Canad. J. Math. Vol. **53** (6), 2001 pp. 1194–1222

Explicit Upper Bounds for Residues of Dedekind Zeta Functions and Values of L-Functions at $s = 1$, and Explicit Lower Bounds for Relative Class Numbers of CM-Fields

Stéphane Louboutin

Abstract. We provide the reader with a uniform approach for obtaining various useful explicit upper bounds on residues of Dedekind zeta functions of numbers fields and on absolute values of values at $s = 1$ of L-series associated with primitive characters on ray class groups of number fields. To make it quite clear to the reader how useful such bounds are when dealing with class number problems for CM-fields, we deduce an upper bound for the root discriminants of the normal CM-fields with (relative) class number one.

1 Introduction

Lately, various class number problems and class groups problems for CM-fields have been solved. These problems include the determinations of the imaginary abelian number fields with class number one (see [CK], [Yam]), relative class number one or class numbers equal to their genus class numbers; the determinations of the non quadratic imaginary cyclic fields of 2-power degrees with cyclic ideal class groups of 2-power orders (see [Lou7]) or with ideal class groups of exponents ≤ 2 (see [Lou3]); the determination of the normal CM-fields of relative class number one with dihedral or dicyclic Galois groups (see [Lef], [LOO], [LO2], [Lou10]); the determination of the non-abelian normal CM-fields of degrees $2n < 48$ of class number one (see [LLO], [LO1], [Lou6], see also [LP]); the determination of the dihedral or quaternion octic CM-fields with ideal class groups cyclic of 2-power orders (see [Lou5], [YK]) or of exponents ≤ 2 (see [LO3], [LYK]).

For solving such problems, there are three obstacles to overcome.

First, one must be able to construct the fields he is going to deal with. Usually this is done by using class field theory (*e.g.* [Lef], [LO2], [LPL]).

Second, one must be able to compute efficiently the relative class numbers of the CM-fields he is going to deal with. This is done by the method developed in [Lou13].

Finally, one must obtain a reasonable upper bound for the absolute values of the discriminants of the CM-fields of a given degree or of a given Galois group with a given relative class number, class number or ideal class group.

Received by the editors April 27, 2001; revised May 14, 2001.
AMS subject classification: 11R42, 11R29.
Keywords: Dedekind zeta functions, L-functions, relative class numbers, CM-fields.

https://doi.org/10.4153/CJM-2001-045-5 Published online by Cambridge University Press

Due to the deep results of [Sta], [Odl] and [Hof] one usually knows beforehand that there are only finitely many such CM-fields. However, these three papers which aimed at proving finiteness results are of little or no practical use when it comes to explicit determinations, for they yield huge bounds on the root discriminants of the CM-fields with small class numbers and small degrees.

In [Lou2], [Lou5], [Lou8], [Lou9] and [Lou11] we developed a wealth of techniques for obtaining lower bounds for relative class numbers of CM-fields, and these lower bounds are particularly good for CM-fields of small degree. The key ingredient of our techniques is the use of explicit upper bounds for residues of Dedekind zeta functions of numbers fields and on absolute values of values at $s = 1$ of L-series associated with primitive characters on ray class groups of number fields.

The aim of this paper is to provide the reader with a uniform approach for proving such useful explicit upper bounds. Not only will we simplify our previous proofs of [Lou8], [Lou9] and [Lou11], but we will also obtain new useful bounds (*e.g.* see (2), (3), (4), (6), (7), (8), (12), (16) and (18)).

2 Upper Bounds for $\mathrm{Res}_{s=1}(\zeta_{\mathbf{K}})$ and $|L(1,\chi)|$

2.1 Notation

To begin with, we set the notation required for understanding the statements of the results given in this section. Let $\mathbf{L}$ be number field of degree $m = r_1 + 2r_2$. Let $\zeta_{\mathbf{L}}$ denote its Dedekind zeta function. We set

$$A_{\mathbf{L}} = \sqrt{d_{\mathbf{L}}/4^{r_2}\pi^m}, \quad \Gamma_{\mathbf{L}}(s) = \Gamma^{r_1}(s/2)\Gamma^{r_2}(s), \quad F_{\mathbf{L}}(s) = A_{\mathbf{L}}^s\Gamma_{\mathbf{L}}(s)\zeta_{\mathbf{L}}(s),$$

$$\lambda_{\mathbf{L}} = \mathrm{Res}_{s=1}(F_{\mathbf{L}}) = \sqrt{d_{\mathbf{L}}/(2\pi)^{2r_2}}\,\mathrm{Res}_{s=1}(\zeta_{\mathbf{L}}),$$

$$\mu_{\mathbf{L}} = \lim_{s\downarrow 1} \frac{1}{\lambda_{\mathbf{L}}}F_{\mathbf{L}} - \frac{1}{s(s-1)}$$

$$B_{\mathbf{L}} = \mu_{\mathbf{L}}\,\mathrm{Res}_{s=1}(\zeta_{\mathbf{L}}).$$

Notice that $\mu_{\mathbf{Q}} = \big(2 + \gamma - \log(4\pi)\big)/2 = 0.023095\cdots$ where $\gamma = 0.577215\cdots$ denotes Euler's constant. We will prove the following results.

2.2 Bounds for Residues of Dedekind Zeta Functions

Theorem 1 *Let* $\mathbf{L}$ *be a number field of degree* $m > 1$*. Set* $e = \exp(1) = 2.718\cdots$.

1. *It holds*

$$\mathrm{Res}_{s=1}(\zeta_{\mathbf{L}}) \le \left(\frac{e\log d_{\mathbf{L}}}{2(m-1)}\right)^{m-1}. \tag{1}$$

2. *Moreover,* $\frac{1}{2} \le \beta < 1$ *and* $\zeta_{\mathbf{L}}(\beta) = 0$ *imply*

$$\mathrm{Res}_{s=1}(\zeta_{\mathbf{L}}) \le (1-\beta)B_{\mathbf{L}}. \tag{2}$$

https://doi.org/10.4153/CJM-2001-045-5 Published online by Cambridge University Press

3. *It holds*

$$B_{\mathbf{L}} \leq \left(\frac{e \log d_{\mathbf{L}}}{2m}\right)^m. \tag{3}$$

Therefore, $\frac{1}{2} \leq \beta < 1$ *and* $\zeta_{\mathbf{L}}(\beta) = 0$ *imply*

$$\mathrm{Res}_{s=1}(\zeta_{\mathbf{L}}) \leq (1-\beta)\left(\frac{e \log d_{\mathbf{L}}}{2m}\right)^m. \tag{4}$$

2.3 Better Bounds for Totally Real Number Fields

Theorem 2

1. *(See [Lou8, Proposition 6]). If* $\mathbf{L}$ *is a real quadratic number field, then*

$$B_{\mathbf{L}} \leq \frac{1}{8} \log^2 d_{\mathbf{L}}, \tag{5}$$

which improves upon (3).

2. *If* $\mathbf{L}$ *is a totally real cubic number field, then*

$$\mathrm{Res}_{s=1}(\zeta_{\mathbf{L}}) \leq \frac{1}{8} \log^2 d_{\mathbf{L}}, \tag{6}$$

which improves upon (1), and

$$B_{\mathbf{L}} \leq \frac{1}{48} \log^3 d_{\mathbf{L}}, \tag{7}$$

which improves upon (3).

3. *More generally, for each integer* $m \geq 2$ *there exists* d_m *effective such that for any totally real number field* $\mathbf{L}$ *of degree* m *we have:*

   (a) *If* $m > 2$, $d_{\mathbf{L}} > d_{m-1}$ *and* $\zeta_{\mathbf{L}}/\zeta$ *is entire, then*

$$\mathrm{Res}_{s=1}(\zeta_{\mathbf{L}}) \leq \frac{1}{2^{m-1}(m-1)!} \log^{m-1} d_{\mathbf{L}}, \tag{8}$$

   which improves upon (1).

   (b) *If* $m > 1$ *and* $d_{\mathbf{L}} > d_m$, *then*

$$B_{\mathbf{L}} \leq \frac{1}{2^m m!} \log^m d_{\mathbf{L}}, \tag{9}$$

   which improves upon (3).

Moreover, if $m \in \{2,3,4,5\}$ *then (8) and (9) are valid without any restriction on* $d_{\mathbf{L}}$.

Remarks 3

1. According to Aramata-Brauer Theorem (see [MM, Th. 3.1]), if $\mathbf{L}/\mathbf{Q}$ is normal, then $\zeta_{\mathbf{L}}/\zeta$ is entire. More generally, according to Uchida-van der Waal Theorem (see [MM, Th. 4.2]), if the Galois group of the normal closure of $\mathbf{L}$ is solvable, then $\zeta_{\mathbf{L}}/\zeta$ is entire.
2. Roughly speaking, the bound (8) is $\sqrt{2\pi(m-1)}$ times smaller than the bound (1) and the bound (9) is $\sqrt{2\pi m}$ times smaller than the bound (3) (use Stirling's formula).
3. The first d_m's are small enough to allow us to use the bounds (8) and (9) for any totally real number field of degree $m \in \{2,3,4,5\}$. It would be nice to be able to prove that the d_m's can be chosen small enough so as to allow us to use these aforementioned two bounds for any totally real number field $\mathbf{L}$ (see Subsection 6.4).

2.4 Bounds for Values at $s = 1$ of L-Functions

Theorem 4 *Let* $\mathbf{L}$ *be a number field of degree* $m \geq 1$. *Let* χ *be a primitive character on some ray class group for* $\mathbf{L}$. *Let* f_χ *denote the norm of the finite part of the conductor of* χ. *Set* $e = \exp(1) = 2.718\cdots$.

1. *Then*

$$|L(1,\chi)| \leq 2\left(\frac{e}{2m}\log(d_{\mathbf{L}} f_\chi)\right)^m. \tag{10}$$

Consequently, if $\mathbf{N}$ *is a* CM*-field of degree* $2m \geq 2$, *then*

$$h_{\mathbf{N}}^- \leq 2Q_{\mathbf{N}} w_{\mathbf{N}} \sqrt{d_{\mathbf{N}}/d_{\mathbf{N}^+}}\left(\frac{e}{2n}\log(d_{\mathbf{N}}/d_{\mathbf{N}^+})\right)^n \tag{11}$$

(see Section 3 for the notations $h_{\mathbf{N}}^-$, $Q_{\mathbf{N}}$, $w_{\mathbf{N}}$ *and* $\mathbf{N}^+$*).*

2. *Moreover,* $\frac{2}{3} \leq \beta < 1$ *and* $L(\beta,\chi) = 0$ *imply*

$$|L(1,\chi)| \leq 4(1-\beta)\left(\frac{e}{2(m+1)}\log(d_{\mathbf{L}} f_\chi)\right)^{m+1}. \tag{12}$$

Theorem 5 *Let* $\mathbf{L}$ *be a given number field,* χ *a non-trivial primitive character* χ *on a ray class group for* $\mathbf{L}$ *which is unramified at all the infinite places of* $\mathbf{L}$, *and* f_χ *the norm of the finite part its conductor. We have*

$$|L(1,\chi)| \leq \frac{1}{2}\mathrm{Res}_{s=1}(\zeta_{\mathbf{L}})\log f_\chi + \begin{cases} 2B_{\mathbf{L}} & \textit{in all cases,} \\ B_{\mathbf{L}} & \textit{if } f_\chi = 1 \textit{ or if } f_\chi \geq e^{2\mu_{\mathbf{L}}}. \end{cases} \tag{13}$$

See also [Lou4] and [Lou11, Th. 7] for similar but less satisfactory results when we drop the assumption that χ is unramified at all the infinite (real) places of $\mathbf{L}$. Since both the upper bounds for $|L(1,\chi)|$ given in Theorem 5 and [Lou11, Th. 7] involve the invariant $B_{\mathbf{L}}$ of $\mathbf{L}$, it was reasonable to determine in Theorem 1 a general upper bound for $B_{\mathbf{L}}$.

Corollary 6 **(Compare with (10))** *Let* $\mathbf{L}$ *be a given number field of degree* $m \geq 2$. *For any non-trivial primitive character* χ *on a ray class group for* $\mathbf{L}$ *which is unramified at all the infinite places of* $\mathbf{L}$ *we have*

$$|L(1,\chi)| \leq \left(\frac{e\log d_{\mathbf{L}}}{2(m-1)}\right)^{m-1} \log(d_{\mathbf{L}} f_\chi). \tag{14}$$

Proof Using (13), (1) and (3), we obtain

$$|L(1,\chi)| \leq \left(\frac{e\log d_{\mathbf{L}}}{2(m-1)}\right)^{m-1} \left(\frac{1}{2}\log f_\chi + a_m \log d_{\mathbf{L}}\right)$$

where $a_m = e(m-1)^{m-1}/m^m \leq 1$ for $m \geq 2$. ■

2.5 Better Bounds for Real Abelian Number Fields

Theorem 7 *Let* χ *be an even primitive Dirichlet character modulo* $f_\chi > 1$.

