $U - U'$. Therefore $|u|$ is subharmonic on U. To show that $P \geq 0$ is necessary, let z be an arbitrary point in R. If we take a sufficiently small regular parametric disk U with z its center, then the Dirichlet problem of $\Delta u = Pu$ is solvable for U with continuous boundary values φ and the solution is nonnegative (positive, resp.) for $\varphi \geq 0$ ($\varphi > 0$, resp.) (cf. Miranda [10]). If u is the solution with $\varphi \equiv 1$, then, since $|u| = u > 0$ is subharmonic, $P(z) = \Delta u(z)/u(z) \geq 0$. Q.E.D.

2. Hereafter, we always assume that $P(z) \geq 0$. For simplicity such a 2-form $P(z)dxdy$, i.e. Hölder continuous and nonnegative, will be referred to as a *density* on R. We denote by $PB(R)$ the subspace of $P(R)$ consisting of solutions u with finite supremum norms:

$$\|u\| = \|u\|_R = \sup_{z \in R} |u(z)| .$$

We also use the standard notation $HB(R)$ for $PB(R)$ with $P \equiv 0$. Then $(PB(R), \|\cdot\|)$ is a Banach space. We wish to determine the Banach space structure of $PB(R)$. We say that R is *hyperbolic* if there exists the harmonic Green's function on R. Nonhyperbolic surfaces are called parabolic. The Ahlfors-Ohtsuka characterization (cf. [19]) says that R is parabolic if and only if there does not exist any nonconstant positive superharmonic function on R. Therefore if R is parabolic and $u \in PB(R)$, then, since $\|u\| - |u|$ is a nonnegative superharmonic function on R, $\|u\| - |u|$ and hence u is a constant. This proves the following Brelot [1, 2]-Ozawa [17] theorem:

LEMMA. *If R is parabolic, then $PB(R) = \{0\}$ for densities $P \not\equiv 0$ and $HB(R) = \boldsymbol{R}$ (the real number field).*

In view of this lemma the Banach space $PB(R)$ is of no interest if R is parabolic, and for this reason, hereafter, we always assume that R is hyperbolic.

3. The *Wiener compactification* R^* of a hyperbolic Riemann surface R is a compact Hausdorff space containing R as its open and dense subset such that $C(R^*) = \{f \,|\, R\,;\, f \in C(R^*)\}$ is the totality of bounded continuous Wiener functions on R. If $F \in C(R^*)$, then $f = F \,|\, R$ is of course defined only on R but we always make the convention that $f(z^*) = F(z^*)$ for $z^* \in R^* - R$. Typical examples of Wiener functions on R are subharmonic functions s on R such that $|s|$ is bounded or more generally dominated by superharmonic functions. Denote by Δ the set of points z^* in R^* such that

$$\liminf_{z \in R, z \to z^*} p(z) = 0$$

for every potential p on R, i.e. a superharmonic function p on R whose greatest harmonic minorant is zero. The set Δ is contained in the Wiener boundary $R^* - R$ of R and referred to as the *Wiener harmonic boundary* of R. For a subset A of R^* we denote by $\overline{A}$ the closure of A in R^* and by ∂A the relative boundary $(\overline{A} - \operatorname{Int} A) \cap R$ with respect to R. Let U be an open subset of R and s be a subharmonic function U bounded from above. The *maximum principle* says that if

$$\limsup_{z \in U, z \to z^*} s(z) \leq M \tag{1}$$

for every z^* in $(\partial U) \cup (\overline{U} \cap \Delta)$, then $s \leq M$ on U.

An open subset W of R will be called *normal* if ∂W is analytic. We do not exclude W with $\partial W = \phi$, i.e. $W = R$, from our family of normal open sets. We denote by Δ^W the open subset $\Delta \cap (\overline{W} - \overline{\partial W})$ of Δ. We can define an operator $\pi_W : C(R^*) \to C(R^*)$ such that $\pi_W f \mid W \in H(W)$, and $\pi_W f \mid (R^* - \overline{W}) \cup (\partial W) \cup \Delta^W = f$ for every $f \in C(R^*)$. For details of Wiener compactification and the operator π_W, we refer to Constantinescu-Cornea [3] or Sario-Nakai [19].

