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A Computation with the Connes–Thom Isomorphism

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Abstract. Let $A \in M_n(\mathbb{R})$ be an invertible matrix. Consider the semi-direct product $\mathbb{R}^n \rtimes \mathbb{Z}$ where the action of $\mathbb{Z}$ on $\mathbb{R}^n$ is induced by the left multiplication by A. Let (α, τ) be a strongly continuous action of $\mathbb{R}^n \rtimes \mathbb{Z}$ on a C^*-algebra B where α is a strongly continuous action of $\mathbb{R}^n$ and τ is an automorphism. The map τ induces a map $\widetilde{\tau}$ on $B \rtimes_\alpha \mathbb{R}^n$. We show that, at the K-theory level, τ commutes with the Connes–Thom map if $\det(A) > 0$ and anticommutes if $\det(A) < 0$. As an application, we recompute the K-groups of the Cuntz–Li algebra associated with an integer dilation matrix.

1 Introduction

In [4], Cuntz and Li initiated the study of C^*-algebras associated with rings. In [6], Cuntz had earlier studied the C^*-algebra associated with the $ax + b$ group over $\mathbb{N}$. These C^*-algebras are unital, purely infinite, and simple. Thus they are classified by their K-groups. In a series of papers, [3,5,11], the K-groups of these algebras associated with number fields and function fields were computed. The main tool used in the K-group computation was the duality result proved in [5] and its variations.

Other approaches and possible generalisations were considered in [1, 10, 13]. The C^*-algebras studied in [10] and in [13] were called Cuntz–Li algebras. Following [10], in [13], the Cuntz–Li algebra associated with a pair $(N \rtimes H, M)$ satisfying certain conditions was studied. Here, M is a normal subgroup of N and $N \rtimes H$ is a semidirect product . The main example considered in [13] is the Cuntz–Li algebra, denoted by $\mathcal{U}_\Gamma$, associated with the pair $(\mathbb{Q}^n \rtimes \Gamma, \mathbb{Z}^n)$, where Γ is a subgroup of $GL_n(\mathbb{Q})$ acting by matrix multiplication on $\mathbb{Q}^n$. In [13], it was proved that $\mathcal{U}_\Gamma$ is Morita-equivalent to $C(X) \rtimes (\mathbb{R}^n \rtimes \Gamma)$ for some compact Hausdorff space X. This is the analog of the Cuntz–Li duality theorem for the algebra $\mathcal{U}_\Gamma$.

A matrix $A \in M_d(\mathbb{Z})$ is called an *integer dilation matrix* if all its eigenvalues are of absolute value greater than 1. In [8], a purely infinite simple C^*-algebra associated with an integer dilation matrix was studied and its K-groups were computed. Their computation depends on realising the C^*-algebra as a Cuntz–Pimsner algebra and by a careful examination of the six term sequence coming from its Toeplitz extension. In [12], a presentation of this algebra was obtained in terms of generators and relations. For the group $\Gamma := \{(A^t)^r : r \in \mathbb{Z}\} \cong \mathbb{Z}$, denote the Cuntz–Li algebra $\mathcal{U}_\Gamma$ by $\mathcal{U}_{A^t}$. The

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presentation given in [12] tells us that the C^*-algebra studied in [8] is the Cuntz–Li algebra $\mathfrak{U}_{A^t}$.

The purpose of this paper is to understand the K-groups of $\mathfrak{U}_{A^t}$ in view of the Cuntz–Li duality theorem. The Cuntz–Li duality theorem in this case says that $\mathfrak{U}_{A^t}$ is Morita-equivalent to a crossed product algebra $(C(X) \rtimes \mathbb{R}^d) \rtimes \mathbb{Z}$ for some compact Hausdorff space. We compute the K-groups using the Pimsner–Voiculescu sequence. I believe that this computation will be of independent interest for the following two reasons:

(a) The K-groups of $\mathfrak{U}_{A^t}$ depends on both d and $\operatorname{sign}(\det(A))$ (*cf.* [8]). The dependence on d is due to the Connes–Thom isomorphism between $K_*(C(X))$ and $K_*(C(X) \rtimes \mathbb{R}^d)$. Also the Connes–Thom map commutes with the action of $\mathbb{Z}$ if $\operatorname{sign}(\det(A)) > 0$ and anticommutes if $\operatorname{sign}(\det(A)) < 0$. This explains the dependence on $\operatorname{sign}(\det(A))$.

(b) It is mentioned in the introduction of [5] that the duality theorem enables one to use homotopy type arguments, which makes it possible to compute the K-groups. We see the same kind of phenomenon here as well (*cf.* Lemma 3.2).

Let $A \in GL_n(\mathbb{R})$. Consider the semidirect product $\mathbb{R}^n \rtimes \mathbb{Z}$ where $\mathbb{Z}$ acts on $\mathbb{R}^n$ by matrix multiplication by A. Let B be a C^*-algebra on which $\mathbb{R}^n \rtimes \mathbb{Z}$ acts. The crossed product $B \rtimes (\mathbb{R}^n \rtimes \mathbb{Z})$ is isomorphic to $(B \rtimes \mathbb{R}^n) \rtimes \mathbb{Z}$. In Sections 2 and 3, we write down the Pimsner–Voiculescu sequence for $(B \rtimes \mathbb{R}^n) \rtimes \mathbb{Z}$ after identifying the crossed product $B \rtimes \mathbb{R}^n$ with B up to KK-equivalence. We show that the Connes–Thom isomorphism commutes with the action of $\mathbb{Z}$ if $\det(A) > 0$ and anticommutes if $\det(A) < 0$. In Sections 4 and 5, the K-groups of $\mathfrak{U}_{A^t}$ are (re)computed.

2 Preliminaries

We use this section to fix notation and recall a few preliminaries. Let $A \in M_n(\mathbb{R})$ be such that $\det(A) \neq 0$. We think of elements of $\mathbb{R}^n$ as column vectors. Thus, the matrix A induces an action of $\mathbb{Z}$ on $\mathbb{R}^n$ by left multiplication. The generator $1 \in \mathbb{Z}$ acts on $\mathbb{R}^n$ by $1.v = Av$ for $v \in \mathbb{R}^n$. Consider the semidirect product $\mathbb{R}^n \rtimes \mathbb{Z}$.

