On Appell's Function $P (\theta, \phi)$

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(Received 11th January, 1932. Read 15th January, 1932.)

§ 1. Appell's functions, $P (\theta, \phi)$, $Q (\theta, \phi)$ and $R (\theta, \phi)$ are defined by the expansion

\[ e^{\theta + j\phi} = P (\theta, \phi) + jQ (\theta, \phi) + j^2R (\theta, \phi) \]

where $j^3 = 1$, affording, both for the third order and the field of two variables, a very direct generalization of the circular functions, as

\[ e^{i\theta} = \cos \theta + i \sin \theta. \]

They can be written as follows:

\[ P (\theta, \phi) = \frac{1}{3} (e^{\theta + \phi} + e^{\theta + j\phi} + e^{\theta + j^2\phi}), \]
\[ Q (\theta, \phi) = \frac{1}{3} (e^{\theta + \phi} + j e^{\theta + j\phi} + j e^{\theta + j^2\phi}), \]
\[ R (\theta, \phi) = \frac{1}{3} (e^{\theta + \phi} + j e^{\theta + j\phi} + j^2 e^{\theta + j^2\phi}), \]

and they satisfy the fundamental relation

\[ P^3 + Q^3 + R^3 - 3PQR = 1. \]

I showed recently that they are of great help to solve numerous problems connected with the equation

\[
\Delta_3 v = \frac{\partial^3 v}{\partial x^3} + \frac{\partial^3 v}{\partial y^3} + \frac{\partial^3 v}{\partial z^3} - 3 \frac{\partial^3 v}{\partial x \partial y \partial z} = 0
\]

and allied equations.\(^2\)

The object of this short note is to state some elementary remarks on the function $P (n\theta, n\phi)$ where $n$ is an integer, and to make more conspicuous the analogy between it and $\cos n\theta$.

§ 2. Let us consider the expression

\[ E = \log (1 - ae^{\theta + \phi}) (1 - ae^{\theta + j\phi}) (1 - ae^{\theta + j^2\phi}), \]

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\(^1\) P. Appell, C. R., 84 (1877), 540.

where \( a \) is an arbitrary constant, and try to expand it in ascending powers of \( a \). We have

\[
E = \log (1 - ae^{\theta + j\phi}) + \log (1 - ae^{\theta - j\phi}) + \log (1 - ae^{j\theta + j\phi})
\]

\[
= - \sum \frac{a^n e^{n(\theta + \phi)}}{n} - \sum \frac{a^n e^{n(j\theta + j\phi)}}{n} - \sum \frac{a^n e^{n(j\theta + j\phi)}}{n}
\]

\[
= - \sum 3a^n P(n\theta, n\phi)/n.
\]

Now we can write, as \( 1 + j + j^2 = 0 \),

\[
E = \log [1 - 3aP(\theta, \phi) + 3a^2P(-\theta, -\phi) - a^3]
\]

So we obtain the function \( P(n\theta, n\phi) \) through the generating function

\[
- \log [1 - 3aP(\theta, \phi) + 3a^2P(-\theta, -\phi) - a^3],
\]

the coefficient of \( a^n \) being \( 3P(n\theta, n\phi)/n \).

The noteworthy analogy with the circular functions arises from the fact that the coefficient of \( a^n \) in the expansion of

\[
- \log [1 - 2a \cos \theta + a^2]
\]

is \((2 \cos n\theta)/n\).

§ 3. The expansion just obtained,

\[
- \log [1 - 3aP(\theta, \phi) + 3a^2P(-\theta, -\phi) - a^3] = \sum \frac{3a^n P(n\theta, n\phi)}{n},
\]

shows that \( P(n\theta, n\phi) \) can be expressed as a \textit{polynomial} with respect to \( P(\theta, \phi) \) and \( P(-\theta, -\phi) \). We observe that

\[
P(-\theta, -\phi) = P^2(\theta, \phi) - Q(\theta, \phi) R(\theta, \phi),
\]

so that \( P(n\theta, n\phi) \) is a polynomial with respect to \( P \) and \( QR \). For instance

\[
P(2\theta, 2\phi) = P^2 + 2QR = 3P^2(\theta, \phi) - 2P(-\theta, -\phi)
\]

\[
P(3\theta, 3\phi) = 1 + 9PQR = 9P^3(\theta, \phi) - 9P(\theta, \phi) P(-\theta, -\phi) + 1.
\]

Our expansion leads readily to the following general result:

\[
\frac{P(n\theta, n\phi)}{n} = \sum_{p} \sum_{q} \frac{(-1)^p 3^{p+q} C_{n+2p+q} C_{n+q-p} P^p(\theta, \phi) P^q(-\theta, -\phi)}{n + 2p + q}
\]

with \( p \leq n, q \leq \frac{1}{3}(n - p) \). The symbol \( C_r \) stands for the number of combinations of \( r \) objects \( s \) at a time. Of course \( \frac{1}{3}(n + 2p + q) \) and \( \frac{1}{3}(n + q - p) \) must be positive integers.
§ 4. Similar formulae can, of course, be written for $Q(n\theta, n\phi)$ and $R(n\theta, n\phi)$. We may use the relations
\[ Q(-\theta, -\phi) = Q^3 - RP \]
\[ R(-\theta, -\phi) = R^3 - PQ. \]

If we take $\phi = 0$, the function $P$ reduces to one of the sines of the third order,
\[ f_1(\theta) = \frac{1}{3} (e^\theta + e^{i\theta} + e^{2i\theta}), \]
and we obtain the expansion
\[ - \log [1 - 3a f_1(\theta) + 3a^2 f_1(-\theta) - a^3] = \sum_{n} 3a^n f_1(n\theta)/n, \]
showing that $f_1(n\theta)$ can be expressed as a polynomial with respect to $f_1(\theta)$ and $f_1(-\theta)$.

We may, perhaps, suggest the following researches:

(a) To express $P(n\theta, n\phi)$ as an hypergeometric function of two variables and of the third order (one of the functions introduced by J. Kampé de Fériet).

(b) To extend the result to sines of the 4th, etc., order, with one or two variables.

(c) To find a generating function for $P(k\theta, k\phi)$. 