4. For a normal open subset W of R (including the case $W = R$) we denote by $PB(W; \partial W)$ the family $\{u \in PB(W) \cap C(R); u \mid R - W = 0\}$. We also use the notation $HB(W; \partial W)$ for $PB(W; \partial W)$ with $P \equiv 0$. If $W = R$, then $PB(W; \partial W) = PB(W; \phi) = PB(W)$. By the same proof as in no. 1 we see that $u \cup 0 = \max(u, 0)$, $-(u \cap 0) = -\min(u, 0)$, and $|u| = u \cup 0 - u \cap 0$ are subharmonic on R for every $u \in PB(W; \partial W)$. Therefore $PB(W; \partial W) \subset C(R^*)$. By the maximum principle (1), $\|u\|_R = \|u\|_\Delta$ for every $u \in PB(W; \partial W)$. For a regular region Ω, i.e. relatively compact and normal region in R, and a continuous function φ in $C(\partial \Omega)$, we denote by P^Ω_φ the function in $P(\Omega) \cap C(\overline{\Omega})$ such that $P^\Omega_\varphi \mid \partial\Omega = \varphi$. We also use the notation H^Ω_φ for P^Ω_φ with $P \equiv 0$. We define a linear operator $\lambda^W_P : PB(W; \partial W) \to PB(R)$ by

$$\lambda^W_P u = \lim_{\Omega \to R} P^\Omega_u \tag{2}$$

for every $u \in PB(W; \partial W)$. Then it satisfies

$$\lambda^W_P u \mid \Delta = u \mid \Delta\,. \tag{3}$$

In particular λ^W_P is isometric and hence injective. We use the notation λ^W_H for λ^W_P with $P \equiv 0$. These operators are referred to as *canonical extensions* (Heins [6]).

To see that (2) is well defined and (3) is valid, set $v_1 = u \cup 0$ and $v_2 = -(u \cap 0)$. Since $\|u\| \geq H^\Omega_{v_i} \geq P^\Omega_{v_i} \geq v_i \geq 0$, $\{P^\Omega_{v_i}\}$ is increasing and thus $P_{v_i}^{\Omega}$ converges to a $u_i \in PB(R)$ as Ω exhausts R $(i = 1, 2)$ and therefore $P_u^{\Omega} = P_{v_1}^{\Omega} - P_{v_2}^{\Omega}$ to $u_1 - u_2$, i.e. (1) is well defined. Similarly $H_{v_i}^{\Omega}$ converges incleasingly to an $h_i \in HB(R)$ which is the least harmonic majorant of the subharmonic function v_i on R and hence $p_i = h_i - v_i$ is a potential on R $(i = 1, 2)$. Since

$$|\lambda_P^W u - u| \leq |u_1 - v_1| + |u_2 - v_2| \leq (h_1 - v_1) + (h_2 - v_2) = p$$

with $p = p_1 + p_2$, a potential on R,

$$|\lambda_P^W u(z^*) - u(z^*)| = \lim_{z \in R, z \to z^*} |\lambda_P^W u(z) - u(z)| \leq \liminf_{z \in R, z \to z^*} p(z) = 0$$

for every point $z^* \in \Delta$, i.e. (3) is valid.

5. Denote by $PB(W; \partial W)^+$ the family $\{u \in PB(W; \partial W); u \geq 0\}$. We maintain that $PB(W; \partial W)^+$ generates $PB(W; \partial W)$; more precisely, for any $u \in PB(W; \partial W)$ there exist $u_i \in PB(W; \partial W)^+$ $(i = 1, 2)$ such that

$$(4) \qquad u = u_1 - u_2\,, \quad u_1|\Delta = u \cup 0|\Delta\,, \quad u_2|\Delta = -(v \cap 0)|\Delta\,.$$

The proof of (4) is similar to that of (2) and (3). Let $v_1 = u \cup 0$ and $v_2 = -(u \cap 0)$. As in no. 4 we see that $H_{v_i}^{\Omega} \geq u_{i\Omega} \geq v_i \geq 0$, where $u_{i\Omega} = P_{v_i}^{W \cap \Omega}$ on $W \cap \Omega$ and $u_{i\Omega} = 0$ on $\Omega - W \cap \Omega$, and that $u_i = \lim_{\Omega \to R} u_{i\Omega}$ exists in $PB(W; \partial W)^+$ and $h_i = \lim_{\Omega \to R} H_{v_i}^{\Omega}$ exists in $HB(R)^+$ $(i = 1, 2)$. Since $u = u_{1\Omega} - u_{2\Omega}$, we deduce $u = u_1 - u_2$. Moreover $0 \leq u_i - v_i \leq h_i - v_i = p_i$ and p_i is a potential $(i = 1, 2)$. Therefore (4) is true.

Reduction operator

6. Since we have assumed that our base Riemann surface R is hyperbolic, there exists the harmonic Green's function $G(z, \zeta) = G_R(z, \zeta)$ on R. Let W be a normal open subset of R. The harmonic Green's function $G_W(z, \zeta)$ on W is defined as follows. Let $W = \bigcup_n W_n$ be the decomposition of W into connected components W_n such that each W_n is a normal region. If both of z and ζ belong to the same W_n, then $G_W(z, \zeta) = G_{W_n}(z, \zeta)$; otherwise $G_W(z, \zeta) = 0$. Including the case $W = R$, we define a linear operator $T_P^W : PB(W; \partial W) \to HB(W; \partial W)$ by

$$(5) \qquad T_P^W u = u + \frac{1}{2\pi} \int_R G_W(\cdot, \zeta) u(\zeta) P(\zeta) d\xi d\eta\,.$$