All the C^*-algebras considered in this paper are assumed to be separable. Let B be a C^*-algebra. A strongly continuous action of $\mathbb{R}^n \rtimes \mathbb{Z}$ on B is equivalent to providing a pair (α, τ), where α is a strongly continuous action of $\mathbb{R}^n$ on B and τ is an automorphism of B such that $\tau\alpha_\xi = \alpha_{A\xi}\tau$ for every $\xi \in \mathbb{R}^n$. If (α, τ) is such a pair, we write $\alpha \rtimes \tau$ for the action of $\mathbb{R}^n \rtimes \mathbb{Z}$. Also, the automorphism τ induces an action, denoted $\widetilde{\tau}$, on the crossed product $B \rtimes_\alpha \mathbb{R}^n$ given by $\widetilde{\tau}(b) := \tau(b)$ for $b \in B$ and $\widetilde{\tau}(U_\xi) := U_{A\xi}$ for $\xi \in \mathbb{R}^n$, where U_ξ denotes the canonical unitary in $\mathcal{M}(B \rtimes_\alpha \mathbb{R}^n)$. Moreover, the crossed product $B \rtimes_{\alpha \rtimes \tau} (\mathbb{R}^n \rtimes \mathbb{Z})$ is isomorphic to $(B \rtimes_\alpha \mathbb{R}^n) \rtimes_{\widetilde{\tau}} \mathbb{Z}$.

The Pimsner–Voiculescu sequence gives the following six-term exact sequence

$$\begin{array}{ccccc}
K_0(B \rtimes_\alpha \mathbb{R}^n) & \xrightarrow{1-\widetilde{\tau}_*} & K_0(B \rtimes_\alpha \mathbb{R}^n) & \longrightarrow & K_0(B \rtimes_{\alpha \rtimes \tau} (\mathbb{R}^n \rtimes \mathbb{Z})) \\
\uparrow & & & & \downarrow \\
K_1(B \rtimes_{\alpha \rtimes \tau} (\mathbb{R}^n \rtimes \mathbb{Z})) & \longleftarrow & K_1(B \rtimes_\alpha \mathbb{R}^n) & \xleftarrow[1-\widetilde{\tau}_*]{} & K_1(B \rtimes_\alpha \mathbb{R}^n).
\end{array}$$

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But by the Connes–Thom isomorphism, we can replace $K_i(B \rtimes_\alpha \mathbb{R}^n)$ by $K_{i+n}(B)$. Let $C_{n,i}: K_i(B) \to K_{i+n}(B \rtimes_\alpha \mathbb{R}^n)$ be the Connes–Thom map. Now we can state our main theorem.

Theorem 2.1 *For $i = 1, 2$, we have $C_{n,i}\tau_* = \epsilon\widetilde{\tau}_* C_{n,i}$, where ϵ is given by*

$$\epsilon := \begin{cases} 1 & \text{if } \det(A) > 0, \\ -1 & \text{if } \det(A) < 0. \end{cases}$$

The following is an immediate corollary of Theorem 2.1.

Corollary 2.2 *Let (α, τ) be a strongly continuous action of $\mathbb{R}^n \rtimes \mathbb{Z}$ on a C^*-algebra B. Then there exists a six term exact sequence*

$$\begin{array}{ccccc} K_n(B) & \xrightarrow{1-\epsilon\tau_*} & K_n(B) & \longrightarrow & K_0(B \rtimes_{\alpha\rtimes\tau} (\mathbb{R}^n \rtimes \mathbb{Z})) \\ \uparrow & & & & \downarrow \\ K_1(B \rtimes_{\alpha\rtimes\tau} (\mathbb{R}^n \rtimes \mathbb{Z})) & \longleftarrow & K_{n+1}(B) & \xleftarrow[1-\epsilon\tau_*]{} & K_{n+1}(B), \end{array}$$

where $\epsilon = \operatorname{sign}(\det(A))$.

3 Proof of Theorem 2.1

We use KK-theory to prove Theorem 2.1. All our algebras are ungraded. We denote the interior Kasparov product

$$KK^{(i)}(A, B) \times KK^{(j)}(B, C) \longrightarrow KK^{(i+j)}(A, C)$$

by $\sharp$ and the external Kasparov product

$$KK^{(i)}(A_1, A_2) \times KK^{(j)}(B_1, B_2) \longrightarrow KK^{(i+j)}(A_1 \otimes A_2, B_1 \otimes B_2)$$

by $\widehat{\otimes}$. We will also identify $K_i(B)$ with $KK^{(i)}(\mathbb{C}, B)$. Also, if $\phi: B_1 \to B_2$ is a C^*-algebra homomorphism, then we denote the KK-element $(B_2, \phi, 0)$ in $KK^{(0)}(B_1, B_2)$ by $[\phi]$.

Under this identification, the Connes–Thom isomorphism is given by $C_n(x) = x \sharp t_\alpha$ where $t_\alpha \in KK^{(n)}(B, B \rtimes_\alpha \mathbb{R}^n)$ is the Thom element.

Now it is immediate that Theorem 2.1 is equivalent to the following theorem.

Theorem 3.1 *One has $[\tau] \sharp t_\alpha = \epsilon\, t_\alpha \sharp [\widetilde{\tau}]$, where $\epsilon = \operatorname{sign}(\det(A))$.*

We now add a bit of notation. If $X \in GL_n(\mathbb{R})$, then X induces an automorphism ϕ_X on $C_0(\mathbb{R}^n)$ given by $(\phi_X f)(v) := f(Xv)$. Let $b_n \in K_n(C_0(\mathbb{R}^n))$ be the Bott element. We denote the image $\phi_{X*}(b_n) \in K_n(C_0(\mathbb{R}^n))$ simply by $X_*(b_n)$.

First let us dispose of the case when the action of $\mathbb{R}^n$ is trivial. For the trivial action the crossed product $B \rtimes_\alpha \mathbb{R}^n$ is isomorphic to $B \otimes C_0(\mathbb{R}^n)$ and $t_{\text{trivial}} = 1_B \widehat{\otimes} b_n$.

Lemma 3.2 *If the action of $\mathbb{R}^n$ is trivial, then $[\tau] \sharp t_{trivial} = \epsilon\, t_{trivial} \sharp [\widetilde{\tau}]$, where* $\epsilon = \operatorname{sign}(\det(A))$.