1. *(See also [Lou1], [Lou12] and [Ram]). Then*

$$|L(1,\chi)| \leq \frac{1}{2}(\log f_\chi + 2\mu_{\mathbf{Q}}) \leq \frac{1}{2}(\log f_\chi + 0.05). \tag{15}$$

2. *Moreover,* $\frac{1}{2} \leq \beta < 1$ *and* $L(\beta,\chi) = 0$ *imply*

$$|L(1,\chi)| \leq \frac{1-\beta}{8}\log^2 f_\chi, \tag{16}$$

which improves upon (12).

Notice that (15) follows from the second bound in (13) applied to $\mathbf{L} = \mathbf{Q}$ and that, for quadratic characters, (16) follows also from (4) and (9). Now, using the fact that the geometric mean is less than or equal to the arithmetic mean and using the conductor-discriminant formula, we obtain:

Corollary 8 *Let* $\mathbf{L}$ *be a real abelian number field of degree* $m > 1$, *discriminant* $d_{\mathbf{L}}$ *and conductor* $f_{\mathbf{L}}$ *(notice that* $d_{\mathbf{L}} \leq f_{\mathbf{L}}^{m-1}$*).*

1. *We have the following improvement on (1) and (6):*

$$\mathrm{Res}_{s=1}(\zeta_{\mathbf{L}}) \leq \left(\frac{1}{2(m-1)}\log d_{\mathbf{L}} + \mu_{\mathbf{Q}}\right)^{m-1}. \tag{17}$$

2. *Moreover,* $\frac{1}{2} \leq \beta < 1$ *and* $\zeta_{\mathbf{L}}(\beta) = 0$ *imply*

$$\mathrm{Res}_{s=1}(\zeta_{\mathbf{L}}) \leq (1-\beta)\frac{\log f_{\mathbf{L}}}{4}\left(\frac{1}{2(m-1)}\log d_{\mathbf{L}} + \mu_{\mathbf{Q}}\right)^{m-1}, \tag{18}$$

which improves upon (4).

Theorem 9 *Let* $\mathbf{L}$ *be a real abelian number field of degree* $m > 1$*, discriminant* $d_{\mathbf{L}}$*. We have the following improvement on (3):*

$$B_{\mathbf{L}} \leq \frac{\log d_{\mathbf{L}}}{4}\left(\frac{1}{2(m-1)}\log d_{\mathbf{L}} + \mu_{\mathbf{Q}}\right)^{m-1}. \tag{19}$$

Notice that (18) is at least $m-1$ times better than the bound which can be deduced from (2) and (19).

2.6 Remarks

Let $\mathbf{L}$ be a real abelian number field of degree $m \geq 1$, and let $X_{\mathbf{N}}$ denote the group of primitive even Dirichlet characters associated with $\mathbf{L}$. Then

$$\mu_{\mathbf{L}} = 1 - \frac{m-2}{2}\gamma - \frac{m}{2}\log(4\pi) + \frac{1}{2}\log d_{\mathbf{L}} + \sum_{\chi \in X_{\mathbf{N}} \setminus \{1\}} \frac{L'(1,\chi)}{L(1,\chi)}.$$

If we assume the Generalized Riemann Hypothesis then $L'(1,\chi)/L(1,\chi)$ is $O(\log\log f_\chi)$ (see [GS, Section 3.1]), and $\mu_{\mathbf{L}}$ is asymptotic to $\frac{1}{2}\log d_{\mathbf{L}}$ as $d_{\mathbf{L}}$ goes to infinity. In this respect, it is worth noticing that the bound (13) can be rewritten in the following form:

$$|L(1,\chi)| \leq \frac{1}{2}\mathrm{Res}_{s=1}(\zeta_{\mathbf{L}})(\log f_\chi + 2\mu_{\mathbf{L}}). \tag{20}$$

3 Lower Bounds for Relative Class Numbers

This section is devoted to giving four examples showing the use of the explicit results of the previous section for obtaining useful explicit lower bounds for relative class number of CM-fields: Theorem 12 which stems from Theorem 1, Theorem 13 which stems from Corollary 8, Theorem 15 which stems from Theorem 5, and Theorem 18 which stems from Corollary 8. We make it clear to the reader that the lowers bounds on relative class numbers for CM-fields which can be obtained by using the results of Section 2 are not good enough to prove that there are only finitely many normal CM-fields of a given relative class number, a result which is known to be true (see [Odl] and [Hof]). However, we can obtain (and we have obtained) various lower bounds for relative class numbers of CM-fields which are much better than the ones given in the aforementioned two papers, provided that we deal with CM-fields of a given degree. In particular, up to now all the determinations of the CM-fields of a given degree (*e.g.* of degree $2m \leq 42$) or of a given Galois group (*e.g.* with Galois group any dihedral group) with a given class number (*e.g.* of class number one) or a given ideal class group (*e.g.* of ideal class groups of exponent ≤ 2) stem from our lower bounds on their relative class numbers. Recall that a number field $\mathbf{N}$ is a CM-field if $\mathbf{N}$ is a totally imaginary number field (of degree $2m$) and $\mathbf{N}$ is a quadratic extension of its maximal totally real subfeld $\mathbf{N}^+$ (of degree m). In that situation, the class number $h_{\mathbf{N}^+}$ of $\mathbf{N}^+$ divides the class number $h_{\mathbf{N}}$ of $\mathbf{N}$. Their ratio $h_{\mathbf{N}}^- = h_{\mathbf{N}}/h_{\mathbf{N}^+}$

https://doi.org/10.4153/CJM-2001-045-5 Published online by Cambridge University Press

is a positive rational integer which is called the relative class number of $\mathbf{N}$. Moreover, let $Q_{\mathbf{N}} \in \{1,2\}$ and $w_{\mathbf{N}} \geq 2$ denote its Hasse unit index and its number of complex roots of unity, respectively. Then

$$h_{\mathbf{N}}^{-} = \frac{Q_{\mathbf{N}} w_{\mathbf{N}}}{(2\pi)^m} \sqrt{\frac{d_{\mathbf{N}}}{d_{\mathbf{N}^+}}} \frac{\operatorname{Res}_{s=1}(\zeta_{\mathbf{N}})}{\operatorname{Res}_{s=1}(\zeta_{\mathbf{N}^+})} \tag{21}$$

(see [Was]). In particular, the upper bound for $h_{\mathbf{N}}^{-}$ given in (11) follows from the upper bound (10) applied to the quadratic character associated with the quadratic extension $\mathbf{N}/\mathbf{N}^+$. For obtaining lower bounds for $h_{\mathbf{N}}^{-}$ we will make use of the upper bounds for $\operatorname{Res}_{s=1}(\zeta_{\mathbf{N}^+})$ (for totally real number fields $\mathbf{N}^+$) given in Section 2 and of the following lower bounds for $\operatorname{Res}_{s=1}(\zeta_{\mathbf{N}})$:

***Proposition 10* (See [Lou2, Proposition A])** *Let $\mathbf{N}$ be a CM-field of degree $2m > 2$. Let $r_{\mathbf{N}} = d_{\mathbf{N}}^{1/2m}$ denote the root discriminant of $\mathbf{N}$ and set $\epsilon_{\mathbf{N}} = \max(\epsilon_{\mathbf{N}}', \epsilon_{\mathbf{N}}'')$ with $\epsilon_{\mathbf{N}}' = 1 - (2\pi m e^{a/2m}/r_{\mathbf{N}})$ and $\epsilon_{\mathbf{N}}'' = \frac{2}{5}\exp(-2\pi m/r_{\mathbf{N}})$. Then, $\frac{1}{2} \leq 1 - (a/\log d_{\mathbf{N}}) \leq \beta < 1$ and $\zeta_{\mathbf{N}}(\beta) \leq 0$ imply*

$$\operatorname{Res}_{s=1}(\zeta_{\mathbf{N}}) \geq \epsilon_{\mathbf{N}}(1-\beta)/e^{a/2}.$$

We refer the reader to [Sta, Lemma 4] and [Hof, Lemma 4] for similar lower bounds for $\operatorname{Res}_{s=1}(\zeta_{\mathbf{N}})$. Notice that the residue at its simple pole $s = 1$ of any Dedekind zeta function $\zeta_{\mathbf{N}}$ is positive (use the analytic class number formula for $\mathbf{N}$, or notice that from its definition we get $\zeta_{\mathbf{N}}(s) \geq 1$ for $s > 1$). Therefore, we have $\lim_{s\to 1^-} \zeta_{\mathbf{N}}(s) = -\infty$ and $\zeta_{\mathbf{N}}\bigl(1 - (a/\log d_{\mathbf{N}})\bigr) \leq 0$ if $\zeta_{\mathbf{N}}$ does not have any real zero in the range $1 - (a/\log d_{\mathbf{N}}) \leq s < 1$.

Proposition 11

1. *The Dedekind zeta function of a number field $\mathbf{E}$ has at most two real zeros in the range $1 - 1/\log d_{\mathbf{E}} \leq s < 1$.*
2. *Let $\mathbf{K}$ be a normal number field and ρ a real simple zero of $\zeta_{\mathbf{K}}$. There exists a quadratic subfield $\mathbf{F} \subseteq \mathbf{K}$ such that $\mathbf{k} \subseteq \mathbf{K}$ and $\zeta_{\mathbf{k}}(\rho) = 0$ if and only if $\mathbf{F} \subseteq \mathbf{k}$.*
3. *If $\mathbf{N}$ is a normal CM-field which does not contain any imaginary quadratic subfield, then either $\zeta_{\mathbf{N}^+}$ has a real zero in the range $1 - 1/\log d_{\mathbf{N}} \leq s < 1$ or $\zeta_{\mathbf{N}}(s) \leq 0$ in this range $1 - 1/\log d_{\mathbf{N}} \leq s < 1$.*
4. *If $\mathbf{N}$ is an imaginary abelian field which does not contain any imaginary quadratic subfield, then either $\zeta_{\mathbf{N}^+}$ has a real zero in the range $1 - 2/\log d_{\mathbf{N}} \leq s < 1$ or $\zeta_{\mathbf{N}}(s) \leq 0$ in this range $1 - 2/\log d_{\mathbf{N}} \leq s < 1$.*

Proof 1. This result is a generalization of [Sta, Lemma 3] and its proof is similar (see [LLO, Lemma 15]).

2. See [Sta, Theorem 3].

3. Assume that there exists s_1 in the range $1 - 1/\log d_{\mathbf{N}} \leq s < 1$ such that $\zeta_{\mathbf{N}}(s_1) > 0$. Since the residue of $\zeta_{\mathbf{N}}$ at $s = 1$ is positive, it holds $\lim_{s\uparrow 1} \zeta_{\mathbf{N}}(s) = -\infty$ and there exists some real zero ρ of $\zeta_{\mathbf{N}}$ of odd multiplicity $n_\rho \geq 1$ in the range $1 - 1/\log d_{\mathbf{N}} \leq s_1 \leq s < 1$ and, according to the first point of this proposition,

in this range we have $n_\rho \le 2$. Hence, $n_\rho = 1$ and, according to the second point of this proposition, ρ is a zero of $\zeta_{\mathbf{F}}$ for some quadratic subfield $\mathbf{F}$ of $\mathbf{N}$. Due to our hypothesis this $\mathbf{F}$ is real, hence is a subfield of $\mathbf{N}^+$ and $\zeta_{\mathbf{N}^+}(\rho) = 0$.

4. For $0 < s < 1$ we have

$$(\zeta_{\mathbf{N}}/\zeta_{\mathbf{N}^+})(s) = \prod_{\chi(-1)=-1} L(s,\chi) = \prod_{\substack{\{\chi,\bar\chi\},\\ \chi(-1)=-1}} |L(s,\chi)|^2 \ge 0$$

where the product is taken over the set of disjoint pairs $\{\chi,\bar\chi\}$ of odd Dirichlet characters of the group $X_{\mathbf{N}}$ of primitive Dirichlet characters associated with the abelian number field $\mathbf{N}$. ■

3.1 The Case of Normal CM-Fields

***Theorem 12* (Compare with [Lou11, Th. 4])** *Let* $\mathbf{N}$ *be a normal* CM*-field of degree* $2m > 2$ *which does not contain any imaginary quadratic subfield. Then*

$$h_{\mathbf{N}}^- \ge \epsilon_{\mathbf{N}} \frac{\sqrt{e}}{2u_m}\left(\frac{\sqrt{r}}{\pi e \log r}\right)^m$$

with $u_m = m^m/(m-1)^{m-1}$ *and* $r = d_{\mathbf{N}}^{1/2m}$ *(the root discriminant of* $\mathbf{N}$*). In particular,* $h_{\mathbf{N}}^- > 1$ *for* $r \ge 40000$*, and* $h_{\mathbf{N}}^- > 1$ *for* $m \ge 10$ *and* $r \ge 14000$*.*

Proof According to Point 3 of Proposition 11, there are two cases to consider.