To see that (5) is well defined, first let $u \in PB(W; \partial W)^+$. Then by the Green formula

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$$H_u^{W\cap\Omega} = u + \frac{1}{2\pi}\int_\Omega G_{W\cap\Omega}(\cdot,\zeta)u(\zeta)P(\zeta)d\xi d\eta\ .$$

Since u is subharmonic, $\lim_{\Omega\to R} H_u^{W\cap\Omega}$ exists in $HB(W;\partial W)$. Observe that $\{G_{W\cap\Omega}(\cdot,\zeta)u(\zeta)\}_\Omega$ is increasing, and therefore the Lebesgue-Fatou theorem yields (5) for $u\geq 0$. The general case follows from the decomposition (4). Similarly, as above,

$$H_u^\Omega = P_u^\Omega + \frac{1}{2\pi}\int_\Omega G_\Omega(\cdot,\zeta)P_u^\Omega(\zeta)P(\zeta)d\xi d\eta$$

for $u\in PB(W;\partial W)^+$ and by making Ω tend to R we deduce

$$\lambda_H^W u = \lambda_P^W u + \frac{1}{2\pi}\int_R G_R(\cdot,\zeta)(\lambda_P^W u)(\zeta)P(\zeta)d\xi d\eta\ .$$

Since $0\leq u\leq \lambda_P^W u$ and $G_W\leq G_R$, we conclude that

$$0\leq T_P^W u - u\leq \frac{1}{2\pi}\int_R G_R(\cdot,\zeta)(\lambda_P^W u)(\zeta)P(\zeta)d\xi d\eta = p$$

with p a potential on R. By the decomposition (4) we can also conclude that $|T_P^W u - u|$ is dominated by a potential on R for general $u\in PB(W;\partial W)$. Therefore we see that

$$(6)\qquad T_P^W u|\Delta = u|\Delta$$

for every $u\in PB(W;\partial W)$. In particular T_P^W is isometric and hence injective.

7. The operator $T_P = T_P^R$ is referred to as a *reduction operator* since T_P reduces the study of $PB(R)$ to that of more tractable class $HB(R)$ (cf. Singer [20]). In this sense it is important to determine when T_P is surjective. As a preparatory observation we state the following: If

$$(7)\qquad \int_W G_R(\cdot,\zeta)P(\zeta)d\xi d\eta < \infty$$

for some $z\in R$ and hence by the Harnack inequality for every $z\in R$, then T_P^W is surjective. To prove this take an $h\in HB(W;\partial W)^+$. Since $0\leq P_h^{W\cap\Omega}\leq h$, $\{P_h^{W\cap\Omega}\}$ is decreasing and converges to a $u\in PB(W:\partial W)^+$. Then

$$h = P_h^{W\cap\Omega} + \frac{1}{2\pi}\int_\Omega G_{W\cap\Omega}(\cdot,\zeta)P_h^{W\cap\Omega}(\zeta)P(\zeta)d\xi d\eta\ .$$

In view of (7) the Lebesgue convergence theorem is applicable to deduce

$$h = u + \frac{1}{2\pi}\int_R G_W(\cdot, \zeta)u(\zeta)P(\zeta)d\xi d\eta ,$$

i.e. $T_P^W u = h$. Since $HB(W; \partial W)^+$ generates $HB(W; \partial W)$, we obtain the required conclusion.

Nondensity points

8. We introduce the set Δ_P of points z^* in Δ such that there exists a neighborhood U^* of z^* in R^* such that

$$\int_{U^* \cap R} G_R(z, \zeta)P(\zeta)d\xi d\eta < \infty \tag{8}$$

for some and hence for every point z in R. Such a point z^* will be referred to as a *nondensity point* of P with the weight G. If we denote by Δ_H for Δ_P with $P \equiv 0$, then $\Delta_H = \Delta$. Clearly the nondensity point set Δ_P is *open*. Since Δ is a Stonean space i.e. every point in Δ has a base of compact and open neighborhoods in Δ, the same is true of Δ_P. Another kind of nondensity point was first introduced in Glasner-Katz [5] (cf. also [15]) for the Royden harmonic boundary. In the definition (8) we can moreover assume that $U^* \cap R$ is a normal open subset of R since we can replace U^* by a smaller neighborhood (cf. [3], [19]). First we remark that

$$u | \Delta - \Delta_P = 0 \tag{9}$$

for every $u \in PB(R)$. In fact, let $z^* \in \Delta$ with $u(z^*) \neq 0$. We can choose a neighborhood U^* of z^* such that $|u(z)| > |u(z^*)|/2$ on $U^* \cap R$. By (5), u is $G_R(\cdot, \zeta)P(\zeta)d\xi d\eta$-integrable on R. Therefore

$$\int_{U^* \cap R} G(z, \zeta)P(\zeta)d\xi d\eta \leq \frac{2}{|u(z^*)|}\int_{U^* \cap R} G(z, \zeta)\,|u(\zeta)|\,P(\zeta)d\xi d\eta < \infty ,$$

i.e. $z^* \in \Delta_P$. This proves (9).