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Proof Note that

$$[\tau] \,\sharp\, t_{\text{trivial}} = [\tau] \mathbin{\widehat{\otimes}} b_n \quad\text{and}\quad t_{\text{trivial}} \,\sharp\, [\widetilde{\tau}] = [\tau] \mathbin{\widehat{\otimes}} A^t_*(b_n).$$

Thus we only need to prove that $A^t_*(b_n) = \epsilon b_n$, where $\epsilon = \operatorname{sign}(\det(A))$.

If $\det(A) > 0$, then A^t is homotopic to identity in $GL_n(\mathbb{R})$. Hence, $A^t_*(b_n) = b_n$.

If $\det(A) < 0$, then A^t is homotopic to $\left(\begin{smallmatrix} -1 & 0 \\ 0 & Id_{n-1}\end{smallmatrix}\right)$ in $GL_n(\mathbb{R})$. It follows that the matrix $\left(\begin{smallmatrix} -1 & 0 \\ 0 & Id_{n-1}\end{smallmatrix}\right)$ sends the Bott element $b_n = b_1 \widehat{\otimes} b_1 \widehat{\otimes} \cdots \widehat{\otimes} b_1$ to $-b_n$. As a consequence, we have $A^t_*(b_n) = -b_n$ if $\det(A) < 0$. This completes the proof. ∎

Now, by the homotopy argument used in [9, Theorem 2], we reduce Theorem 3.1 to Lemma 3.2.

For $s \in [0,1]$, let α^s be the action of $\mathbb{R}^n$ on B defined by $\alpha^s_\xi(b) := \alpha_{s\xi}(b)$. Note that $\alpha^1 = \alpha$, and α^0 gives the trivial action. Observe that $\tau\alpha^s_\xi = \alpha^s_{A\xi}\tau$. For $s \in [0,1]$, denote the automorphism τ by τ^s and the automorphism induced by τ on $B \rtimes_{\alpha^s} \mathbb{R}^n$ by $\widetilde{\tau^s}$.

Let $IB := C[0,1] \otimes B$. Consider the action $\underline{\alpha}$ of $\mathbb{R}^n$ and the automorphism $\underline{\tau}$ on IB defined by

$$\underline{\alpha}_\xi(f)(s) := \alpha^s_\xi(f(s)), \quad \underline{\tau}(f)(s) := \tau(f(s)).$$

Observe that for $\xi \in \mathbb{R}^n$, $\underline{\tau}\,\underline{\alpha}_\xi = \underline{\alpha}_{A\xi}\,\underline{\tau}$. The automorphism $\underline{\tau}$ induces an automorphism on $IB \rtimes_{\underline{\alpha}} \mathbb{R}^n$ and we denote it by $\widetilde{\underline{\tau}}$.

For $s \in [0,1]$, let $\epsilon_s\colon IB \to B$ be the evaluation map. Then $\epsilon_s\colon (IB, \underline{\alpha}) \to (B, \alpha^s)$ is equivariant. We denote the induced map from $IB \rtimes_{\underline{\alpha}} \mathbb{R}^n$ to $B \rtimes_{\alpha^s} \mathbb{R}^n$ by $\widehat{\epsilon_s}$. Also for $s \in [0,1]$, $\widehat{\epsilon_s} \circ \widetilde{\underline{\tau}} = \widetilde{\tau^s} \circ \widehat{\epsilon_s}$.

Lemma 3.3 *For $s \in [0,1]$, the element $[\widehat{\epsilon_s}] \in KK^{(0)}(IB \rtimes_{\underline{\alpha}} \mathbb{R}^n, B \rtimes_{\alpha^s} \mathbb{R}^n)$ is a KK-equivalence.*

Proof Observe that $t_{\underline{\alpha}} \,\sharp\, [\widehat{\epsilon_s}] = [\epsilon_s] \,\sharp\, t_{\alpha^s}$. Since $[\epsilon_s] \in KK^{(0)}(IB, B)$ and the Thom elements are KK-equivalences, it follows that $[\widehat{\epsilon_s}]$ is a KK-equivalence. This completes the proof. ∎

Proposition 3.4 *The following are equivalent. Recall that $\epsilon = \operatorname{sign}(\det(A))$.*

(i) *For every $s \in [0,1]$, $[\tau^s] \,\sharp\, t_{\alpha^s} = \epsilon\, t_{\alpha^s} \,\sharp\, [\widetilde{\tau^s}]$.*

(ii) *There exists $s \in [0,1]$ such that $[\tau^s] \,\sharp\, t_{\alpha^s} = \epsilon\, t_{\alpha^s} \,\sharp\, [\widetilde{\tau^s}]$.*

(iii) *The Kasparov product $[\underline{\tau}] \,\sharp\, [t_{\underline{\alpha}}] = \epsilon\, t_{\underline{\alpha}} \,\sharp\, [\widetilde{\underline{\tau}}]$.*

Proof Let $s \in [0,1]$ be given. Observe the following.

$$\begin{aligned}
&[\underline{\tau}] \,\sharp\, t_{\underline{\alpha}} = \epsilon\, t_{\underline{\alpha}} \,\sharp\, [\widetilde{\underline{\tau}}] \\
&\quad\Leftrightarrow [\underline{\tau}] \,\sharp\, t_{\underline{\alpha}} \,\sharp\, [\widehat{\epsilon_s}] = \epsilon\, t_{\underline{\alpha}} \,\sharp\, [\widetilde{\underline{\tau}}] \,\sharp\, [\widehat{\epsilon_s}] \text{ (Since } [\widehat{\epsilon_s}] \text{ is a } KK\text{-equivalence.)} \\
&\quad\Leftrightarrow [\underline{\tau}] \,\sharp\, [\epsilon_s] \,\sharp\, t_{\alpha^s} = \epsilon\, t_{\underline{\alpha}} \,\sharp\, [\widehat{\epsilon_s} \circ \widetilde{\underline{\tau}}] \\
&\quad\Leftrightarrow [\epsilon_s \circ \underline{\tau}] \,\sharp\, t_{\alpha^s} = \epsilon\, t_{\underline{\alpha}} \,\sharp\, [\widetilde{\tau^s} \circ \widehat{\epsilon_s}] \\
&\quad\Leftrightarrow [\tau^s \circ \epsilon_s] \,\sharp\, t_{\alpha^s} = \epsilon\, t_{\underline{\alpha}} \,\sharp\, [\widehat{\epsilon_s}] \,\sharp\, [\widetilde{\tau^s}] \\
&\quad\Leftrightarrow [\epsilon_s] \,\sharp\, [\tau^s] \,\sharp\, t_{\alpha^s} = \epsilon\, [\epsilon_s] \,\sharp\, t_{\alpha^s} \,\sharp\, [\widetilde{\tau^s}] \\
&\quad\Leftrightarrow [\tau^s] \,\sharp\, [t_{\alpha^s}] = \epsilon\, t_{\alpha^s} \,\sharp\, [\widetilde{\tau^s}] \text{ (Since } [\epsilon_s] \text{ is a } KK\text{-equivalence.)}
\end{aligned}$$