First, assume that $\zeta_{\mathbf{N}^+}$ has no real zero in the range $1 - 1/\log d_{\mathbf{N}} \le s < 1$. Then $\zeta_{\mathbf{N}}\big(1-(1/\log d_{\mathbf{N}})\big) \le 0$ and using Proposition 10 with $a = 1$ we obtain

$$\mathrm{Res}_{s=1}(\zeta_{\mathbf{N}}) \ge \epsilon_{\mathbf{N}}/\sqrt{e}\log d_{\mathbf{N}}.$$

Using (1) we conclude that

$$\mathrm{Res}_{s=1}(\zeta_{\mathbf{N}})/\,\mathrm{Res}_{s=1}(\zeta_{\mathbf{N}^+}) \ge \epsilon_{\mathbf{N}}/\sqrt{e}\left(\frac{e\log d_{\mathbf{N}^+}}{2(m-1)}\right)^{m-1}\log d_{\mathbf{N}}. \tag{22}$$

Second, assume that $\zeta_{\mathbf{N}^+}$ has a real zero β in the range $1-1/\log d_{\mathbf{N}} \le s < 1$. Then $\zeta_{\mathbf{N}}(\beta) = 0 \le 0$ and using Proposition 10 with $a = 1$ we obtain

$$\mathrm{Res}_{s=1}(\zeta_{\mathbf{N}}) \ge \epsilon_{\mathbf{N}}(1-\beta)/\sqrt{e}.$$

Using (4) we conclude that

$$\mathrm{Res}_{s=1}(\zeta_{\mathbf{N}})/\,\mathrm{Res}_{s=1}(\zeta_{\mathbf{N}^+}) \ge \epsilon_{\mathbf{N}}/\sqrt{e}\left(\frac{e\log d_{\mathbf{N}^+}}{2m}\right)^{m}. \tag{23}$$

https://doi.org/10.4153/CJM-2001-045-5 Published online by Cambridge University Press

Finally, since (23) is always greater than or equal to (22) (for $d_{\mathbf{N}} \geq d_{\mathbf{N}^+}^2$), we conclude that (22) is valid in both cases. Using (21) and (22) we obtain

$$h_{\mathbf{N}}^- \geq \epsilon_{\mathbf{N}} \frac{Q_{\mathbf{N}} w_{\mathbf{N}} \sqrt{d_{\mathbf{N}}/d_{\mathbf{N}^+}}}{2\pi\sqrt{e}(\frac{\pi e}{m-1}\log d_{\mathbf{N}^+})^{m-1}\log d_{\mathbf{N}}}. \tag{24}$$

To deduce the desired lower bound, we use $\sqrt{d_{\mathbf{N}}/d_{\mathbf{N}^+}} \geq d_{\mathbf{N}}^{1/4} = r^{m/2}$, $\log d_{\mathbf{N}^+} \leq \frac{1}{2}\log d_{\mathbf{N}} = m\log r$, $\log d_{\mathbf{N}} = 2m\log r$, $Q_{\mathbf{N}} w_{\mathbf{N}} \geq 2$, $\epsilon'_{\mathbf{N}} \geq 1 - (2\pi m e^{1/2m}/r)$ and $\epsilon''_{\mathbf{N}} \geq \frac{2}{5}\exp(-2\pi m/r)$. ∎

3.2 The Case of Imaginary Abelian Fields

Theorem 13 *Let* $\mathbf{N}$ *be an abelian CM-field of degree* $2m > 2$ *which does not contain any imaginary quadratic subfield. Then*

$$h_{\mathbf{N}}^- \geq \frac{\epsilon_{\mathbf{N}}}{eu_m}\left(\frac{\sqrt{r}}{\pi\log r + 0.146}\right)^m$$

with $u_m = m^m/(m-1)^{m-1}$ *and* $r = d_{\mathbf{N}}^{1/2m}$ *(the root discriminant of* $\mathbf{N}$*). In particular,* $h_{\mathbf{N}}^- > 1$ *for* $r \geq 10000$*, and* $h_{\mathbf{N}}^- > 1$ *for* $m \geq 10$ *and* $r \geq 1200$*.*

Proof It is easily verified that

$$h_{\mathbf{N}}^- \geq \frac{\epsilon_{\mathbf{N}} Q_{\mathbf{N}} w_{\mathbf{N}} \sqrt{d_{\mathbf{N}}/d_{\mathbf{N}^+}}}{\pi e(\frac{\pi}{(m-1)}\log d_{\mathbf{N}^+} + 2\pi\mu_{\mathbf{Q}})^{m-1}\log d_{\mathbf{N}}}. \tag{25}$$

The proof of (25) is similar to the proof of (24): Point 4 of Proposition 11 allows us to use Proposition 10 with $a = 2$ and we use (17) and (18) (instead of using Point 3 of Proposition 11, (1) and (4)). ∎

We refer the reader to [CK] for the solution of the relative class number one problem for the imaginary abelian fields, a solution based on refinements of the lower bound given in Theorem 13.

3.3 Remarks

The reader can easily check that our proofs and statements of Theorems 12 and 13 are still valid under the hypothesis that if $\mathbf{N}$ contains an imaginary quadratic field $\mathbf{k}$ then $\zeta_{\mathbf{k}}(s) < 0$ for $0 < s < 1$. Now, according to [Hor, Th. 1] (for the abelian case) and to [Oka] (for the non-abelian case), if $\mathbf{k} \subseteq \mathbf{N}$ are CM-fields then $h_{\mathbf{k}}^-$ divides $4h_{\mathbf{N}}^-$ (see also [LOO, Point (iii) of Theorem 5] and [Lem, Theorem 2 and Corollary 1]). In particular, if $h_{\mathbf{N}}^- = 1$ then $h_{\mathbf{k}}^- \in \{1, 2, 4\}$. According to [Arn], all the imaginary quadratic fields $\mathbf{k}$ of class numbers 1, 2 and 4 are known and it is only a matter of computation to verify that we have $\zeta_{\mathbf{k}}(s) < 0$ in the range $0 < s < 1$ for all the imaginary quadratic fields $\mathbf{k}$ of class numbers 1, 2 or 4. Therefore, we are allowed to use our lower bounds and we obtain:

Theorem 14 *The root discriminant of a normal* CM*-field* $\mathbf{N}$ *(respectively, of an imaginary abelian field* $\mathbf{N}$*) of degree* $2m \geq 20$ *with relative class number one is less than or equal to* 14000 *(respectively, less than or equal to* 1200*).*

We will prove in Theorem 15 that the use of Theorem 5 may sometimes drastically reduce these bounds on root discriminants: we will prove that the root discriminant of a dihedral CM-field $\mathbf{N}$ of degree $2m \geq 20$ with relative class number one is less than or equal to 3400. It may be worth noticing that if $\mathbf{N}$ ranges over CM-fields of degrees $2m$ going to infinity, then as we have $r_{\mathbf{N}} \geq r_{\mathbf{N}^+}$ and as $\mathbf{N}^+$ is a totally real field of degree m, Odlyzko's bounds for discriminants yield $\liminf r_{\mathbf{N}} \geq 8\pi e^{\gamma+\pi/2} > 215$ under the assumption of the generalized Riemann hypothesis (see [Ser]). In particular, our upper bound for the root discriminants of the normal CM-fields with relative class number one is not sharp enough to prove that there are only finitely many such normal CM-fields, a result which is a corollary of [Odl, Theorem 2, p. 279] (see also [Hof, Corollaries 1 and 2, p. 47]). However, our bounds are sharp enough to prove that there are only finitely many normal CM-fields of a given degree with relative class number one and that there are only finitely many normal CM-fields with relative class number one in any family of normal CM-fields whose root discriminants go to infinity with their discriminants (*e.g.*, the family of the imaginary abelian number fields, or the family of the dihedral CM-fields). Moreover, it must be pointed out that our lower bounds for relative class numbers become better and easier to use than those given in these papers when the degree of $\mathbf{N}$ is not too large, say $2m = [\mathbf{N} : \mathbf{Q}] \leq 50$.

3.4 The Case of Dihedral CM-Fields

Let $\mathbf{N}$ be a normal CM-field of degree $2m = 4n \geq 12$ with m odd, and assume that its Galois group $\mathrm{Gal}(\mathbf{N}/\mathbf{Q})$ is isomorphic to the dihedral group D_{2m} of order $2m$. Let $\mathbf{M}$ denote the maximal abelian subfield of $\mathbf{N}$. Hence, $\mathbf{M}$ is an imaginary biquadratic bicyclic field. We have the following theorem which allows us to improve upon the bounds given in Theorem 14:

Theorem 15 *Fix an imaginary biquadratic bicyclic field* $\mathbf{M}$*, assume that* $\zeta_{\mathbf{M}}(s) \leq 0$ *for* $0 < s < 1$*, let* $\mathbf{L}$ *denote its real quadratic subfield and set* $\rho_{\mathbf{L}} = \mathrm{Res}_{s=1}(\zeta_{\mathbf{L}})$ *and* $\mu_{\mathbf{L}}' = \max(0, 2\mu_{\mathbf{L}} - \frac{1}{2}\log d_{\mathbf{L}})$*. Let* $\mathbf{N}$ *range over the dihedral* CM*-fields of degree* $2m = 4n \geq 12$*, n odd, containing* $\mathbf{M}$*. It holds*

$$h_{\mathbf{N}}^- \geq \epsilon_{\mathbf{N}} \frac{2}{emu_m} \left(\frac{r}{4\pi^2\rho_{\mathbf{L}}(\log r + \mu_{\mathbf{L}}')}\right)^{m/2}$$

with $u_m = \big(n/(n-1)\big)^{n-1} = \big(m/(m-2)\big)^{(m-2)/2}$ *and where* $r = d_{\mathbf{N}}^{1/2m}$ *(the root discriminant of* $\mathbf{N}$*). In particular,* $m \geq 6$ *and* $h_{\mathbf{N}}^- = 1$ *imply* $r \leq 4500$*, and* $m \geq 10$ *and* $h_{\mathbf{N}}^- = 1$ *imply* $r \leq 3400$*.*

Proof First, since

$$(\zeta_{\mathbf{N}}/\zeta_{\mathbf{M}})(s) = \prod_{\chi^2 \neq 1} L(s,\chi) = \prod_{\{\chi,\bar{\chi}\},\chi^2\neq 1} |L(s,\chi)|^2 \geq 0 \quad (0 < s < 1)$$

https://doi.org/10.4153/CJM-2001-045-5 Published online by Cambridge University Press

(where χ ranges over the $2n-2$ non quadratic characters associated with the cyclic extension $\mathbf{N}/\mathbf{L}$ of degree $2n$), we conclude that $\zeta_{\mathbf{M}}(s) \leq 0$ for $0 < s < 1$ implies $\zeta_{\mathbf{N}}\big(1 - (2/\log d_{\mathbf{N}})\big) \leq 0$ and $\operatorname{Res}_{s=1}(\zeta_{\mathbf{N}}) \geq 2\epsilon_{\mathbf{N}}/e \log d_{\mathbf{N}}$. Second, using (13), $d_{\mathbf{N}^+}/d_{\mathbf{L}}^n = \prod_{\chi_+\neq 1} f_{\chi_+}$ and $d_{\mathbf{N}^+} \leq \sqrt{d_{\mathbf{N}}} = r^{2n}$ we obtain

$$\begin{aligned}
\operatorname{Res}_{s=1}(\zeta_{\mathbf{N}^+}) &= \rho_{\mathbf{L}} \prod_{\chi_+\neq 1} |L(1,\chi_+)| \\
&\leq \rho_{\mathbf{L}} \Big(\frac{1}{n-1} \sum_{\chi_+\neq 1} |L(1,\chi_+)|\Big)^{n-1} \\
&\leq \rho_{\mathbf{L}} \Big(\frac{\rho_{\mathbf{L}}}{2(n-1)} \log(d_{\mathbf{N}^+}/d_{\mathbf{L}}^n) + 2B_{\mathbf{L}}\Big)^{n-1} \\
&\leq \rho_{\mathbf{L}} \Big(\frac{\rho_{\mathbf{L}}}{2(n-1)} \log(r^{2n}/d_{\mathbf{L}}^n) + \frac{2n}{n-1}\rho_{\mathbf{L}}\mu_{\mathbf{L}}\Big)^{n-1} \\
&\leq u_m \rho_{\mathbf{L}}^n (\log r + \mu'_{\mathbf{L}})^{n-1}
\end{aligned}$$