9. Let K be a compact and open set in Δ_P. We can find a neighborhood W^* of K in R^* such that $W = W^* \cap R$ is normal in R and (7) is valid since K is compact. Choose a $\varphi \in C(R^*)$ such that $\varphi | K = 1$ and $\varphi | (R^* - W^*) \cup (\Delta - K) = 0$. Such a φ exists since K is open and compact in Δ and $K \cap (R^* - W^*) = \phi$. Observe that $\pi_W \varphi \in HB(W; \partial W)$ with $\pi_W \varphi | K = 1$ and $\pi_W \varphi | K = 0$ (cf. no. 3). By (7), $T_P^W; PB(W; \partial W) \to HB(W; \partial W)$ is surjective and hence $(T_P^W)^{-1} \circ \pi_W \varphi \in PB(W; \partial W)$ with

$(T_P^W)^{-1}\circ\pi_W\varphi|K=1$ and $(T_P^W)^{-1}\circ\pi_W\varphi|\varDelta-K=0$ (cf. (6)). Finally set $e_K = \lambda_P^W\circ(T_P^W)^{-1}\circ\pi_W\varphi\in PB(R)$. By (3), $e_K|K=1$ and $e_K|\varDelta-K=0$. Put

$$e_P=\sup_{K\subset\varDelta_P} e_K ,$$

where K runs over all compact open subsets of $\varDelta_P$. The conditionally monotone compactness of $PB(R)$ assures that $e_P\in PB(R)$. Clearly $0\leq e_P\leq 1$ on $\varDelta$. Since every point $z^*\in\varDelta_P$ has such a K with $z^*\in K$, we see that $e_P|\varDelta_P=1$. With (9) we now conclude that

$$(10)\qquad e_P\in PB(R)\ ,\quad 0\leq e_P\leq 1\ ,\quad e_P|\varDelta_P=1\ ,\quad e_P|\varDelta-\varDelta_P=0\ .$$

The function e_P will be refered to as the *P-unit* (cf. Singer [21]). It is easy to see that e_P is the greatest function in $PB(R)$ dominated by 1 and actually

$$e_P=\lim_{\Omega\to R} P_1^{\Omega}\ .$$

Therefore e_P is the *P-elliptic measure* in the terminology of Royden [18]. Since $e_P|\varDelta\in C(\varDelta)$ and $e_P|\varDelta$ is the characteristic function of $\varDelta_P$, we obtain the following

THEOREM. *The nondensity point set $\varDelta_P$ is compact and open in $\varDelta$.*

10. We denote by $C(\varDelta;\varDelta_P)$ the family $\{\varphi\in C(\varDelta);\varphi|\varDelta-\varDelta_P=0\}$. Clearly $C(\varDelta;\varDelta_P)$ is isomorphic to $C(\varDelta_P)$ as Banach spaces by the natural correspondence $\tau_P: C(\varDelta;\varDelta_P)\to C(\varDelta_P)$ given by $\tau_P u=u|\varDelta_P$. We define an operator, the *boundary restriction*, $\rho_P: PB(R)\to C(\varDelta;\varDelta_P)$ by $\rho_P u=u|\varDelta$ for every $u\in PB(R)$. We write ρ_H for ρ_P with $P\equiv 0$. By (1), ρ_P is isometric. We prove

THEOREM. *The boundary restriction ρ_P is surjective.*

Proof. Let $\varphi\in C(\varDelta;\varDelta_P)$ and W^* be an open neighborhood of $\varDelta_P$ in R^* such that $W=W^*\cap R$ is normal and (7) is valid for W. We can extend φ to R^* so that $\varphi\in C(R^*)$ with $\varphi(R^*-W^*)\cup(\varDelta-\varDelta_P)=0$. Then as in no. 9 $u=\lambda_W^P\circ(T_P^W)^{-1}\circ\pi_W\varphi$ belongs to $PB(R)$ and $u|\varDelta=\varphi$, i.e. $\rho_P u=\varphi$.
Q.E.D.