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The proof is now complete. ■

Theorem 3.1 now follows from Proposition 3.4 and Lemma 3.2.

4 The Cuntz–Li Algebra Associated with an Integer Dilation Matrix

As an application of Corollary 2.2, we recompute the K-theory of the C^*-algebra associated with an integer dilation matrix studied in [8]. Let us recall the C^*-algebra considered in [8].

For the rest of the paper, we let $A \in M_d(\mathbb{Z})$ denote an integer dilation matrix *i.e.*, all the eigenvalues of A are of absolute value greater than 1. The matrix A acts on $\mathbb{R}^d$ by matrix multiplication and leaves $\mathbb{Z}^d$ invariant. Denote the resulting endomorphism on $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$ by σ_A. The map σ_A is surjective and has finite fibres. Denote the map $C(\mathbb{T}^d) \ni f \to f \circ \sigma_A \in C(\mathbb{T}^d)$ by α_A. Consider the transfer operator $L\colon C(\mathbb{T}^d) \to C(\mathbb{T}^d)$ defined by

$$L(f)(x) := \frac{1}{|\sigma_A^{-1}(x)|} \sum_{\sigma_A(y)=x} f(y).$$

Then L satisfies the condition $L(\alpha_A(f)g) = fL(g)$ for $f, g \in C(\mathbb{T}^d)$. In [8], the Exel Crossed product $C(\mathbb{T}^d) \rtimes_{\alpha_A,L} \mathbb{N}$ was viewed as a Cuntz–Pimsner algebra $\mathcal{O}(M_L)$ of a suitable Hilbert $C(\mathbb{T}^d)$ bimodule M_L.

By a careful examination of the six term sequence (and the maps involved) associated with the exact sequence $0 \to \text{Ker}(Q) \to \mathcal{T}(M_L) \to \mathcal{O}(M_L) \to 0$, the K-groups of $C(\mathbb{T}^d) \rtimes_{\alpha_A} \mathbb{N}$ were computed in [8]. Here Q denotes the quotient map $\mathcal{T}(M_L) \to \mathcal{O}(M_L)$.

For our purposes, the following description of $\mathcal{O}(M_L)$ in terms of generators and relations is more relevant. Let us recall the following proposition from [12, Proposition 3.3].

Proposition 4.1 ([12]) *The Exel crossed product $C(\mathbb{T}^d) \rtimes_{\alpha_A,L} \mathbb{N}$ is the universal C^*-algebra generated by an isometry v and unitaries $\{u_m : m \in \mathbb{Z}^d\}$ satisfying the following relations:*

$$u_m u_n = u_{m+n}, \quad v u_m = u_{A^t m} v, \quad \sum_{m \in \Sigma} u_m v v^* u_m^{-1} = 1.$$

Here Σ denotes a set of distinct coset representatives of the group $\mathbb{Z}^d/A^t\mathbb{Z}^d$.

The first two relations are called $(E1)$ and the last relation is called $(E3)$ in [12]. In [12], the following relation

$$v^* u_m v := \begin{cases} 0 & \text{if } m \notin A^t\mathbb{Z}^d, \\ u_{(A^t)^{-1}m} & \text{if } m \in A^t\mathbb{Z}^d, \end{cases}$$

called $(E2)$ is also considered. But it is superfluous as $(E1)$ and $(E3)$ imply $(E2)$. For if $m \notin A^t\mathbb{Z}^d$, the projections vv^* and $u_m v v^* u_m^{-1}$ are orthogonal by $(E3)$. Hence $v^* u_m v = 0$ if $m \notin A^t\mathbb{Z}^d$. If $m \in A^t\mathbb{Z}^d$, then using $v u_m = u_{A^t m} v$, one obtains $v^* u_m v = u_{(A^t)^{-1}m}$. Thus, $(E1)$ and $(E3)$ imply $(E2)$.

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The following setup was initially considered in [10]. Consider a semi-direct product $N \rtimes H$ and let M be a normal subgroup of N. Let $P := \{a \in H : aMa^{-1} \subset M\}$. Then P is a semigroup containing the identity e. For $a \in P$, let $M_a = aMa^{-1}$. Assume that the following hold:

(C1) The group $H = PP^{-1} = P^{-1}P$.
(C2) For every $a \in P$, the subgroup aMa^{-1} is of finite index in M.
(C3) The intersection $\bigcap_{a \in P} aMa^{-1} = \{e\}$, where e denotes the identity element of G.

Definition 4.2 The Cuntz–Li algebra associated with the pair $(N \rtimes H, M)$ is the the universal C^*-algebra generated by a set of isometries $\{s_a : a \in P\}$ and a set of unitaries $\{u(m) : m \in M\}$ satisfying the relations

$$s_a s_b = s_{ab}, \qquad u(m)u(n) = u(mn),$$
$$s_a u(m) = u(ama^{-1})s_a, \qquad \sum_{k \in M/M_a} u(k)e_a u(k)^{-1} = 1,$$

where e_a denotes the final projection of s_a. We denote the Cuntz–Li algebra associated with the pair $(N \rtimes H, M)$ by $\mathfrak{A}[N \rtimes H, M]$.