(where χ_+ ranges over the $n-1$ non trivial characters associated with the cyclic extension $\mathbf{N}^+/\mathbf{L}$ of degree n). Using (21) we obtain the desired lower bound for $h_{\mathbf{N}}^-$. Now, the relative class number $h_{\mathbf{M}}^-$ of $\mathbf{M}$ divides the class number $h_{\mathbf{N}}^-$ of $\mathbf{N}$ (see [LOO, Theorem 5]). Hence, $h_{\mathbf{N}}^- = 1$ implies $h_{\mathbf{M}}^- = 1$. However, it is known that there are only 147 imaginary biquadratic bicyclic fields $\mathbf{M}$ with relative class number one, and it is only a matter of computation to verify that $\zeta_{\mathbf{M}}(s) \leq 0$ in the range $0 < s < 1$ for these 147 fields. Finally, the computation (based on [Lou9] and carried out in [LL, Section 7]) of $\rho_{\mathbf{L}}$ and $\mu_{\mathbf{L}}$ for the real quadratic subfields $\mathbf{L}$ of the 147 imaginary biquadratic bicyclic fields $\mathbf{M}$ with relative class number one yields $\rho_{\mathbf{L}} \leq 4.213$ (for $d_{\mathbf{L}} = 65689$) and $\mu'_{\mathbf{L}} \leq 1.787$ (for $d_{\mathbf{L}} = 1608$) whenever $h_{\mathbf{M}}^- = 1$. Using these bounds and our lower bound on $h_{\mathbf{N}}^-$, we deduce the last assertion. ■

Notice that exponent $m/2$ of $\log r$ in Theorem 15 is half as large as the one $2m$ in Theorem 12. We refer the reader to [LO2] and [Lef] for the solution of the class number one problem for the dihedral CM-fields, solution based on refinements of the lower bound given in Theorem 15.

3.5 The Case of Some Non-Normal Sextic CM-Fields

Proposition 16 *Let* $\mathbf{F}$ *be a real cyclic cubic field and* $\mathbf{K}$ *be a non-normal* CM*-sextic field with maximal totally real subfield* $\mathbf{F}$. *The normal closure* $\mathbf{N}$ *of* $\mathbf{K}$ *is a* CM*-field of degree* 24 *with Galois group* $\operatorname{Gal}(\mathbf{N}/\mathbf{Q})$ *isomorphic to the direct product* $\mathcal{A}_4 \times C_2$, $\mathbf{N}^+$ *is a normal subfield of* $\mathbf{N}$ *of degree* 12 *and Galois group* $\operatorname{Gal}(\mathbf{N}^+/\mathbf{Q})$ *isomorphic to* $\mathcal{A}_4$, *the compositum maximal abelian subfield* $\mathbf{A}$ *of* $\mathbf{N}$ *is an imaginary sextic field containing* $\mathbf{F}$,

$$\zeta_{\mathbf{N}}/\zeta_{\mathbf{N}^+} = (\zeta_{\mathbf{A}}/\zeta_{\mathbf{F}})(\zeta_{\mathbf{K}}/\zeta_{\mathbf{F}})^3, \tag{26}$$

and $d_{\mathbf{N}}$ *divides* $d_{\mathbf{K}}^{12}$, *and* $\big[1 - (1/12\log d_{\mathbf{K}}), 1\big) \subseteq \big[1 - (1/\log d_{\mathbf{N}}), 1\big)$.

Proof Let us only prove (26). Set $\mathbf{K}_0 = \mathbf{A}$ and let $\mathbf{K}_i$ denote the three conjugate fields of $\mathbf{K}$. Since the Galois group of the abelian extension $\mathbf{N}/\mathbf{F}$ is the elementary 2-group $\mathcal{C}_2 \times \mathcal{C}_2 \times \mathcal{C}_2$, using abelian L-functions we obtain $\zeta_{\mathbf{N}}/\zeta_{\mathbf{N}^+} = \prod_{i=0}^{3}(\zeta_{\mathbf{K}_i}/\zeta_{\mathbf{F}})$. Since the three $\mathbf{K}_i$'s with $1 \leq i \leq 3$ are isomorphic to $\mathbf{K}$, we have $\zeta_{\mathbf{K}_i} = \zeta_{\mathbf{K}}$ for $1 \leq i \leq 3$, and we obtain (26). Finally, since $\mathbf{N} = \mathbf{K}_1\mathbf{K}_2\mathbf{K}_3$ and since the three $\mathbf{K}_i$'s are pairwise isomorphic, we conclude that $d_{\mathbf{N}}$ divides $d_{\mathbf{K}}^{12}$ (see [Sta, Lemma 7]). ■

Lemma 17 **(See [LLO, Lemma 15])** *The Dedekind zeta function of a number field* $\mathbf{M}$ *has at most two real zeros in the range* $1 - (1/\log d_{\mathbf{M}}) \leq s < 1$.

Theorem 18 *Let* $\mathbf{K}$ *be a non-normal sextic* CM*-field with maximal totally real subfield a real cyclic cubic field* $\mathbf{F}$ *of conductor* $f_{\mathbf{F}}$. *Set* $r = d_{\mathbf{K}}^{1/6}$ *(the root discriminant of* $\mathbf{K}$*) and* $\epsilon_{\mathbf{K}} = 1 - (6\pi e^{1/72}/r)$. *We have*

$$h_{\mathbf{K}}^- \geq \frac{\epsilon_{\mathbf{K}}}{6e^{1/24}\pi^3}\left(\frac{\sqrt{r}}{3\log r + 0.1}\right)^3. \tag{27}$$

Therefore, $h_{\mathbf{K}}^- = 1$ *implies* $r \leq 33000$.

Proof There are two cases to consider.

First, assume that $\zeta_{\mathbf{F}}$ does not have any real zero in $\left[1 - (1/12\log d_{\mathbf{K}}), 1\right[$. According to (26) any real zero β in $\left[1 - (1/12\log d_{\mathbf{K}}), 1\right[$ of $\zeta_{\mathbf{K}}$ would be a triple zero of $\zeta_{\mathbf{N}}$ in $\left[1 - (1/\log d_{\mathbf{N}}), 1\right[$, which would contradict Lemma 17. Hence, $\zeta_{\mathbf{K}}$ does not have any real zero in $\left[1 - (1/\log 12 d_{\mathbf{K}}), 1\right[$, and $\zeta_{\mathbf{K}}\big(1 - (1/12\log d_{\mathbf{K}})\big) \leq 0$. Using Proposition 10 with $a = 12$ we obtain

$$\mathrm{Res}_{s=1}(\zeta_{\mathbf{K}}) \geq \epsilon_{\mathbf{K}}/12e^{1/24}\log d_{\mathbf{K}}.$$

Using (17), (21) and $Q_{\mathbf{K}}w_{\mathbf{K}} \geq 2$, we conclude that

$$h_{\mathbf{K}}^- \geq \epsilon_{\mathbf{K}}\frac{1}{12e^{1/24}\pi^3}\frac{\sqrt{d_{\mathbf{K}}/d_{\mathbf{F}}}}{(\log f_{\mathbf{F}} + 0.05)^2\log d_{\mathbf{K}}}. \tag{28}$$

Second, assume that $\zeta_{\mathbf{F}}$ has a real zero β in $\left[1 - (1/12\log d_{\mathbf{K}}), 1\right[$. Then $\zeta_{\mathbf{K}}(\beta) = 0$. Using Proposition 10 with $a = 12$ we obtain

$$\mathrm{Res}_{s=1}(\zeta_{\mathbf{K}}) \geq \epsilon_{\mathbf{K}}(1-\beta)/e^{1/24}.$$

Using (18), (21) and $Q_{\mathbf{K}}w_{\mathbf{K}} \geq 2$, we conclude that

$$h_{\mathbf{K}}^- \geq \epsilon_{\mathbf{K}}\frac{4}{e^{1/24}\pi^3}\frac{\sqrt{d_{\mathbf{K}}/d_{\mathbf{F}}}}{(\log f_{\mathbf{F}})(\log f_{\mathbf{F}} + 0.05)^2}. \tag{29}$$

Finally, since $d_{\mathbf{K}} \geq d_{\mathbf{F}}^2 \geq f_{\mathbf{F}}^4$, the lower bound (29) is always better than the lower bound (28). Hence, the lower bound (28) always holds.

To deduce the desired lower bound, we use $\sqrt{d_{\mathbf{K}}/d_{\mathbf{F}}} \geq d_{\mathbf{K}}^{1/4} = r^{3/2}$, $\log f_{\mathbf{F}} = \frac{1}{2}\log d_{\mathbf{F}} \leq \frac{1}{4}\log d_{\mathbf{K}} = \frac{3}{2}\log r$ and $\log d_{\mathbf{K}} = 6\log r$. ■

We refer the reader to [BouL] for the solution of the class number one problem for these non-normal sextic CM-fields (there are 19 non-isomorphic such sextic CM-fields), a solution based on refinements of the lower bound given in Theorem 18. In [Bou], point 2 of Theorem 2 is also used for settling the class number one problem for the sextic CM-fields whose real cubic subfields are non normal.

3.6 Other Lower Bounds

We refer the reader to [LPP, Section 4] for other explicit lower bounds for relative class numbers of non-normal CM-fields. It would also be possible to derive from [Mur] effective lower bounds for relative class numbers of non-normal CM-fields.

4 Integral Representations

This section is devoted to proving Theorems 22 and 23 below which will then allow us to prove the explicit results given in Section 2 of this paper.

4.1 Assumptions on f and Definitions of A_f, Γ_f and F_f

Let us stage the framework in which Dedekind zeta functions, L-functions and their products will fit nicely. Let

$$f(s) = \sum_{n\geq 1} a_n(f) n^{-s} \tag{30}$$

be a given Dirichlet series.

Hypothesis (i) We assume that $\sum_{n\geq 1} a_n(f) n^{-s}$ is absolutely convergent in the half plane $\{s = \sigma + it; \sigma = \Re(s) > 1\}$. In particular, for any $\alpha > 1$ we have $|f(s)| = O(1)$ in the open half-plane $\{s; \Re(s) > \alpha\}$.

Hypothesis (ii) We assume that there exist some positive constant $A_f > 0$ and some Gamma factor

$$\Gamma_f(s) = \Gamma^a(s/2)\Gamma^b\big((s+1)/2\big)\Gamma^c(s)$$

(a, b and c non negative rational integers) such that

$$F_f(s) = A_f^s \Gamma_f(s) f(s)$$

extends to a meromorphic function on the complex plane with only two poles, at $s = 1$ and $s = 0$, satisfying the functional equation

$$F_f(1-s) = W_f \overline{F_f(\bar{s})} = W_f A_f^s \Gamma_f(s) \tilde{f}(s) \tag{31}$$

(for some complex number W_f of absolute value equal to one) where

$$\tilde{f}(s) := \overline{f(\bar{s})} = \sum_{n\geq 1} \overline{a_n(f)} n^{-s}$$

for $\Re(s) > 1$ (in particular, $\tilde{f} = f$ if all the $a_n(f)$ are real for $n \geq 1$).

We let $n_f \geq 0$ denote the order of $s = 1$ and $s = 0$ as poles of F_f.

Notice that according to (56) below, (a, b, c) and A_f are not uniquely determined by f (in fact, we could assume $c = 0$ and in that case (a, b) and A_f would be uniquely determined by f).

Hypothesis (iii) We assume that

$$s \mapsto \Lambda_f(s) = \big(s(s-1)\big)^{n_f} F_f(s)$$

is an entire function of finite order, thus such that there exists $\alpha > 0$ and $r_\alpha > 0$ such that $|s| \geq r_\alpha$ implies $|\Lambda_f(s)| \leq \exp(|s|^\alpha)$.