11. The surjectiveness question of the canonical extension λ_P^W can now be settled in terms of $\varDelta_P$:

THEOREM. *The canonical extension λ_P^W is surjective if and only if $\overline{W} - \overline{\partial W}$ is a neighborhood of Δ_P in R^*.*

Proof. Suppose λ_P^W is surjective. Then there exists a $u \in PB(W; \partial W)$ such that $\lambda_P^W u = e_P$. Since $u|\Delta = \lambda_P^W u|\Delta = e_P|\Delta$, we see that $u|\Delta_P = 1$. In view of $u|R^* - \overline{W} = 0$, $\overline{W} - \overline{\partial W}$ is a neighborhood of Δ_P. Conversely if $\overline{W} - \overline{\partial W}$ is a neighborhood of Δ_P, then we can choose an open neighborhood W_0^* of Δ_P such that $\overline{W} - \overline{\partial W} \supset W_0^*$, $W_0 = W_0^* \cap R$ is normal in R, and (7) is valid for W_0. The canonical extension $\lambda_P^{W_0,W}: PB(W_0; \partial W_0) \to PB(W; \partial W)$ relative to W can be defined by

$$\lambda_P^{W_0,W} u = \lim_{\Omega \to R} P_u^{\Omega \cap W}$$

on W and 0 on $R - W$. As in no. 4, $\lambda_P^{W_0,W} u|\Delta = u|\Delta$. Let $v \in PB(R)$ and $\varphi \in C(R^*)$ with $\varphi|(R^* - \overline{W}_0) \cup (\Delta - \Delta_P) = 0$ and $\varphi|\Delta_P = 1$. Then $u = (T_P^{W_0})^{-1} \circ \pi_{W_0}(\varphi v) \in PB(W_0, \partial W_0)$ and $u|\Delta = v|\Delta$. Thus $\lambda_P^{W_0}: PB(W_0; \partial W_0) \to PB(R)$ is surjective. Since $\lambda_P^{W_0} = \lambda_P^W \circ \lambda_P^{W_0,W}$ and $\lambda_P^{W_0}$

$$\begin{array}{ccc} PB(W_0; \partial W_0) & \xrightarrow{\lambda_P^{W_0}} & PB(R) \\ \downarrow{\scriptstyle \lambda_P^{W_0,W}} & \nearrow{\scriptstyle \lambda_P^W} & \\ PB(W; \partial W) & & \end{array}$$

is surjective, λ_P^W must be surjective. Q.E.D.

Canonical isomorphisms

12. Let P and Q be two densities on a hyperbobic Riemann surface R. A linear isomorphism $T_{Q,P}$ of $PB(R)$ onto $QB(R)$ will be referred to as a *canonical isomorphism* if $|T_{Q,P}u - u|$ is dominated by a potential $p = p_u$ on R for every $u \in PB(R)$. This is equivalent to that

$$T_{Q,P} u|\Delta = u|\Delta \tag{11}$$

for every $u \in PB(R)$ (cf. Constantinescu-Cornea [3]). By (1) we see that $T_{Q,P}$ is an isometry and thus $T_{Q,P}$ is a special Banach space isomorphism of $PB(R)$ onto $QB(R)$. In such a case we say that $PB(R)$ and $QB(R)$ are *canonically isomorphic*. We are ready to prove one of our main result in this paper:

THEOREM. *Banach spaces $PB(R)$ and $QB(R)$ are canonically isomorphic if and only if nondensity point sets Δ_P and Δ_Q are identical.*

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Proof. Suppose $PB(R)$ and $QB(R)$ are canonically isomorphic. Let $z^* \in \varDelta_P$. There exists a $\varphi \in C(\varDelta; \varDelta_P)$ with $\varphi(z^*) = 1$. By Theorem in no. 10, there exists a $u \in PB(R)$ with $\rho_P u = \varphi$. Then $T_{Q,P}u | \varDelta = u | \varDelta$ shows that $(T_{Q,P}u)(z^*) = u(z^*) = \varphi(z^*) = 1$ and (9) yields that $z^* \in \varDelta_Q$, i.e. $\varDelta_P \subset \varDelta_Q$. Since the reverse inclusion can be shown similarly, we conclude that $\varDelta_P = \varDelta_Q$. Conversely assume that $\varDelta_P = \varDelta_Q$. Then the operator $T = \rho_Q^{-1}\rho_P$: $PB(R) \to QB(R)$ can be defined as a bijective mapping and $Tu | \varDelta = u | \varDelta$ for every $u \in PB(R)$. Therefore T fulfills the condition of canonical isomorphism and $T = T_{Q,P}$, i.e. $PB(R)$ and $QB(R)$ are canonically isomorphic. Q.E.D.

$$\begin{array}{ccc} PB(R) & \xrightarrow{\ T\ } & QB(R) \\ \downarrow{\scriptstyle \rho_P} & & \uparrow{\scriptstyle \rho_Q^{-1}} \\ C(\varDelta;\varDelta_P) & \xrightarrow[\text{id.}]{} & C(\varDelta;\varDelta_Q) \end{array}$$

13. We simply say that $PB(R)$ and $QB(R)$ are *isomorphic* if there exists a Banach space isomorphism (i.e. bijective linear isometry) of $PB(R)$ onto $QB(R)$. Then we obtain another of our main result:

