POSITIVE DERIVATIONS ON PARTIALLY ORDERED STRONGLY REGULAR RINGS

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Abstract

The author presents a proof that a partially ordered strongly regular ring $S$ which has the additional property that the square of each member of $S$ is greater than or equal to zero cannot have nontrivial positive derivations.

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1. Introduction

In recent years efforts have been made to study positive derivations on certain classes of partially ordered rings, see Colville [1], and also on partially ordered linear algebras, see Dai [2]. This note is concerned exclusively with the investigation of positive derivations on partially ordered strongly regular rings. It will be shown that a p.o. ring $(S, +, \cdot, \leq)$ which is strongly regular and has the additional property that $x^2 \geq 0$ for each $x \in S$ cannot have nontrivial positive derivations. Also, an example is included to complement the given theorem.

Recall that a ring $(S, +, \cdot)$ is strongly regular if for each $x \in S$ there exists a $y \in S$ such that $x^2 y = x$. Terminology and background material on p.o. rings needed for this article may be found in Fuchs [3].
DEFINITION. The statement that $\delta$ is a positive derivation on a p.o. ring $(S, +, \cdot, \leq)$ means that $\delta$ is a map from $S$ into $S$ such that (1) $\delta(x + y) = \delta(x) + \delta(y)$ for each $x, y \in S$, (2) $\delta(xy) = x\delta(y) + \delta(x)y$ for each $x, y \in S$, and (3) $\delta(x) \geq 0$ for each $x \in S$, with $x \geq 0$.

2. Main results

Before proving the main theorem, we need the following lemma.

**Lemma.** Suppose $(S, +, \cdot, \leq)$ is a partially ordered strongly regular ring such that $x^2 \geq 0$ for each $x \in S$. If $\delta$ is a positive derivation defined on $S$ and $x \in S$, with $x \geq 0$, then $\delta(x) = 0$.

**Proof.** Let $x \in S$ with $x \geq 0$. Since the ring is strongly regular, there exists a $y \in S$ such that $x^2y = x$. Using the fact that $xyx = x$, it follows that $\delta(x) = \delta(xyx) = \delta(xy)x + (xy)\delta(x)$. Thus, $x\delta(x) = x\delta(xy)x + x\delta(x)$ and this implies that $x\delta(xy)x = 0$. Hence $[\delta(xy)]^2 = 0$ and consequently $x\delta(xy) = 0$, since $(S, +, \cdot, \leq)$ contains no nonzero nilpotent elements. In a similar manner, $\delta(xy)x = 0$ and thus $\delta(x) = xy\delta(x)$. Recalling that it can be shown that $y$ will commute with $x$ in a strongly regular ring, we have that $\delta(xy) = \delta(yx) = y\delta(x) + \delta(y)x$ and this implies, with $x\delta(xy) = 0$, that $0 = xy\delta(x) + x\delta(y)x$. Hence $xy\delta(x) = -x\delta(y)x$. Therefore $\delta(x) = -x\delta(y)x$. From $0 \leq xy^2$, we have $0 \leq \delta(xy^2) = \delta(xy)y + (xy)\delta(y)$. Multiplying on the left by $x$, gives $0 \leq x\delta(xy)y + x\delta(y) = 0 + x\delta(y) = x\delta(y)$. Next, multiplying on the right by $x$, we obtain $0 \leq x\delta(y)x = -\delta(x)$. Hence $\delta(x) \leq 0$. Thus $\delta(x) = 0$, since it is also true that $\delta(x) \geq 0$.

**Theorem.** Suppose $(S, +, \cdot, \leq)$ is a partially ordered strongly regular ring such that $x^2 \geq 0$ for each $x \in S$. If $\delta$ is a positive derivation defined on $S$, then $\delta(x) = 0$ for each $x \in S$.

**Proof.** Suppose $x_1 \in S$ such that $\delta(x_1) \neq 0$. Consider $x_1^2$. From the preceding lemma, $\delta(x_1^2) = 0$, since $x_1^2 \geq 0$. Appealing again to strong regularity, there exists a $y_1 \in S$ such that $x_1^2y_1 = x_1$. Thus, $\delta(x_1) = \delta(x_1^2)y_1 + x_1^2\delta(y_1) = x_1^2\delta(y_1)$. Multiplying the preceding equation by $y_1$ on the left and using the fact that $y_1x_1^2 = x_1$ is also true, we obtain $y_1\delta(x_1) = x_1\delta(y_1)$. Consequently, $\delta(x_1) = x_1[x_1\delta(y_1)] = x_1y_1\delta(x_1) = y_1x_1\delta(x_1)$. Hence $x_1\delta(x_1) \neq 0$ since $\delta(x_1) \neq 0$. Thus $[x_1\delta(x_1)]^2 \neq 0$, since the ring contains no nonzero nilpotent elements. Next, $0 = \delta(x_1^2) = x_1\delta(x_1) + \delta(x_1)x_1$ implies that $x_1\delta(x_1) = -\delta(x_1)x_1$. Multiplying the preceding equation on the left by $x_1$ and on the right by $\delta(x_1)$, gives $0 \leq x_1^2[\delta(x_1)]^2 = -[x_1\delta(x_1)]^2$. 

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Hence, by the hypothesis on the ring, $[x_i \delta(x_i)]^2 = 0$ and this contradicts the earlier observation that $[x_i \delta(x_i)]^2 \neq 0$. Therefore $\delta(x_i) = 0$. Consequently $\delta(x) = 0$ for each $x \in S$.

3. An example

We conclude this note with an example of a partially ordered strongly regular ring $(S, +, \cdot, \leq)$ such that (1) the positive cone $P \neq \{0\}$, (2) contains at least one element $x \in S$ such that $x^2 \not\in 0$, and (3) has associated with it a positive derivation $\delta$ such that $\delta(x) \neq 0$.

Consider the following construction. Let $(S, +, \cdot)$ denote the ring of all formal truncated Laurent series $\sum_{i=n}^{\infty} a_i x^i$, where $a_i$ is a real number and $n$ is an arbitrary integer, with ordinary addition and multiplication of series as the two operations defined on $S$. Let the positive cone $P$, for the partial ordering on $S$, consist of all series of the form $\sum_{i=0}^{\infty} a_i x^i$ with $a_i \geq 0$ for $0 \leq i \leq \infty$. Note that the sum and product of two members of $P$ is again a member of $P$ and $P \neq \{0\}$. This ring is a field and thus $(S, +, \cdot, \leq)$ is a partially ordered strongly regular ring. Now define a mapping $\delta$ from $S$ into $S$ by $\delta[\sum_{i=n}^{\infty} a_i x^i] = \sum_{i=n}^{\infty} ia_i x^{i-1}$. It is easily seen that $\delta$ is a positive derivation and is obviously nontrivial. Last, we observe that the series $g(x) = 1 - x$ has the property that $[g(x)]^2 \neq 0$.

References


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