Let $A \in M_d(\mathbb{Z})$ be a dilation matrix. Then A acts on $\mathbb{Q}^d$ by left multiplication. Consider the semidirect product $\mathbb{Q}^d \rtimes \mathbb{Z}$ and the normal subgroup $\mathbb{Z}^d$ of $\mathbb{Q}^d$. For this pair $(\mathbb{Q}^d \rtimes \mathbb{Z}, \mathbb{Z}^d)$, $P = \{A^r : r \geq 0\} \cong \mathbb{N}$. Moreover, conditions (C1)–(C3) are satisfied. (See [13, Example 2.6, p. 3].) Let us denote the Cuntz–Li algebra $\mathfrak{A}[\mathbb{Q}^d \rtimes \mathbb{Z}, \mathbb{Z}^d]$ simply by $\mathfrak{U}_A$. By using the presentation (*cf.* Prop. 4.1) of the Exel's Crossed product $C(\mathbb{T}^d) \rtimes_{\alpha_A, L} \mathbb{N}$ given in terms of isometries and unitaries, it follows that $C(\mathbb{T}^d) \rtimes_{\alpha_A, L} \mathbb{N}$ is isomorphic to $\mathfrak{U}_{A^t}$.

First we recall the Cuntz–Li duality result proved in [13]. Let $N_A := \bigcup_{r=0}^{\infty} A^{-r}\mathbb{Z}^d$. Then N_A is a subgroup of $\mathbb{R}^d$. Let $\mathbb{Z}$ act on $\mathbb{R}^d$ by left multiplication by A^t and consider the semidirect product $\mathbb{R}^d \rtimes \mathbb{Z}$. The semidirect product $\mathbb{R}^d \rtimes \mathbb{Z}$ acts on the C^*-algebra $C^*(N_A)$. The action α of $\mathbb{R}^d$ and the automorphism τ, corresponding to the action of $\mathbb{Z}$, are given by

$$\alpha_\xi(\delta_\nu) := e^{-2\pi i \langle \xi, \nu \rangle} \delta_\nu \text{ for } \xi \in \mathbb{R}^d, \quad \tau(\delta_\nu) := \delta_{A^{-1}\nu}$$

where $\{\delta_\nu : \nu \in N_A\}$ denotes the canonical unitaries in $C^*(N_A)$ and $\langle \cdot, \cdot \rangle$ denotes the usual inner product on $\mathbb{R}^d$. Note that $\tau\alpha_\xi = \alpha_{A^t\xi}\tau$ for $\xi \in \mathbb{R}^d$.

The following proposition was proved in [13] (*cf.* [13, Theorem 8.2 and Proposition 8.6]).

Proposition 4.3 *The C^*-algebra $\mathfrak{U}_{A^t}$ is Morita-equivalent to*

$$C^*(N_A) \rtimes_{\alpha \rtimes \tau} (\mathbb{R}^d \rtimes_{A^t} \mathbb{Z}).$$

Now using the Morita-equivalence in Proposition 4.3 and using the version of Pimsner–Voiculescu exact sequence established in Corollary 2.2, the K-groups of $\mathfrak{U}_{A^t}$ can be computed.

5 K-groups of the Cuntz–Li algebra $\mathfrak{U}_{A^t}$

Recall that $N_A := \bigcup_{r=0}^{\infty} A^{-r}\mathbb{Z}^d$. Set $N_A^{(r)} := A^{-r}\mathbb{Z}^d$. Then $\{N_A^{(r)}\}_{r=0}^{\infty}$ forms an increasing sequence of subgroups, each isomorphic to $\mathbb{Z}^d$ and $N_A = \bigcup_{r=0}^{\infty} N_A^{(r)}$. Thus, $C^*(N_A)$ is the inductive limit of $C^*(N_A^{(r)}) \cong C(\mathbb{T}^d)$. Hence $K_*(C^*(N_A))$ can be computed as the inductive limit of the K-groups of $C^*(N_A^{(r)}) \cong C(\mathbb{T}^d)$.

Let us first recall the K-theory of $C(\mathbb{T}^d) \cong C^*(\mathbb{Z}^d)$. It is well known and can be proved by the Kunneth formula that as a $\mathbb{Z}_2$ graded ring, $K_*(C^*(\mathbb{Z}^d))$ is isomorphic to the exterior algebra $\Lambda^*(\mathbb{Z}^d)$. The map $\mathbb{Z}^d \ni e_i \to \delta_{e_i} \in K_1(C^*(\mathbb{Z}^d))$ extends to a graded ring isomorphism from $\Lambda^*(\mathbb{Z}^d)$ to $K_*(C^*(\mathbb{Z}^d))$.

Remark 5.1 The isomorphism $K_*(C(\mathbb{T}^d)) \cong \Lambda^*\mathbb{Z}^d$ was also used in [8].

Let us now fix some notations. For $0 \le n \le d$, let A_n be the map on $\Lambda^n(\mathbb{Z}^d)$ induced by A. Thus $A_0 = 1$, $A_1 = A$ and $A_d = \det(A)$. For a subset I of $\{1,2,\dots,d\}$, of cardinality n (arranged in increasing order), $I = \{i_1 < i_2 < \cdots < i_n\}$, let $e_I := e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n}$. Then $\{e_I : |I| = n\}$ is a basis for $\Lambda^n(\mathbb{Z}^d)$. For subsets J,K of $\{1,2,\dots,d\}$ of size n, let $A_{J,K}$ be the submatrix of A obtained by considering the rows coming from J and columns coming from K. With respect to the basis $\{e_I : |I| = n\}$, the $(J,K)^{th}$ entry of the matrix corresponding to A_n is $\det(A_{J,K})$.

Note that for $n \ge 1$, A_n is again an integer dilation matrix. For if we upper triangualise A, then with respect to the basis $\{e_I : |I| = n\}$, arranged in lexicographic order, A_n is upper triangular and the eigenvalues of A_n are product of eigenvalues of A. Thus the eigenvalues of A_n are of absolute value greater than 1. This fact was used in [8] (*cf.* [8, Proposition 4.6]).

Let $n \in \{0,1,2,\dots,d\}$. Consider $\Lambda^n(\mathbb{Z}^d)$ as a subgroup of $\Lambda^n(\mathbb{Q}^d)$. Then A_n is invertible on $\Lambda^n(\mathbb{Q}^d)$. Set $\Gamma_n := \bigcup_{r=0}^{\infty} A_n^{-r}(\Lambda^n(\mathbb{Z}^d))$.