Recall that according to Stirling's formula, in any strip $\alpha \leq \sigma = \Re(s) \leq \beta$ and $|t| = |\Im(s)| \geq 1$ we have

$$\Gamma(s) = O(e^{-\pi|t|/2}|t|^{\sigma-1/2}) \quad \text{and} \quad \frac{1}{\Gamma(s)} = O(e^{\pi|t|/2}|t|^{-(\sigma-1/2)})$$

(see [Rad, Section 21]), which yields

$$\Gamma_f(s) = O(e^{-(a+b+2c)\pi|t|/4}|t|^{((a+b+2c)\sigma-(a+c))/2}) \tag{32}$$

and

$$\frac{\Gamma_f(s)}{\Gamma_f(1-s)} = O(|t|^{(a+b+2c)(\sigma-1/2)}). \tag{33}$$

In particular, let $\alpha > 1$ be given. For $\Re(s) = \alpha$ and $|t| = |\Im(s)| \geq 1$, we have $|f(s)| = O(1)$, $|\tilde{f}(s)| = O(1)$ and

$$|f(1-s)| = A_f^{2\alpha-1}|\Gamma_f(s)/\Gamma_f(1-s)|\,|\tilde{f}(s)| = O(|t|^{(a+b+2c)(\alpha-1/2)}).$$

Hence, according to the Phragmen-Lindelöf Theorem (see [Lan, Chapter XIII, §5] and [Rad, Section 33]) and to (32), we obtain:

Lemma 19 *Assume that f satisfies Hypotheses (i), (ii) and (iii) above. For a given $\alpha > 1$, there exists $M \geq 0$ such that $f(s) = O(|t|^M)$ and $F_f(s) = O(e^{-(a+b+2c)\pi|t|/4}|t|^M)$ in the range $1 - \alpha \leq \Re(s) \leq \alpha$ and $|t| = |\Im(s)| \geq 1$.*

4.2 Properties of Mellin Tranforms

Let $a < b$ and $0 < \beta \leq \pi$ be given. Let M_1 denote the set of the functions Ψ holomorphic in the strip $a < \Re(s) < b$ which satisfy for all $\epsilon > 0$ and $\delta > 0$

$$|\Psi(s)| \leq C_{\epsilon,\delta}e^{-(\beta-\delta)|t|}, \quad a + \epsilon \leq \Re(s) \leq b - \epsilon$$

(*e.g.* $\Psi(s) = \Gamma(s)$ with $0 = a < b$ and $\beta = \pi/2$), and let M_2 denote the set of the functions Φ holomorphic in the sector $\{s; |\arg(s)| < \beta \text{ and } s \neq 0\}$ which satisfy for all $\epsilon > 0$ and $\delta > 0$

$$|\Phi(s)| \leq C'_{\epsilon,\delta}|s|^{-c}, \quad |\arg(s)| \leq \beta - \delta, \quad a + \epsilon \leq c \leq b - \epsilon$$

(*e.g.* $\Phi(s) = e^{-s}$ with $0 = a < b$ and $\beta = \pi/2$).

Theorem 20 **(See [Mel] or [Rad, Section 27])** *If $\Psi \in M_1$ then for $a < \alpha < b$*

$$s \mapsto M^{-1}\Psi(x) = \frac{1}{2\pi i}\int_{\Re(s)=\alpha} \Psi(s)x^{-s}\,ds \in M_2,$$

if $\Phi \in M_2$ then

$$s \mapsto M\Phi(s) = \int_0^\infty \Phi(t)t^s\frac{dt}{t} \in M_1,$$

and we have $M^{-1}M\Phi = \Phi$ and $MM^{-1}\Psi = \Psi$. Finally, the inverse Mellin transform $\Phi = M^{-1}\Psi$ of a product $\Psi = \prod_{i=1}^r \Psi_i$ of functions Ψ_i in M_1 is equal to the convolution $\Phi_1 \star \Phi_2 \star \cdots \star \Phi_r$ of the inverse Mellin transforms $\Phi_i = M^{-1}\Psi_i$ (where $\Phi_1 \star \Phi_2(x) = \int_0^\infty \Phi_1(x/t)\Phi_2(t)\,dt$), and the convolution of positive functions is positive.

4.3 Definitions of $H_f(x)$, $S_f(x)$ and $h_f(x)$

We have assumed f regular enough to warrant the forthcoming calculations with Mellin and inverse Mellin transforms (in particular, the Mellin inversion formula is valid for F_f). We set

$$H_f(x) := M^{-1}\Gamma_f(x) = \frac{1}{2\pi i}\int_{\Re(s)=\alpha} \Gamma_f(s)x^{-s}\,ds \quad (\alpha > 0 \text{ and } x > 0)$$

(inverse Mellin transform of Γ_f) and

$$S_f(x) := M^{-1}F_f(x)$$

$$(34) \qquad = \frac{1}{2\pi i}\int_{\Re(s)=\alpha} F_f(s)x^{-s}\,ds \quad (\alpha > 1 \text{ and } x > 0)$$

$$(35) \qquad = \sum_{n\geq 1} a_n(f)H_f(nx/A_f)$$

(inverse Mellin transform of F_f). The most important observation is that we have $H_f(x) > 0$ for $x > 0$ (use the last assertion of Theorem 20).

Let p_k, $0 \leq k \leq n_f$, be the complex numbers such that

$$(36) \qquad F_f(s) = \sum_{k=0}^{n_f} \frac{p_k}{(s-1)^k} + O(s-1).$$

Then,

$$(37) \qquad \operatorname{Res}_{s=1}\big(F_f(s)x^{-s}\big) = \frac{1}{x}\sum_{k=1}^{n_f} \frac{(-1)^{k-1}p_k}{(k-1)!}\log^{k-1} x = \frac{1}{x}P_f(\log x).$$

Using (31), we obtain

$$(38) \qquad \operatorname{Res}_{s=0}\big(F_f(s)x^{-s}\big) = -W_f\sum_{k=1}^{n_f} \frac{\bar{p}_k}{(k-1)!}\log^{k-1} x = -W_f\overline{P_f(-\log x)}.$$

Now, Lemma 19 allows us to shift the line of integration in (34) leftwards to the line $\Re(s) = 1-\alpha < 0$. We pick up residues at $s = 1$ and $s = 0$ (see formulae (37) and (38)), and using the functional equation (31) to come back to the line $\Re(s) = \alpha > 1$, we obtain the functional equation

$$\frac{1}{x}S_f\left(\frac{1}{x}\right) = W_f\overline{S_f(x)} + h_f(x) \tag{39}$$

where

$$h_f(x) := P_f(-\log x) - \frac{W_f}{x}\overline{P_f(\log x)}. \tag{40}$$

We have

$$\begin{aligned} F_f(s) &= MM^{-1}F_f(s) = MS_f(s) \\ &= \int_0^\infty S_f(x)x^s\frac{dx}{x} = \int_1^\infty S_f(x)x^s\frac{dx}{x} + \int_1^\infty S_f\left(\frac{1}{x}\right)x^{-s}\frac{dx}{x}. \end{aligned} \tag{41}$$

4.4 Definitions of R_f and I_f and Integral Representation of F_f

Using (41) and (39) we obtain:

Theorem 21 *Set*

$$R_f(s) := \int_1^\infty h_f(x)x^{-s}\,dx = \sum_{k=1}^{n_f}\left(\frac{p_k}{(s-1)^k} + W_f\frac{\bar{p}_k}{(-s)^k}\right) \tag{42}$$

and

$$I_f(s) := \int_1^\infty S_f(x)x^{s-1}\,dx + W_f\int_1^\infty \overline{S_f(x)}x^{-s}\,dx \tag{43}$$

(which defines an entire function). Then $R_f(1-s) = W_f\overline{R_f(\bar{s})} = W_fR_{\bar{f}}(s)$, $I_f(1-s) = W_f\overline{I_f(\bar{s})} = W_fI_{\bar{f}}(s)$ *and for any s in the complex plane it holds*

$$F_f(s) = R_f(s) + I_f(s).$$

Moreover, according to (36) and (42), we have

$$I_f(1) = \lim_{s\downarrow 1}(F_f - R_f)(s) = p_0 - W_f\sum_{k=1}^{n_f}(-1)^k\bar{p}_k. \tag{44}$$

4.5 General Upper Bounds

Theorem 22 *Let f_1 and f_2 be two Dirichlet series for which*

$$|a_n(f_1)| \leq a_n(f_2) \quad (\text{for all } n \geq 1).$$

For $\frac{1}{2} \leq \beta \leq 1 < s$ we have

$$|I_{f_1}(\beta)| \leq 2A_{f_1}^s \Gamma_{f_1}(s) f_2(s).$$

Proof Since $H_{f_1}(x) > 0$ for $x > 0$ and $|a_n(f_1)| \leq a_n(f_2)$ for all $n \geq 1$, we obtain

$$|S_{f_1}(x)| \leq \sum_{n \geq 1} a_n(f_2) H_{f_1}(x)$$

(for $x > 0$), and using (43) we obtain

$$\begin{aligned}
|I_{f_1}(\beta)| &\leq \sum_{n\geq 1} a_n(f_2) \int_1^{\infty} H_{f_1}(nx/A_{f_1})(x^{\beta-1} + x^{-\beta})\,dx \\
&\leq \sum_{n\geq 1} a_n(f_2) \int_1^{\infty} H_{f_1}(nx/A_{f_1})(x^{s-1} + x^{-s})\,dx \\
&\leq 2\sum_{n\geq 1} a_n(f_2) \int_1^{\infty} H_{f_1}(nx/A_{f_1})x^{s-1}\,dx \\
&\leq 2\sum_{n\geq 1} a_n(f_2) \int_0^{\infty} H_{f_1}(nx/A_{f_1})x^{s-1}\,dx \\
&= 2A_{f_1}^s \sum_{n\geq 1} a_n(f_2) n^{-s} \int_1^{\infty} H_{f_1}(x)x^{s-1}\,dx \\
&= 2A_{f_1}^s \Gamma_{f_1}(s) f_2(s). \qquad \blacksquare
\end{aligned}$$

Theorem 23 *Let f_1 and f_2 be two Dirichlet series for which $\Gamma_{f_1} = \Gamma_{f_2}$,*

$$|a_n(f_1)| \leq a_n(f_2) \quad (\text{for all } n \geq 1)$$

and $W_{f_2} = 1$. Set $d = A_{f_1}/A_{f_2}$ and $n_2 = n_{f_2}$. Let p_k denote the coefficients associated with F_{f_2} defined in (36). Assume $\frac{1}{2} \leq \beta \leq 1$. It holds

$$|I_{f_1}(\beta)| \leq J_{f_2}(d) := (d+1)I_{f_2}(1) + d\int_1^d h_{f_2}(x)\frac{dx}{x} + \int_1^d h_{f_2}(x)\,dx \tag{45}$$

and

$$J_{f_2}(d) = \operatorname{Res}_{s=1}\left(s \mapsto F_{f_2}(s)\left(\frac{1}{s} + \frac{1}{s-1}\right)(d^s + d^{1-s})\right) \tag{46}$$

$$= \sum_{i=0}^{n_2}\left(p_i + \sum_{k=i+1}^{n_2}(-1)^{k-(i+1)}p_k\right)\frac{d+(-1)^i}{i!}\log^i d \tag{47}$$

$$= d\frac{p_{n_2}}{n_2!}\log^{n_2} d + d\frac{p_{n_2} + p_{n_2-1}}{(n_2-1)!}\log^{n_2-1} d + O(d\log^{n_2-1} d). \tag{48}$$

https://doi.org/10.4153/CJM-2001-045-5 Published online by Cambridge University Press

If $d = 1$ *then we have the better bound*

$$|I_{f_1}(\beta)| \le I_{f_2}(\beta) \le I_{f_2}(1) = p_0 + \sum_{k=1}^{n_{f_2}} (-1)^{k-1} p_k. \tag{49}$$

It also holds

$$|I'_{f_1}(\beta)| \le (d+1)(\log d) I_{f_2}(1) + (d-1) I'_{f_2}(1) + K_{f_2}(d) + R_{f_2}(d) \tag{50}$$

$$\text{where } K_{f_2}(d) = d\int_1^d h_{f_2}(x)\log(d/x)\frac{dx}{x} + \int_1^d h_{f_2}(x)\log(d/x)\,dx$$

$$\text{and } R_{f_2}(d) = d\int_d^\infty S_{f_2}(x)\big(\log(x/d)\big)\,\frac{dx}{x} + \int_d^\infty S_{f_2}(x)\big(\log(x/d)\big)\,dx.$$

Proof Since $H_{f_1}(x) = H_{f_2}(x) > 0$ for $x > 0$ and since $|a_n(f_1)| \le a_n(f_2)$ for all $n \ge 1$, we obtain (use (35)):

$$|S_{f_1}(x)| \le S_{f_2}(x/d) \quad \text{for } x > 0.$$

Since $\beta \mapsto x^{\beta-1} + x^{-\beta}$ increases with $\beta \ge \frac{1}{2}$ for $x \ge 1$ and since S_{f_2} satisfies the functional equation $\frac{1}{x}S_{f_2}(\frac{1}{x}) = S_{f_2}(x) + h_{f_2}(x)$ (see (39)), using (43) we obtain

$$|I_{f_1}(\beta)| \le \int_1^\infty S_{f_2}(x/d)(x^{\beta-1} + x^{-\beta})\,dx \tag{51}$$

$$\begin{aligned}
&\le \int_1^\infty S_{f_2}(x/d)\,dx + \int_1^\infty S_{f_2}(x/d)\frac{dx}{x} \qquad (52)\\
&= d\int_{1/d}^\infty S_{f_2}(x)\,dx + \int_{1/d}^\infty S_{f_2}(x)\frac{dx}{x}\\
&= d\int_1^\infty S_{f_2}(x)\,dx + d\int_1^d \frac{1}{x}S_{f_2}\Big(\frac{1}{x}\Big)\frac{dx}{x}\\
&\quad + \int_1^\infty S_{f_2}(x)\frac{dx}{x} + \int_1^d \frac{1}{x}S_{f_2}\Big(\frac{1}{x}\Big)\,dx\\
&\le (d+1)\Big(\int_1^\infty S_{f_2}(x)\,dx + \int_1^\infty S_{f_2}(x)\frac{dx}{x}\Big)\\
&\quad + d\int_1^d h_{f_2}(x)\frac{dx}{x} + \int_1^\infty h_{f_2}(x)\,dx,
\end{aligned}$$

which provides us with the desired bounds (45) and (49) (for if $d = 1$ then the right hand sides of (51) and (52) are equal to $I_{f_2}(\beta)$ and $I_{f_2}(1)$, respectively).