THEOREM. *Banach spaces $PB(R)$ and $QB(R)$ are isomorphic if and only if nondensity point sets $\varDelta_P$ and $\varDelta_Q$ are homeomorphic.*

Actually we can prove a bit more general assertion without adding any elaboration. Let $P(Q$, resp.) be a density on a hyperbolic Riemann surface $R(S$, resp.). We can speak of isomorphisms of $PB(R)$ onto $QB(S)$ as Banach spaces and also nondensity point sets $\varDelta_P$ and $\varDelta_Q$ relative to Wiener compactifications R^* and S^* of R and S, respectively. The above theorem is, then, a special case, i.e. $R = S$, of the following:

THEOREM. *Banach spaces $PB(R)$ and $QB(S)$ are isomorphic if and only if nondensity point sets $\varDelta_P$ and $\varDelta_Q$ are homeomorphic.*

Proof. Suppose there exists a homeomorphism α of $\varDelta_P$ onto $\varDelta_Q$. Then $\varphi \to A\varphi = \varphi \circ \alpha^{-1}$ is a Banach space isomorphism of $C(\varDelta_P)$ onto $C(\varDelta_Q)$.

$$\begin{array}{ccc} PB(R) & \xrightarrow{\ T\ } & QB(S) \\ \downarrow{\scriptstyle \rho_P} & & \uparrow{\scriptstyle \rho_Q^{-1}} \\ C(\varDelta;\varDelta_P) & & C(\varDelta;\varDelta_Q) \\ \downarrow{\scriptstyle \tau_P} & & \uparrow{\scriptstyle \tau_Q^{-1}} \\ C(\varDelta_P) & \xrightarrow{\ A\ } & C(\varDelta_Q) \end{array}$$

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Then $T = \rho_P^{-1} \circ \tau_Q^{-1} \circ A \circ \tau_P \circ \rho_P$ is a Banach space isomorphisms of $PB(R)$ onto $QB(S)$. Conversely assume that there exists a Banach space isomorphism T of $PB(R)$ onto $QB(R)$. Then $A = \tau_Q \circ \rho_Q \circ T \circ \rho_P^{-1} \circ \tau_P^{-1}$ is a Banach space isomorphism of $C(\Delta_P)$ onto $C(\Delta_Q)$. In such a case there exists a

$$\begin{array}{ccc} PB(R) & \xrightarrow{T} & QB(S) \\ \uparrow \rho_P^{-1} & & \downarrow \rho_Q \\ C(\Delta\,;\Delta_P) & & C(\Delta\,;\Delta_Q) \\ \uparrow \tau_P^{-1} & & \downarrow \tau_Q \\ C(\Delta_P) & \xrightarrow{A} & C(\Delta_Q) \end{array}$$

homeomorphism α of Δ_P and Δ_Q and an $a \in C(\Delta_Q)$ with $|a| = 1$ such that $A\varphi = a \cdot \varphi \circ \alpha^{-1}$ (cf. e.g. Dunford-Schwartz [4]); in particular Δ_P and Δ_Q are homeomorphic. Q.E.D.

14. Suppose there exists a neighborhood U^* of $z^* \in \Delta$ such that

$$\int_U G_U(z, \zeta) P(\zeta) d\xi d\eta < \infty \qquad (U = U^* \cap R) \tag{12}$$

for every z in U. We can replace U^* by a $W^* \subset U^*$ which is also a neighborhood of z^* such that $W = W^* \cap R$ is normal and (12) with U replaced by W is valid. Let $h \in HB(W\,;\partial W)$ such that $0 \leq h \leq 1$ and $h(z^*) = 1$. Then as in no. 7 $T_P^W : PB(W\,;\partial W) \to HB(W\,;\partial W)$ is surjective and $T_P^W v \,|\, \tilde{\Delta} = v \,|\, \tilde{\Delta}$ for every $v \in PB(W\,;\partial W)$, where $\tilde{\Delta}$ is the Wiener harmonic boundary of W. There exists a projection p of the Wiener compactification $\tilde{W}$ of W onto the closure $\overline{W}$ of W in R^* such that $p\,|\,W$ is an identity, $p^{-1}(\Delta^W) \subset \tilde{\Delta}$, and $p : W \cup p^{-1}(\Delta^W) \to W \cup \Delta^W$ is homeomorphic (cf. e.g. Constantinescu-Cornea [1] and [19]), where $\Delta^W = \Delta \cap (\overline{W} - \partial W)$. Therefore $T_P^W v \,|\, \Delta^W = v \,|\, \Delta^W$. Since $T_P^W v \,|\, \partial W = v \,|\, \partial W = 0$, $T_P^W v \,|\, \partial W = v \,|\, \partial W = 0$ and hence $T_P^W v \,|\, \partial W = v \,|\, \partial W = 0$ and hence $T_P^W v \,|\, \Delta = v \,|\, \Delta$. In particular, if $v = (T_P^W)^{-1} h$, then $v(z^*) = (T_P^W v)(z^*) = h(z^*) = 1$ and $z^* \in \Delta_P$. Thus we have shown the following