Proposition 5.2 *The K-groups of the C^*-algebra $C^*(N_A)$ are given by*

$$K_0(C^*(N_A)) \cong \bigoplus_{\substack{n\ even\\ 0\le n\le d}} \Gamma_n \quad \text{and} \quad K_1(C^*(N_A)) \cong \bigoplus_{\substack{n\ odd\\ 0\le n\le d}} \Gamma_n.$$

Proof Since $C^*(N_A)$ is the inductive limit of $C^*(N_A^{(r)}) \cong C(\mathbb{T}^d)$, it follows that $K_*(C^*(N_A))$ is the inductive limit of $K_*(C^*(N_A^{(r)})) \cong K_*(C(\mathbb{T}^d))$. Identify $K_*(C^*(N_A^{(r)}))$ with $\Lambda^*(\mathbb{Z}^d)$ via the map $\delta_{A^{-r}(e_i)} \to e_i$. With this identification, the inclusion map $C^*(N_A^{(r)}) \to C^*(N_A^{-(r+1)})$ induces the map $\oplus_{0\le n\le d} A_n$ at the K-theory level. (Reason: If we write e_j as a linear combination of $\{A^{-1}e_i\}$ the matrix involved is just A.)

Thus we are left to show that the inductive limit of

$$\Big(\bigoplus_n \Lambda^n\mathbb{Z}^d, \bigoplus_n A_n\Big)_{r=0}^{\infty}$$

is $\bigoplus_n \Gamma_n$. Again, it is enough to show that the inductive limit of $(\Lambda^n\mathbb{Z}^d, A_n)_{r=0}^{\infty}$ is isomorphic to Γ_n. Let $H_r = \Lambda^n\mathbb{Z}^d$. If $v \in \Gamma_n$, write v as $v = A_n^{-r}w$ with $w \in \Lambda^n(\mathbb{Z}^d)$.

The map $\Gamma_n \ni v \to w \in H_r$ is an isomorphism between Γ_n and $\lim_{r\to\infty}(\Lambda^n(\mathbb{Z}^d), A_n)$. This completes the proof. ∎

Now let us calculate the automorphism τ on $C^*(N_A)$. Recall that τ on the generating unitaries is given by $\tau(\delta_v) = \delta_{A^{-1}v}$. Thus, it is immediate that τ induces the map $\bigoplus_n A_n^{-1}$ on $\bigoplus_n \Gamma_n$ when one identifies $K_*(C^*(N_A))$ with $\bigoplus_n \Gamma_n$ (together with their $\mathbb{Z}_2$ grading).

For $r \geq 0$, let $j_r\colon C^*(\mathbb{Z}^d) \to C^*(N_A)$ be the inclusion given by $j_r(\delta_v) = \delta_{A^{-r}v}$. Proposition 5.2 implies that the maps j_r induce injective maps at the K-theory level. Let $\widetilde{\tau}\colon C^*(\mathbb{Z}^d) \to C^*(\mathbb{Z}^d)$ be the homomorphism given on the generators by $\widetilde{\tau}(\delta_v) = \delta_{Av}$. Observe that for $r \geq 0$, $j_{r+1}\widetilde{\tau} = j_r$, and $\tau j_r = j_{r+1}$.

Theorem 5.3 *The K-groups of the Cuntz–Li algebra $\mathfrak{U}_{A^t}$ fit into an exact sequence as follows:*

$$\begin{array}{ccccc}
K_d(C^*(\mathbb{Z}^d)) & \xrightarrow{1-\epsilon\widetilde{\tau}_*} & K_d(C^*(\mathbb{Z}^d)) & \longrightarrow & K_0(\mathfrak{U}_{A^t}) \\
\uparrow & & & & \downarrow \\
K_1(\mathfrak{U}_{A^t}) & \longleftarrow & K_{d+1}(C^*(\mathbb{Z}^d)) & \xleftarrow[1-\epsilon\widetilde{\tau}_*]{} & K_{d+1}(C^*(\mathbb{Z}^d)),
\end{array}$$

where $\epsilon := \operatorname{sign}(\det(A))$.

Proof Our main tool is Corollary 2.2 and the Morita equivalence between $\mathfrak{U}_{A^t}$ and $C^*(N_A) \rtimes (\mathbb{R}^n \rtimes \mathbb{Z})$. By Corollary 2.2, we have the six term sequence

$$\begin{array}{ccccc}
K_d(C^*(N_A)) & \xrightarrow{1-\epsilon\tau_*} & K_d(C^*(N_A)) & \longrightarrow & K_0(\mathfrak{U}_{A^t}) \\
\delta\uparrow & & & & \downarrow\sigma \\
K_1(\mathfrak{U}_{A^t}) & \longleftarrow & K_{d+1}(C^*(N_A)) & \xleftarrow[1-\epsilon\tau_*]{} & K_{d+1}(C^*(N_A)).
\end{array}$$

By Proposition 5.2, we know that $K_*(C^*(N_A)) = \bigoplus_{0\leq n\leq d} \Gamma_n$ as $\mathbb{Z}_2$ graded abelian groups. Also $1 - \epsilon\tau_* = \bigoplus_{0\leq n\leq d}(1 - \epsilon A_n^{-1})$. But A_n is an integer dilation matrix for $1 \leq n \leq d$. Hence neither 1 nor -1 is an eigenvalue of A_n for $1 \leq n \leq d$. Thus, $\operatorname{Ker}(1-\epsilon\tau_*)$ is $\mathbb{Z}$ if $\epsilon = 1$ and 0 if $\epsilon = -1$. In any case, $\operatorname{Ker}(1-\epsilon\tau_*) \subset j_{0*}(K_*(C^*(\mathbb{Z}^d)))$. Thus the image of both σ and δ is contained in $j_{0*}(K_*(C^*(\mathbb{Z}^d))$. Also, j_{0*} is injective and hence we can replace $K_*(C^*(N_A))$ appearing at the corners of the two rows of the six term sequence by $j_{0*}K_*(C^*(\mathbb{Z}^d)) \cong K_*(C^*(\mathbb{Z}^d))$. The automorphism τ maps $j_0(C^*(\mathbb{Z}^d))$ into $j_1(C^*(\mathbb{Z}^d))$. Thus, $K_*(C^*(N_A))$ appearing in the middle of the two rows can be replaced by $j_{1*}K_*((C^*(\mathbb{Z}^d)) \cong K_*(C^*(\mathbb{Z}^d))$.