Since all the $a_n(f_2)$ are real and since $W_{f_2} = +1$, we conclude that the p_k's are real for $0 \le k \le n_{f_2}$ and, according to (37), (38) and (40), we have

$$h_{f_2}(x) = \sum_{k=1}^{n_{f_2}} \frac{p_k}{(k-1)!} g_k(x)$$

where $g_k(x) = \big(1 + (-1)^k/x\big) \log^{k-1} x$ is such that

$$d\int_1^d g_k(x)\frac{dx}{x} + \int_1^d g_k(x)\,dx$$
$$= \frac{d+(-1)^k}{k}\log^k d + (k-1)!\sum_{i=1}^{k-1}(-1)^{k-1-i}\frac{d+(-1)^i}{i!}\log^i d,$$

which, together with (44), yields (47). Finally, the reader will check that using (36) he can compute the residue (46) and gets that this residue (46) is indeed equal to (47).

The proof of (50) is similar to that of (45): using (43) we have

$$\begin{aligned}
|I'_{f_1}(\beta)| &\le \int_1^\infty S_{f_2}(x/d)(\log x)(x^{\beta-1}+x^{-\beta})\,dx\\
&\le \int_1^\infty S_{f_2}(x/d)(\log x)\,dx + \int_1^\infty S_{f_2}(x/d)(\log x)\frac{dx}{x}\\
&= d\int_{1/d}^\infty S_{f_2}(x)\big(\log(dx)\big)\,dx + \int_{1/d}^\infty S_{f_2}(x)\big(\log(dx)\big)\frac{dx}{x}\\
&= d\int_1^\infty S_{f_2}(x)\big(\log(dx)\big)\,dx + d\int_1^d \frac{1}{x}S_{f_2}\Big(\frac{1}{x}\Big)\big(\log(d/x)\big)\frac{dx}{x}\\
&\quad + \int_1^\infty S_{f_2}(x)\big(\log(dx)\big)\frac{dx}{x} + \int_1^d \frac{1}{x}S_{f_2}\Big(\frac{1}{x}\Big)\big(\log(d/x)\big)\,dx\\
&\le (d+1)(\log d)\Big(\int_1^\infty S_{f_2}(x)\,dx + \int_1^\infty S_{f_2}(x)\frac{dx}{x}\Big)\\
&\quad + (d-1)\Big(\int_1^\infty S_{f_2}(x)(\log x)\,dx - \int_1^\infty S_{f_2}(x)(\log x)\frac{dx}{x}\Big) + R_{f_2}(d)
\end{aligned}$$

(recall the functional equation $\frac{1}{x}S_{f_2}\big(\frac{1}{x}\big) = S_{f_2}(x) + h_{f_2}(x)$)

$$= (d+1)(\log d)I_{f_2}(1) + (d-1)I'_{f_2}(1) + K_{f_2}(d) + R_{f_2}(d). \qquad \blacksquare$$

5 Useful Lemmas

As in [Lou11], to obtain neat bounds in Section 2 of this paper, we will use the following Lemma:

Lemma 24 **(See [Lou11, Lemma 9])** *Let ζ denote the Riemann zeta function and Γ denote the Euler Gamma function, and set*

$$\Lambda(s) = s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s) \quad \textit{and} \quad G(s) = \frac{\sqrt{\pi}}{s}\frac{\Gamma\big((s+1)/2\big)}{\Gamma(s/2)}.$$

Then Λ *and* G *are positive,* $\log\Lambda$ *and* $\log G$ *are convex on the open interval* $(0,+\infty)$, $\Lambda(1)=G(1)=1$, $\Lambda(6)=4\pi^3/63=1.968\cdots$ *and* $G(6)=5\pi/32=0.490\cdots$.

The following coarse lower bounds for discriminants of number fields will be used for proving our results:

Lemma 25 *Let* $\mathbf{L}$ *be a number field of degree* $m=r_1+2r_2\geq 1$. *Then*

$$d_{\mathbf{L}}\geq e^{2(m-1)/3}\geq e^{2(m-1)/5}.$$

Moreover, if $m>1$ *then*

$$d_{\mathbf{L}}\geq e^{2(m+1)/5}\geq e^{2m/5}.$$

Hence, if χ *a primitive character on a ray class group of a number field* $\mathbf{L}$ *of degree* m *then*

$$d_{\mathbf{L}}f_\chi\geq e^{2(m+1)/5}\geq e^{2m/5}.$$

Proof According to Minkowski's geometric bounds for discriminants we have

$$d_{\mathbf{L}}\geq (m^m/m!)^2(\pi/4)^{2r_2}\geq u_m:=(m^m/m!)^2(\pi/4)^m.$$

Now,

$$u_{m+1}/u_m=\frac{\pi}{4}\big((1+1/m)^m\big)^2\geq\frac{\pi}{4}2^2=\pi$$

and $u_1=\pi/4$ yield $u_m\geq\frac{1}{4}\pi^m$, and we obtain $u_m\geq e^{2(m-1)/3}$ for $m\geq 2$ and $u_m\geq e^{2(m+1)/5}$ for $m\geq 3$. If $m=2$ then $d_{\mathbf{L}}\geq 5\geq e^{2(m+1)/5}$ and if $m=1$ then $d_{\mathbf{L}}=1\geq e^{2(m-1)/3}$. Now, if $m\geq 2$ then $d_{\mathbf{L}}f_\chi\geq d_{\mathbf{L}}\geq e^{2(m+1)/5}$ and if $m=1$ then $d_{\mathbf{L}}=1$, $f_\chi\geq 3$ and $d_{\mathbf{L}}f_\chi\geq 3\geq e^{2(m+1)/5}$. ■

The following Lemma will allow us to use Theorems 22 and 23:

Lemma 26 *Let* $\mathbf{L}$ *be a number field of degree* $m\geq 1$, *let* ζ *denote the Riemann zeta function and let* χ *be a character on a ray class group for* $\mathbf{L}$. *For any positive rational integer* $n\geq 1$ *we have*

$$0\leq\big|a_n\big(s\mapsto L(s,\chi)\big)\big|\leq a_n(\zeta_{\mathbf{L}}), \tag{53}$$

$$0\leq a_n(\zeta_{\mathbf{L}})\leq a_n(\zeta^m) \tag{54}$$

and

$$|a_n(\zeta_{\mathbf{L}}/\zeta)|\leq a_n(\zeta^{m-1}). \tag{55}$$

Here, the $a_n(f)$*'s are the coefficients of the Dirichlet series expansions of the considered functions, as defined in (30).*

https://doi.org/10.4153/CJM-2001-045-5 Published online by Cambridge University Press

Proof The bound (53) follows from the fact that for any integral ideal $\mathcal{I}$ of $\mathbf{L}$ we have $|\chi(\mathcal{I})| \leq 1$ and from the fact that $a_n\big(s \mapsto L(s,\chi)\big) = \sum_{\mathcal{I}} \chi(\mathcal{I})$ where $\mathcal{I}$ range over the integral ideals of $\mathbf{L}$ of norm n. Since $n \mapsto a_n(\zeta_{\mathbf{L}})$, $n \mapsto a_n(\zeta^m)$ and $n \mapsto a_n(\zeta_{\mathbf{L}}/\zeta)$ are multiplicative, it suffices to prove that (54) and (55) are valid for $n = p^k$ any prime-power. Let $(p) = \prod_{i=1}^{g} \mathcal{P}_i^{e_i}$ be the prime ideal factorization of (p), let $p^{f_i} = N(\mathcal{P}_i)$ denote the norm of the prime ideal $\mathcal{P}_i$ and set $E_g = \{(x_1,\dots,x_g); \sum_{i=1}^{g} f_i x_i = k\}$ and $F_g = \{(x_1,\dots,x_g); \sum_{i=1}^{g} x_i = k\}$ (where the x_i's denote nonnegative rational integers). Let us first prove (54). Since $(x_1,\dots,x_g) \in E_g \mapsto (f_1x_1,\dots,f_gx_g) \in F_g$ is injective, we do obtain $0 \leq a_{p^k}(\zeta_{\mathbf{L}}) = \#E_g \leq \#F_g = \binom{g-1+k}{k} \leq \binom{m-1+k}{k} = a_{p^k}(\zeta^m)$. Let us now prove (55). To begin with, notice that $a_{p^k}(\zeta_{\mathbf{L}}/\zeta) = a_{p^k}(\zeta_{\mathbf{L}}) - a_{p^{k-1}}(\zeta_{\mathbf{L}})$. Now, if $g = m$ then $a_{p^k}(\zeta_{\mathbf{L}}/\zeta) = a_{p^k}(\zeta_{\mathbf{L}}) - a_{p^{k-1}}(\zeta_{\mathbf{L}}) = \binom{m-1+k}{k} - \binom{m-1+k-1}{k-1} = \binom{m-2+k}{k} = a_{p^k}(\zeta^{m-1})$ (which also follows from the fact that the Euler factor of $\zeta_{\mathbf{L}}/\zeta$ and ζ^{m-1} are equal), and if $g \leq m-1$ then $0 \leq a_{p^k}(\zeta_{\mathbf{L}}) \leq \binom{g-1+k}{k}$, $0 \leq a_{p^{k-1}}(\zeta_{\mathbf{L}}) \leq \binom{g-1+k-1}{k-1} \leq \binom{g-1+k}{k}$ and we also obtain $|a_{p^k}(\zeta_{\mathbf{L}}/\zeta)| = |a_{p^k}(\zeta_{\mathbf{L}}) - a_{p^{k-1}}(\zeta_{\mathbf{L}})| \leq \binom{g-1+k}{k} \leq \binom{m-2+k}{k} = a_{p^k}(\zeta^{m-1})$. ■

Remarks 27 Suppose we want to refer to this Lemma 26 to allow us to use Theorem 23 with $f_1(s) = L(s,\chi)$ and $f_2 = \zeta_{\mathbf{L}}$, with $f_1 = \zeta_{\mathbf{L}}$ and $f_2 = \zeta^m$, or with $f_1 = \zeta_{\mathbf{L}}/\zeta$ and $f_2 = \zeta^{m-1}$. Then the Gamma factors which arise in the functional equations of f_1 and f_2 must be equal. This clearly amounts to asking that χ be unramified at all the infinite places of $\mathbf{L}$ or that $\mathbf{L}$ be totally real, respectively, thus explaining the assumptions made in Theorems 2 and 5, together with the assumption that χ be even in Theorem 7.

We will finally make use of the following functional equation:

$$\Gamma(s) = \pi^{-1/2} 2^{s-1} \Gamma(s/2) \Gamma\big((s+1)/2\big). \tag{56}$$

6 The Proofs

We are now in a position to prove all the results stated in Section 2. For each function f which we will introduce, we refer the reader to subsection 4.1 for the definitions of the contant $A_f > 0$, the Gamma factor Γ_f and the meromorphic function F_f. We then refer the reader to subsection 4.4 for the definitions of the rational function R_f and the integral function I_f.