THEOREM. *A point z^* in Δ belongs to Δ_P if and only if* (12) *is valid.*

Applications

15. A subset $K \subset R$ is said to be *B-negligible* if there exists a po-

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tential p such that $p \geq 1$ on K. In this case, since $\liminf_{z \in R, z \to z^*} p(z) = 0$ for $z^* \in \Delta$, we see that $\bar{K} \cap \Delta = \phi$. Conversely, if $\bar{K} \cap \Delta = \phi$, then there exists a $\varphi \in C(R^*)$ such that $0 \leq \varphi \leq 1$, $\varphi | \bar{K} = 1$, and $\varphi | \Delta = 0$. Then there exists a potential p with $\varphi \leq p$ on R (cf. [3]). Hence $p \geq 1$ on K and therefore K is B-negligible. Thus we have the following characterization: *A subset $K \subset R$ is B-negligible if and only if $R^* - \bar{K}$ is a neighborhood of Δ.* Compact sets in R are trivial examples of B-negligible sets. From Theorem in no. 12 the following criterion of Royden follows at once:

COROLLARY (ORDER COMPARISON THEOREM). *If there exists a constant $c \in [1, \infty)$ such that $c^{-1}P \leq Q \leq cP$ on R except possibly for a B-negligible set K, then $PB(R)$ and $QB(R)$ are canonically isomorphic.*

In general, $Q \leq cP$ on $R - K$ implies

$$\int_{R-\bar{K}} G(z, \zeta) Q(\zeta) d\xi d\eta \leq c \int_{R-\bar{K}} G(z, \zeta) P(\zeta) d\xi d\eta .$$

Since $\overline{R - \bar{K}} - (\partial(\overline{R - \bar{K}}))$ is a neighborhood of Δ, the above inequality implies that $\Delta_P \subset \Delta_Q$. In passing we insert here a consequence of $\Delta_P \subset \Delta_Q$, i.e. a consequence of $Q \leq cP$ on $R - K$ with B-negligible K. Since Δ_P is also compact and open in Δ_Q, the function $\tau\varphi$ given by $\tau\varphi = \varphi$ on Δ_P and $\tau\varphi = 0$ on $\Delta_Q - \Delta_P$ belongs to $C(\Delta_Q)$ for every $\varphi \in C(\Delta_P)$, i.e. $\tau : C(\Delta_P) \to C(\Delta_Q)$ is a linear isometry. Then $T = \rho_Q^{-1} \circ \tau_Q^{-1} \circ \tau \circ \tau_P \circ \rho_P$ is a linear isometry of $PB(R)$ into $QB(R)$ with $Tu | \Delta = u | \Delta$ for every $u \in PB(R)$. Returning to the above corollary, we also see that $\Delta_P \supset \Delta_Q$ from $c^{-1}P \leq Q$. Thus $\Delta_P = \Delta_Q$; $PB(R)$ and $QB(R)$ are canonically isomorphic. This criterion was obtained by Royden [18] for compact exceptional set K and by Loeb [8] in an abstract setting. The present formulation is stated in [16].

16. Let $G^P(z, \zeta)$ be the Green's function on R for the equation $\Delta u = Pu$ whose existence is always assured for any R (even for compact R) if $P \not\equiv 0$ (Myrberg [11, 12, 13]). In the present case, since we have assumed that R is hyperbolic, $G^P(z, \zeta)$ exists for every density P including $P \equiv 0$; as before we write $G(z, \zeta)$ for $G^P(z, \zeta)$ with $P \equiv 0$. Consider conditions

$$\int_{R-K} |P(\zeta) - Q(\zeta)| \, d\xi d\eta < \infty \; ; \tag{13}$$

(14) $$\int_{R-K} G(z,\zeta)\,|P(\zeta) - Q(\zeta)|\,d\xi d\eta < \infty\ ;$$

(15) $$\int_{R-K} (G^P(z,\zeta) + G^Q(z,\zeta))\,|P(\zeta) - Q(\zeta)|\,d\xi d\eta < \infty\ ;$$

(16) $$\int_{R-K} (G^P(z,\zeta)Q(\zeta) + G^Q(z,\zeta)P(\zeta))d\xi d\eta < \infty\ .$$

Here K is a B-negligible set in R and (14)–(16) are assumed to be valid for one and hence by the Harnack inequality for every z in S. Since

$$G^P(z,\zeta) \leq G(z,\zeta) \quad \text{and} \quad \int_R G^P(z,\zeta)P(\zeta)d\xi d\eta < \infty$$

because of

$$e_P = 1 - \frac{1}{2\pi}\int_R G^P(\cdot,\zeta)P(\zeta)d\xi d\eta$$

(see the proof of the corollary below) it is clear that the following implications are valid: (13) $\to$ (14) $\to$ (15) $\rightleftarrows$ (16).