Now the commutation relation $\tau j_0 = j_1$ and $j_1\widetilde{\tau} = j_0$ implies that we can replace $1 - \epsilon\tau_*$ appearing in the exact sequence by $\widetilde{\tau}_* - \epsilon 1$. We can multiply any morphism appearing in the exact sequence by $-\epsilon$ without changing the exactness of the sequence.

Thus we obtain the following six term sequence

$$\begin{array}{ccccc} K_d(C^*(\mathbb{Z}^d)) & \xrightarrow{1-\epsilon\widetilde{\tau}_*} & K_d(C^*(\mathbb{Z}^d)) & \longrightarrow & K_0(\mathcal{U}_{A^t}) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{U}_{A^t}) & \longleftarrow & K_{d+1}(C^*(\mathbb{Z}^d)) & \xleftarrow[1-\epsilon\widetilde{\tau}_*]{} & K_{d+1}(C^*(\mathbb{Z}^d)). \end{array}$$

This completes the proof. ∎

The above six term sequence can be used to obtain a formula for the K-theory of $\mathcal{U}_{A^t}$. We illustrate for the case when $\det(A) > 0$ and d is even. We leave the other cases to the reader. Thus, assume $\det(A) > 0$ and d is even. In this case, the exact sequence of Theorem 5.3 reads as

$$\begin{array}{ccccc} \bigoplus_{\substack{n \text{ even} \\ 0\le n\le d}} \Lambda^n(\mathbb{Z}^d) & \xrightarrow{1-\oplus_n A_n} & \bigoplus_{\substack{n \text{ even} \\ 0\le n\le d}} \Lambda^n(\mathbb{Z}^d) & \longrightarrow & K_0(\mathcal{U}_{A^t}) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{U}_{A^t}) & \longleftarrow & \bigoplus_{\substack{n \text{ odd} \\ 0\le n\le d}} \Lambda^n(\mathbb{Z}^d) & \xleftarrow[1-\oplus_n A_n]{} & \bigoplus_{\substack{n \text{ odd} \\ 0\le n\le d}} \Lambda^n(\mathbb{Z}^d). \end{array}$$

Now for $n \ge 1$, A_n is a dilation matrix and thus $\ker(1 - A_n) = 0$ if $n \ge 1$. Hence we conclude from the above six term sequence that

$$K_0(\mathcal{U}_{A^t}) \equiv \bigoplus_{\substack{n \text{ even} \\ 0\le n\le d}} \operatorname{coker}(1 - A_n).$$

Now $A_0 = 1$ and hence $\oplus_{n \text{ even}} \ker(1 - A_n) = \mathbb{Z}$. Again the six term sequence gives the following short exact sequence.

$$0 \longrightarrow \bigoplus_{\substack{n \text{ odd} \\ 0\le n\le d}} \operatorname{coker}(1 - A_n) \longrightarrow K_1(\mathcal{U}_{A^t}) \longrightarrow \mathbb{Z} \longrightarrow 0.$$

Since $\mathbb{Z}$ is free, it follows that

$$K_1(\mathcal{U}_{A^t}) \equiv \bigoplus_{\substack{n \text{ odd} \\ 0\le n\le d}} \operatorname{coker}(1 - A_n) \oplus \mathbb{Z}.$$

Remark 5.4 If d is odd, one obtains the same formula, but the roles of K_0 and K_1 are interchanged. If $\det(A) < 0$, we obtain a formula for $K_*(\mathcal{U}_{A^t})$ in terms of the cokernels of $1 + A_n$. Here the direct summand $\mathbb{Z}$ will not appear as $\operatorname{Ker}(1 + A_n) = 0$ for $0 \le n \le d$.

The rest of this section is devoted to reconciling our computation of the K-groups of $\mathcal{U}_{A^t}$ with the result obtained in [8]. More precisely with [8, Theorem 4.9]. Let us recall the notations as in [8].

For a subset $K = \{k_1 < k_2 < \cdots < k_n\}$ of $\{1, 2, \ldots, d\}$, denote the complement arranged in increasing order by $K^{'}$ and let $K^{'} = \{k_{n+1} < k_{n+2} < \cdots < k_d\}$. Denote the permutation $i \to k_i$ by τ_K. For a permutation σ, $\operatorname{sign}(\sigma)$ is 1 if σ is even and -1 if

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$\operatorname{sign}(\sigma)$ is odd. Also recall that if K and J are subsets of size n, then $A_{K,J}$ is the matrix obtained from A by considering the rows from K and columns from J.

For $0 \le n \le d$, let $\widetilde{B}_n$ be the $\binom{d}{n} \times \binom{d}{n}$ matrix defined as follows. (We index the columns and rows by subsets of $\{1, 2, \dots, d\}$ of size n). The (K, L)-th entry of $\widetilde{B}_n$ is $\operatorname{sign}(\tau_K \tau_L) \det(A_{K', L'})$.

The matrices B_n as defined in [8, Prop. 4.6] are then given by $B_n = \operatorname{sign}(\det(A))\widetilde{B}_n$. Denote the matrix whose (K, L)-th entry is $\det(A_{K',L'})$ by C_n. By convention, $\widetilde{B}_d = 1 = C_d$. Note that $\widetilde{B}_n$ and C_n are conjugate over $\mathbb{Z}$. For the matrix $\operatorname{diag}(\operatorname{sign}(\tau_K))$ conjugates $\widetilde{B}_n$ to C_n.

Let $U_n \colon \Lambda^n(\mathbb{Z}^d) \to \Lambda^{d-n}(\mathbb{Z}^d)$ be defined by $U_n e_I := e_{I'}$. Then U_n is invertible and $U_n C_n U_n^{-1} = A_{d-n}$. Since $\operatorname{sign}(\det(A))C_n$ is conjugate (over $\mathbb{Z}$) to B_n, it follows that $\operatorname{sign}(\det(A))A_{d-n}$ is conjugate (over $\mathbb{Z}$) to B_n.