6.1 Proof of Theorem 1

Choose $f(s) = \zeta_{\mathbf{L}}(s) = \sum_{n \geq 1} a_n(\mathbf{L}) n^{-s}$ and change the notation accordingly, that is to say change all the indices $\bullet_f$ in $\bullet_{\mathbf{L}}$. We have $A_{\mathbf{L}} = \sqrt{d_{\mathbf{L}}/4^{r_2}\pi^m}$ and $\Gamma_{\mathbf{L}}(s) = \Gamma^{r_1}(s/2)\Gamma^{r_2}(s)$, $W_{\mathbf{L}} = 1$. Set $\lambda_{\mathbf{L}} \overset{\text{def}}{=} \operatorname{Res}_{s=1}(F_{\mathbf{L}}) = A_{\mathbf{L}}\Gamma_{\mathbf{L}}(1) \operatorname{Res}_{s=1}(\zeta_{\mathbf{L}})$. According to Theorem 21 we have

$$F_{\mathbf{L}}(s) = R_{\mathbf{L}}(s) + I_{\mathbf{L}}(s) = \frac{\lambda_{\mathbf{L}}}{s(s-1)} + \int_1^\infty S_{\mathbf{L}}(x)(x^{s-1} + x^{-s})\,dx. \tag{57}$$

1. First, since $a_n(\mathbf{L}) \geq 0$ for $n \geq 1$ we obtain $S_{\mathbf{L}}(x) \geq 0$ for $x > 0$ and

$$\frac{\lambda_{\mathbf{L}}}{s(s-1)} \leq F_{\mathbf{L}}(s) = A_{\mathbf{L}}^s \Gamma_{\mathbf{L}}(s)\zeta_{\mathbf{L}}(s) \leq A_{\mathbf{L}}^s \Gamma_{\mathbf{L}}(s)\zeta^m(s) \quad (s > 1)$$

and we finally rewrite this inequality as

$$\mathrm{Res}_{s=1}(\zeta_{\mathbf{L}}) \leq \frac{d_{\mathbf{L}}^{(s-1)/2}}{(s-1)^{m-1}} g_1(s) \quad (s > 1)$$

where

$$g_1(s) = s^{-(r_1+r_2-1)}\Lambda^m(s)G^{r_2}(s)$$

with Λ and G as in Lemma 24 (use (56)).
Now, to get the term $d_{\mathbf{L}}^{(s-1)/2}/(s-1)^{m-1}$ as small as possible we choose

$$s = s_{\mathbf{L}} = 1 + \frac{2(m-1)}{\log d_{\mathbf{L}}} \in [1, 6]$$

(use Lemma 25). Since Λ and G are log-convex on the open interval $(0, +\infty)$, we deduce that g_1 is convex on $[1, 6]$ and, for $(r_1, r_2) \neq (0, 1)$, we obtain

$$g_1(s_{\mathbf{L}}) \leq \max\big(g_1(1), g_1(6)\big) = \max\big(1, 6(2\pi^3/189)^{r_1}(5\pi^7/47628)^{r_2}\big) = 1,$$

whereas for $(r_1, r_2) = (0, 1)$, *i.e.*, for $\mathbf{k}$ an imaginary quadratic field, we have $d_{\mathbf{L}} \geq 3$, $s_{\mathbf{L}} \leq 3$, $g_1(3) = \Lambda^2(3)G(3) = 0.878\cdots \leq 1$ and $g_1(s_{\mathbf{L}}) \leq \max\big(g_1(1), g_1(3)\big) = 1$, thus proving the first point of Theorem 1.

2. Second, since $s \mapsto x^{s-1} + x^{-s}$ increases with $s \geq \frac{1}{2}$ for $x \geq 1$ and since $S_{\mathbf{L}}(x) \geq 0$ for $x > 0$, we deduce that $\frac{1}{2} \leq \beta < 1$ and $\zeta_{\mathbf{L}}(\beta) = 0$ imply $F_{\mathbf{L}}(\beta) = 0$, and using (57) we obtain

$$\lambda_{\mathbf{L}} = \beta(1-\beta)I_{\mathbf{L}}(\beta) \leq (1-\beta)I_{\mathbf{L}}(\beta) \leq (1-\beta)I_{\mathbf{L}}(1) = (1-\beta)\lambda_{\mathbf{L}}\mu_{\mathbf{L}},$$

thus proving the second point of Theorem 1.

3. Third, since $s \mapsto x^{s-1} + x^{-s}$ increases with $s \geq \frac{1}{2}$, for $s > 1$ we have

$$\lambda_{\mathbf{L}}\mu_{\mathbf{L}} = I_{\mathbf{L}}(1) \leq I_{\mathbf{L}}(s) \leq \frac{\lambda_{\mathbf{L}}}{s(s-1)} + I_{\mathbf{L}}(s) = F_{\mathbf{L}}(s) \leq A_{\mathbf{L}}^s \Gamma_{\mathbf{L}}(s)\zeta^m(s)$$

which we write

$$B_{\mathbf{L}} \leq \frac{d_{\mathbf{L}}^{(s-1)/2}}{(s-1)^m} g_2(s) \quad (s > 1)$$

where

$$g_2(s) = s^{-(r_1+r_2)}\Lambda^m(s)G^{r_2}(s)$$

with Λ and G as in Lemma 24.

https://doi.org/10.4153/CJM-2001-045-5 Published online by Cambridge University Press

Now, to get the term $d_{\mathbf{L}}^{(s-1)/2}/(s-1)^m$ as small as possible we choose

$$s = s_{\mathbf{L}} = 1 + \frac{2m}{\log d_{\mathbf{L}}} \in [1, 6]$$

(use Lemma 25). Since Λ and G are log-convex on the open interval $(0, +\infty)$, we deduce that g_2 is convex on $[1, 6]$ and we obtain

$$g_2(s_{\mathbf{L}}) \leq \max\bigl(g_2(1), g_2(6)\bigr) = \max\bigl(1, (2\pi^3/189)^{r_1}(5\pi^7/47628)^{r_2}\bigr) = 1,$$

thus proving the third point of Theorem 1.

6.2 Proof of Theorem 4

Let b denote the number of real places of $\mathbf{L}$ at which χ is ramified and set $a = r_1 - b$.

6.2.1 Proof of the First Point of Theorem 4

Choose $f_1(s) = L(s, \chi)$ and $f_2(s) = \zeta^m(s)$.

We have $A_{f_1} = \sqrt{d_{\mathbf{L}} f_\chi / 4^{r_2}\pi^m}$, $\Gamma_{f_1}(s) = \Gamma^a(s/2)\Gamma^b\bigl((s+1)/2\bigr)\Gamma^{r_2}(s)$. Since F_{f_1} is entire, we have $R_{f_1}(s) = 0$ and $F_{f_1}(s) = I_{f_1}(s)$. Applying Theorem 22, we have

$$|F_{f_1}(1)| = |I_{f_1}(1)| \leq 2A_{f_1}^s \Gamma_{f_1}(s)\zeta^m(s) \quad (s > 1)$$

and we rewrite this inequality as

$$|L(1, \chi)| \leq 2\frac{(d_{\mathbf{L}} f_\chi)^{(s-1)/2}}{(s-1)^m} g_3(s) \quad (s > 1)$$

where

$$g_3(s) = s^{-(a+r_2)}\Lambda^m(s)G^{b+r_2}(s)$$

with $\Lambda(s)$ and $G(s)$ as in Lemma 24 (use (56)).

Now, to get the term $(d_{\mathbf{L}} f_\chi)^{(s-1)/2}/(s-1)^m$ as small as possible we choose

$$s = s_\chi = 1 + \frac{2m}{\log(d_{\mathbf{L}} f_\chi)} \in [1, 6]$$

(use Lemma 25). Since Λ and G are log-convex on the open interval $(0, +\infty)$, we deduce that g_3 is convex on $[1, 6]$ and we obtain

$$g_3(s_\chi) \leq \max\bigl(g_3(1), g_3(6)\bigr) = \max\bigl(1, (2\pi^3/189)^{a+r_2}(5\pi^4/504)^{b+r_2}\bigr) = 1,$$

thus proving the first point of Theorem 4.

https://doi.org/10.4153/CJM-2001-045-5 Published online by Cambridge University Press

6.2.2 Proof of the Second Point of Theorem 4

Choose $f_1(s) = L(s,\chi)\zeta(s)$ and $f_2(s) = \zeta^{m+1}(s)$.

We have $A_{f_1} = \sqrt{d_{\mathbf{L}} f_\chi / 4^{r_2}\pi^{m+1}}$ and $\Gamma_{f_1}(s) = \Gamma^{a+1}(s/2)\Gamma^b\big((s+1)/2\big)\Gamma^{r_2}(s)$. Set $\lambda_1 = \mathrm{Res}_{s=1}(F_{f_1})$ and $\lambda_0 = \mathrm{Res}_{s=0}(F_{f_1}) = -W_\chi\bar{\lambda}_1$. We have $R_{f_1}(s) = \frac{\lambda_1}{s-1} + \frac{\lambda_0}{s}$. Since $L(\beta,\chi) = 0$ implies $F_{f_1}(\beta) = 0$ and $R_{f_1}(\beta) = -I_{f_1}(\beta)$, according to Theorem 22 we have

$$|R_{f_1}(\beta)| = \left|\frac{\lambda_1}{1-\beta} - \frac{\lambda_0}{\beta}\right| = |I_{f_1}(\beta)| \le 2A^s_{f_1}\Gamma_{f_1}(s)\zeta^{m+1}(s) \quad (s>1).$$

Since $2/3 \le \beta < 1$, in setting $\lambda = |\lambda_1| = |\lambda_0|$, we have

$$\frac{\lambda}{2(1-\beta)} \le \lambda\left(\frac{1}{1-\beta} - \frac{1}{\beta}\right) = \left|\left|\frac{\lambda_1}{1-\beta}\right| - \left|\frac{\lambda_0}{\beta}\right|\right| \le \left|\frac{\lambda_1}{1-\beta} - \frac{\lambda_0}{\beta}\right|,$$

and we obtain

$$\lambda \le 4(1-\beta)A^s_{f_1}\Gamma_{f_1}(s)\zeta^{m+1}(s) \quad (s>1)$$

which we write

$$|L(1,\chi)| \le 4\frac{(d_{\mathbf{L}} f_\chi)^{(s-1)/2}}{(s-1)^{m+1}} g_4(s) \quad (s>1)$$

where

$$g_4(s) = s^{-(a+r_2+1)}\Lambda^{m+1}(s)G^{b+r_2}(s)$$

with $\Lambda(s)$ and $G(s)$ as in Lemma 24.

Now, to get the term $(d_{\mathbf{L}} f_\chi)^{(s-1)/2}/(s-1)^{m+1}$ as small as possible we choose

$$s = s_\chi = 1 + \frac{2(m+1)}{\log(d_{\mathbf{L}} f_\chi)} \in [1,6]$$

(use Lemma 25). Since Λ and G are log-convex on the open interval $(0,+\infty)$, we deduce that g_4 is convex on $[1,6]$ and we obtain

$$g_4(s_\chi) \le \max\big(g_4(1), g_4(6)\big) = \max\big(1, (2\pi^3/189)^{a+r_2+1}(5\pi^4/504)^{b+r_2}\big) = 1,$$

thus proving the second point of Theorem 4.

6.3 Proof of Theorem 2

Let us first prove the bound (6). Choose $f_1(s) = \zeta_{\mathbf{L}}/\zeta_{\mathbf{Q}}$ (which is entire) and $f_2(s) = \zeta^2(s)$. We have $\Gamma_{f_1}(s) = \Gamma_{f_2}(s) = \Gamma^2(s/2)$, $d = A_{f_1}/A_{f_2} = \sqrt{d_{\mathbf{L}}}$,

$$F_{f_2}(s) = \pi^{-s}\Gamma^2(s/2)\zeta^2(s) = p_2(s-1)^{-2} + p_1(s-1)^{-1} + p_0 + O\big((s-1)\big)$$

with $p_2 = 1$, $p_1 = \gamma - \log(4\pi) = -1.953808\cdots$ and $p_0 = 2.954838\cdots$. Hence, according to Lemma 26 and to (47) in Theorem 23 in which $d = \sqrt{d_{\mathbf{L}}}$, we have

$$\begin{aligned} dB_{\mathbf{L}} = \lambda_{\mathbf{L}}\mu_{\mathbf{L}} &= I_{f_1}(1) \\ &\leq p_2 \frac{d+1}{2}\log^2 d + (p_1+p_2)(d-1)\log d + (p_0+p_1-p_2)(d+1) \\ &\leq \frac{d+1}{2}\log^2 d - 0.953(d-1)\log d + 0.002(d+1) \end{aligned}$$

which is less than $\frac{1}{2}d\log^2 d$ for $d_{\mathbf{L}} = d^2 \geq 2^2$, which is always the case.