COROLLARY (INTEGRAL COMPARISON THEOREM). *If one of conditions* (13)–(16) *is valid, then $PB(R)$ and $QB(R)$ are canonically isomorphic.*

This was obtained in [14] for $K = \phi$ and in the present form in [16] (cf. also Maeda [9]). The fact that (14) and hence (13) implies $\Delta_P = \Delta_Q$ is entirely clear. To show (15) or (16) implies $\Delta_P = \Delta_Q$, we may assume that $K = \phi$ in (15) since we can replace R by its normal open subset W whose $\tilde{\Delta}$ contains Δ (cf. no. 14). The Green formula yields

$$Q_1^\Omega = P_1^\Omega + \frac{1}{2\pi}\int_\Omega G_\Omega^Q(\cdot,\zeta)P_1^\Omega(\zeta)(P(\zeta) - Q(\zeta))d\xi d\eta\ .$$

Since $e_P = \lim_{\Omega\to R} P_1^\Omega$ and $e_Q = \lim_{\Omega\to R} Q_1^\Omega$, (15) and the Lebesgue convergence theorem imply

$$e_Q = e_P + \frac{1}{2\pi}\int_R G^Q(\cdot,\zeta)e_P(\zeta)(P(\zeta) - Q(\zeta))d\xi d\eta\ .$$

Similarly

$$Q_1^\Omega = 1 - \frac{1}{2\pi}\int_\Omega G_\Omega^Q(\cdot,\zeta)Q(\zeta)d\xi d\eta$$

and the Lebesgue-Fatou theorem yield

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$$e_Q = 1 - \frac{1}{2\pi}\int_R G^Q(\cdot,\zeta)Q(\zeta)d\xi d\eta\ .$$

Set

$$h = T_P e_P = e_P + \frac{1}{2\pi}\int_R G(\cdot,\zeta)e_P(\zeta)P(\zeta)d\xi d\eta\ .$$

Observe that

$$\frac{1}{2\pi}\int_R G^Q(\cdot,\zeta)e_P(\zeta)Q(\zeta)d\xi d\eta \le \frac{1}{2\pi}\int_R G^Q(\cdot,\zeta)Q(\zeta)d\xi d\eta = 1 - e_Q$$

and

$$\frac{1}{2\pi}\int_R G^Q(\cdot,\zeta)e_P(\zeta)P(\zeta)d\xi d\eta \le \frac{1}{2\pi}\int_R G(\cdot,\zeta)e_P(\zeta)P(\zeta)d\xi d\eta = h - e_P\ .$$

Therefore $|e_Q - e_P| \le (h - e_P) + (1 - e_Q)$. Since $h|\varDelta = T_P e_P|\varDelta = e_P|\varDelta$,

$$|e_Q - e_P| \le 1 - e_Q$$

on $\varDelta$. If $z^* \in \varDelta_Q$, then $e_Q(z^*) = 1$ and hence $e_P(z^*) = 1$, i.e. $z^* \in \varDelta_P$. Thus we conclude that $\varDelta_Q \subset \varDelta_P$, and similarly $\varDelta_P \subset \varDelta_Q$, i.e. $\varDelta_P = \varDelta_Q$.

17. Each of the conditions (14)–(16) takes the following form for $Q \equiv 0$:

$$\int_{R-K} G(z,\zeta)P(\zeta)d\xi d\eta < \infty\ , \tag{17}$$

where again K is a B-negligible set. Clearly $\varDelta_P = \varDelta$ is equivalent to (17) for some B-negligible K. Thus we have

COROLLARY. *Banach spaces $PB(R)$ and $HB(R)$ are canonically isomorphic, i.e. the reduction operator $T_P: PB(R) \to HB(R)$ is surjective, if and only if* (17) *is valid for some B-negligible set K.*

The sufficiency of (17) for $K = \phi$ was obtained in [14]. The condition (17) for $K = \phi$ may not be necessary is remarked by Lahtinen [7]. The assertion in the present form is stated in [16].

18. Let h_P be the least harmonic majorant of the P-unit e_P (the P-elliptic measure). Clearly $h_P = T_P e_P$. Then $e_P|\varDelta = h_P|\varDelta$. Therefore $\varDelta_P = \varDelta_Q$ if and only if $h_P = h_Q$ and we have the following

COROLLARY. *Banach spaces $PB(R)$ and $QB(R)$ are canonically iso-*

morphic if and only if $h_P = h_Q$. *In particular* $PB(R)$ *and* $HB(R)$ *are canonically isomorphic if and only if* $h_P = 1$.

That $h_P = h_Q$ is sufficient and that $h_P = 1$ is necessary and sufficient are recent results of Lahtinen [7], in which he also studies the class $PB(R)$ for not necessarily $P \geq 0$ (cf. also Myrberg [13]).

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