Now the formula for the K-groups of $\mathcal{U}_{A^t}$ can be restated in terms of the matrices B_n's. For example, if d is even and $\det(A) > 0$, we obtain

$$K_0(\mathcal{U}_{A^t}) \cong \bigoplus_{\substack{n \text{ even} \\ 0 \le n \le d}} \operatorname{coker}(1 - B_n),$$
$$K_1(\mathcal{U}_{A^t}) \cong \bigoplus_{\substack{n \text{ odd} \\ 0 \le n \le d}} \operatorname{coker}(1 - B_n) \oplus \mathbb{Z}.$$

For the exact formulas of the K-groups in terms of the matrices B_n's, we refer the reader to [8, Theorem 4.9].

Cuntz and Vershik have also computed the K-groups of $\mathcal{U}_{A^t}$ as a special case of their considerations in [7]. In particular, a six term exact sequence similar to, but different from, the one in Theorem 5.3 is obtained. We finish this paper with a few remarks about their method for the case of $\mathcal{U}_{A^t}$. Let us denote the matrix A^t by B in what follows.

Let

$$\overline{M} := \Big\{ (v_r) \in \prod_{r=0}^{\infty} \frac{\mathbb{Z}^d}{B^r \mathbb{Z}^d} : v_{r+1} \equiv v_r \mod B^r \mathbb{Z}^d \Big\}.$$

Then $\overline{M}$ is a compact abelian group when given the subspace topology inherited from the product topology on $\prod_{r=0}^{\infty} \frac{\mathbb{Z}^d}{B^r \mathbb{Z}^d}$. The map $\mathbb{Z}^d \ni v \to (v, v, v, \dots) \in \overline{M}$ is an injective homomorphism and we identify $\mathbb{Z}^d$ with a subgroup of $\overline{M}$. Moreover, $\mathbb{Z}^d$ is dense in $\overline{M}$. The abelian group $\mathbb{Z}^d$ acts on $\overline{M}$ by translation. The semigroup $\mathbb{N}$ acts on $\mathbb{Z}^d$ by $1.v = Bv$ for $v \in \mathbb{Z}^n$. This action extends to an action of $\mathbb{N}$ on $\overline{M}$. Thus, one obtains an injective action of the semigroup $\mathbb{Z}^d \rtimes \mathbb{N}$ on $\overline{M}$. Then $\mathcal{U}_{A^t}$ is isomorphic to the semigroup crossed product $\mathcal{B} \rtimes \mathbb{N}$, where $\mathcal{B} := C(\overline{M}) \rtimes \mathbb{Z}^n$. Denote the action of $\mathbb{N}$ on $\mathcal{B}$ by τ.

Remark 5.5 For the proof of the assertions made in the previous paragraph, the reader is referred to [13, Corollary 6.6] and [7, Theorem 2.6]. The isomorphism $\mathcal{U}_{A^t} \cong (C(\overline{M}) \rtimes \mathbb{Z}^d) \rtimes \mathbb{N}$ forms the basis for the Cuntz–Li duality theorem.

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By Pimsner–Voiculescu sequence, one obtains the six term sequence

$$\begin{array}{ccccc} K_0(\mathcal{B}) & \xrightarrow{1-\tau_*} & K_0(\mathcal{B}) & \longrightarrow & K_0(\mathcal{B} \rtimes \mathbb{N}) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{B} \rtimes \mathbb{N}) & \longleftarrow & K_1(\mathcal{B}) & \xleftarrow[1-\tau_*]{} & K_1(\mathcal{B}). \end{array}$$

$K_*(\mathcal{B})$ is replaced by $K_*(C^*(\mathbb{Z}^d))$ in [7, Theorem 3.1]. The idea involved is as follows.

The r-th projection $\overline{M} \ni (v_i) \to v_r \in \mathbb{Z}^d / B^r \mathbb{Z}^d$ is onto and has kernel $B^r \overline{M}$ ([13, Lemma 6.1]). Thus, $\bigcap_{r=0}^{\infty} B^r \overline{M} = \{0\}$ and $\overline{M}/B^r\overline{M} \cong \mathbb{Z}^d / B^r \mathbb{Z}^d$. One has natural inclusions $C(\overline{M}/B^r\overline{M}) \subset C(\overline{M}/B^{r+1}\overline{M}) \subset C(\overline{M})$. Since the intersection $\bigcap_{r=0} B^r \overline{M} = \{0\}$, the union of $C(\overline{M}/B^r\overline{M})$ is dense in $C(M)$. Moreover, the inclusions are $\mathbb{Z}^d$ invariant. Thus, $\mathcal{B} := C(\overline{M}) \rtimes \mathbb{Z}^d$ is the inductive limit of $C(\mathbb{Z}^d / B^r \mathbb{Z}^d) \rtimes \mathbb{Z}^d$.

By imprimitivity, $C(\mathbb{Z}^d / B^r \mathbb{Z}^d) \rtimes \mathbb{Z}^d \cong M_{N(r)}(C^*(B^r \mathbb{Z}^d)) \cong M_{N(r)}(C^*(\mathbb{Z}^d))$, where $N(r) = |\det(B)|^r$. Thus, the K-theory of $\mathcal{B}$ can be computed from the K-theory of $C^*(\mathbb{Z}^d)$. By examining the connecting maps

$$M_{N(r)}(C^*(\mathbb{Z}^d)) \to M_{N(r+1)}(C^*(\mathbb{Z}^d)),$$

one can replace $K_*(\mathcal{B})$ by $K_*(C^*(\mathbb{Z}^d))$. This is proved precisely in [7, Theorem 3.1]. In our notation, the map $1 - \tau_*$ becomes $1 - \epsilon \oplus_n C_n$ (see [7, example 3.12]), where $\epsilon = \operatorname{sign}(\det(A))$ The main difference between our approach and the approach pursued in [7] is the starting point. Cuntz and Vershik rely on the explicit description of $\mathcal{U}_{A^t}$ as a (semigroup) crossed product, where we appeal to the Morita equivalence provided by the Cuntz–Li duality theorem. The advantage of Cuntz and Vershik's method is that it is direct and applies to other examples as well. The one advantage of our method is that it explains conceptually why the sign of $\det(A)$ and the parity of d play a role in the formula of the K-groups.

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