Let us now prove point 2 of Theorem 2, from which (7) will follow. To prove the bound (8), we choose $f_1(s) = \zeta_{\mathbf{L}}$ and $f_2(s) = \zeta^m(s)$. We have $\Gamma_{f_1}(s) = \Gamma_{f_2}(s) = \Gamma^m(s/2)$, $d = A_{f_1}/A_{f_2} = \sqrt{d_{\mathbf{L}}}$. To prove the bound (9), we choose $f_1(s) = \zeta_{\mathbf{L}}/\zeta_{\mathbf{Q}}$ (which is entire) and $f_2(s) = \zeta^{m-1}(s)$. We have $\Gamma_{f_1}(s) = \Gamma_{f_2}(s) = \Gamma^{m-1}(s/2)$, $d = A_{f_1}/A_{f_2} = \sqrt{d_{\mathbf{L}}}$. Now, in both cases we deduce the desired result from (48) in Theorem 23 once we notice that since for a given $m \geq 1$ we have

$$F_m(s) := \left(\pi^{-s/2}\Gamma(s/2)\zeta(s)\right)^m = \frac{1}{(s-1)^m} + \frac{c_m}{(s-1)^{m-1}} + \cdots$$

where $1 + c_m = 1 - n\left(\log(4\pi) - \gamma\right)/2$ is less than 0 for $m \geq 2$, there exists $d_m > 0$ effective such that for $d \geq \sqrt{d_m}$ we have $J_m(d) \leq \frac{d}{m!}\log^m d$ where

$$J_m(d) := \operatorname{Res}_{s=1}\left(s \mapsto F_m(s)\left(\frac{1}{s} + \frac{1}{s-1}\right)(d^s + d^{1-s})\right).$$

As for the last assertion of point 2 of Theorem 2, using Maple for computing the expressions (47) of $J_m(d)$ and plotting the graphs of the $d \mapsto J_m(d)$ for $m \in \{2,3,4,5\}$, the reader can check that the lower bound $d = \sqrt{d_{\mathbf{L}}} \geq e^{m/5}$ (see Lemma 25) yields $J_m(d) \leq \frac{d}{m!}\log^m d$ for $m \in \{2,3,4,5\}$.

6.4 An Open Problem

It would be rather desirable to have an explicit expression (depending on m only) for such d_m's. Numerical investigations suggest that such d_m's can be chosen small enough so that the condition $d_{\mathbf{L}} \geq d_m$ will always be satisfied for any totally real number field of degree m. More precisely, according to Lemma 25, for totally real number fields of degree m we have $d_{\mathbf{L}} \geq e^{2m/5}$ and it seems that for any $m \geq 2$ we have $J_m(d) \leq \frac{d}{m!}\log^m d$ for $d \geq e^{m/5}$.

6.5 Proof of Theorem 5

Choose $f_1(s) = L(s,\chi)$ and $f_2(s) = \zeta_{\mathbf{L}}(s)$.

We have $\Gamma_{f_1} = \Gamma_{f_2} = \Gamma_{\mathbf{L}}$, $R_{f_1}(s) = 0$ and $F_{f_1}(s) = I_{f_1}(s)$, $n_{f_2} = 1$, $p_1 = \lambda_{\mathbf{L}}$ and $p_0 = \lambda_{\mathbf{L}}\mu_{\mathbf{L}} - \lambda_{\mathbf{L}}$, $d = A_{f_1}/A_{f_2} = \sqrt{f_\chi}$. According to (47) of Theorem 23, we have

$$d\lambda_{\mathbf{L}}|L(1,\chi)|/\operatorname{Res}_{s=1}(\zeta_{\mathbf{L}}) = |F_{f_1}(1)| = |I_{f_1}(1)| \leq \lambda_{\mathbf{L}}\left((d+1)\mu_{\mathbf{L}} + (d-1)\log d\right).$$

Hence,

$$d|L(1,\chi)| \le \mathrm{Res}_{s=1}(\zeta_{\mathbf{L}})\big((d-1)\log d + (d+1)\mu_{\mathbf{L}}\big),$$

and the desired result follows (for if $f_\chi = 1$ then $d = 1$ and we use (49) to obtain $|I_{f_1}(1)| \le I_{f_2}(1) = p_0 + p_1 = \lambda_{\mathbf{L}}\mu_{\mathbf{L}}$).

6.6 Proof of Theorems 7 and 9

6.6.1 The Key Proposition

***Proposition 28* (Corollary to Theorem 23)** *Let χ be a primitive even Dirichlet character of conductor $f_\chi > 1$. Set $\Gamma_\chi = \Gamma(s/2)$, $A_\chi = \sqrt{f_\chi/\pi}$ and $F_\chi(s) = A_\chi^s\Gamma_\chi(s)L(s,\chi)$. Then, for $\frac{1}{2} \le \beta \le 1$ we have*

$$|F_\chi(\beta)|/\sqrt{f_\chi} \le \frac{1}{2}\log f_\chi + \mu_{\mathbf{Q}}$$

and

$$|F'_\chi(\beta)|/\sqrt{f_\chi} \le \frac{1}{2}\Big(\frac{1}{4}\log^2 f_\chi - \mu_{\mathbf{Q}}^2\Big) \le \frac{1}{8}\log^2 f_\chi.$$

Proof Choose $f_1(s) = L(s,\chi)$ and $f_2(s) = \zeta(s)$. We have $R_{f_1} = 0$, $I_{f_1} = F_{f_1} = F_\chi$, $d = \sqrt{f_\chi}$, $n_{f_2} = 1$ and

$$F_{f_2}(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s) = \frac{p_1}{s-1} + p_0 + p_{-1}(s-1) + O\big((s-1)^2\big)$$

with $p_1 = 1$, $p_0 = \big(\gamma - \log(4\pi)\big)/2 = -0.976904\cdots$, $p_{-1} = 1.000248\cdots$ and $\mu_{\mathbf{Q}} = p_0 + p_1$. Then (47) in Theorem 23 gives

$$|I_{f_1}(\beta)| \le (p_0+p_1)(d+1) + p_1(d-1)\log d = d(\log d + \mu_{\mathbf{Q}}) + (\mu_{\mathbf{Q}} - \log d),$$

which yields the first assertion, and (50) in Theorem 23 gives

$$\begin{aligned}|I'_{f_1}(\beta)| &\le p_1\frac{d-1}{2}\log^2 d + p_0(d+1)\log d + (p_1+p_{-1})(d-1) + R_{f_2}(d)\\ &\le \frac{d-1}{2}\log^2 d + (\mu_{\mathbf{Q}}-1)(d+1)\log d + (p_1+p_{-1})(d-1) + \frac{\pi}{6d}e^{-\pi d^2},\end{aligned}$$

for $(\frac{d}{x}+1)\log(\frac{x}{d}) \le \frac{x}{d}$ for $x \ge d$ and $S_{f_2}(x) = 2\sum_{n\ge1} e^{-\pi n^2x^2}$ yield

$$R_{f_2}(d) \le \frac{1}{d}\int_d^\infty xS_{f_2}(x)\,dx = \frac{1}{\pi d}\sum_{n\ge1}\frac{1}{n^2}e^{-\pi n^2d^2} \le \frac{e^{-\pi d^2}}{\pi d}\sum_{n\ge1}\frac{1}{n^2} = \frac{\pi}{6d}e^{-\pi d^2}.$$

The desired second assertion follows. ■

6.6.2 Proof of Theorem 7

We have $F_\chi(1) = \sqrt{f_\chi}L(1,\chi)$ so that the first point of Proposition 28 yields (15). If $L(\beta,\chi) = 0$ then $F_\chi(\beta) = 0$, and using Proposition 28 we obtain

$$\begin{aligned}|L(1,\chi)| &= |F_\chi(1)|/\sqrt{f_\chi} = |F_\chi(1) - F_\chi(\beta)|/\sqrt{f_\chi} \\ &\leq (1-\beta) \sup_{t\in]\beta,1[} |F'_\chi(t)|/\sqrt{f_\chi} \leq \frac{1-\beta}{8}\log^2 f_\chi.\end{aligned}$$

6.6.3 Proof of Theorem 9

Let $X_{\mathbf{L}}$ be the group of primitive even Dirichlet characters associated with $\mathbf{N}$. Then $d_{\mathbf{L}} = \prod_{\chi\in X_{\mathbf{L}}\setminus\{1\}} f_\chi$, $F_{\mathbf{L}}(s) = F_{\mathbf{Q}}(s)\prod_{\chi\in X_{\mathbf{L}}\setminus\{1\}} F_\chi(s)$, $\mu_{\mathbf{L}} = \mu_{\mathbf{Q}} + \sum_{\chi\in X_{\mathbf{L}}\setminus\{1\}}(F'\chi/F_\chi)(1)$ and $\sqrt{d_{\mathbf{L}}}\operatorname{Res}_{s=1}(\zeta_{\mathbf{L}}) = \prod_{\chi\in X_{\mathbf{L}}\setminus\{1\}} F_\chi(1)$. Hence,

$$\begin{aligned}B_{\mathbf{L}} &= \mu_{\mathbf{L}} \prod_{\psi\in X_{\mathbf{L}}\setminus\{1\}} \frac{F_\psi(1)}{\sqrt{f_\psi}} \\ &= \mu_{\mathbf{Q}} \prod_{\psi\in X_{\mathbf{L}}\setminus\{1\}} \frac{F_\psi(1)}{\sqrt{f_\psi}} + \sum_{\chi\in X_{\mathbf{L}}\setminus\{1\}} \frac{F'_\chi(1)}{\sqrt{f_\chi}}\Big(\prod_{\psi\in X_{\mathbf{L}}\setminus\{1,\chi\}} \frac{F_\psi(1)}{\sqrt{f_\psi}}\Big).\end{aligned}$$

Using both points of Proposition 28 and noticing that the geometric mean is less than or equal to the arithmetic mean, we obtain

$$\begin{aligned}&\frac{F'_\chi(1)}{\sqrt{f_\chi}} \prod_{\psi\in X_{\mathbf{L}}\setminus\{1,\chi\}} \frac{F_\psi(1)}{\sqrt{f_\psi}} \\ &\quad\leq \frac{1}{2}\left(\frac{1}{2}\log f_\chi - \mu_{\mathbf{Q}}\right)\left(\frac{1}{2}\log f_\chi + \mu_{\mathbf{Q}}\right) \prod_{\psi\in X_{\mathbf{L}}\setminus\{1,\chi\}} \left(\frac{1}{2}\log f_\psi + \mu_{\mathbf{Q}}\right) \\ &\quad\leq \frac{1}{2}\left(\frac{1}{2}\log f_\chi - \mu_{\mathbf{Q}}\right)\left(\frac{1}{2(m-1)}\log d_{\mathbf{L}} + \mu_{\mathbf{Q}}\right)^{m-1}\end{aligned}$$

and

$$\begin{aligned}B_{\mathbf{L}} &\leq \left(\mu_{\mathbf{Q}} + \frac{1}{2}\sum_{\chi\in X_{\mathbf{L}}\setminus\{1\}} \left(\frac{1}{2}\log f_\chi - \mu_{\mathbf{Q}}\right)\right)\left(\frac{1}{2(m-1)}\log d_{\mathbf{L}} + \mu_{\mathbf{Q}}\right)^{m-1} \\ &= \frac{1}{4}\big(\log d_{\mathbf{L}} + 2(3-m)\mu_{\mathbf{Q}}\big)\left(\frac{1}{2(m-1)}\log d_{\mathbf{L}} + \mu_{\mathbf{Q}}\right)^{m-1} \\ &\leq \frac{\log d_{\mathbf{L}}}{4}\left(\frac{1}{2(m-1)}\log d_{\mathbf{L}} + \mu_{\mathbf{Q}}\right)^{m-1} \text{ for } m \geq 3.\end{aligned}$$

Finally, if $m = 2$ then 9 provides us with a better bound than the one we want to obtain.

References

[Arn] S. Arno, *The imaginary quadratic fields of class number* 4. Acta Arith. **60**(1992), 321–334.

[BarL] P. Barrucand and S. Louboutin, *Majoration et minoration du nombre de classes d'idéaux des corps réels purs de degré premier.* Bull. London Math. Soc. **25**(1993), 533–540.

[Bou] G. Boutteaux, *Corps a multiplication complexe non galoisiens de degré* 6 *principaux.* Ph.D. Thesis, Univ. Caen, in preparation.

[BouL] G. Boutteaux and S. Louboutin, *The class number one problem for some non-normal sextic* CM*-fields.* To appear.

[CK] K.-Y. Chang and S.-H. Kwon, *Class numbers of imaginary abelian number fields.* Proc. Amer. Math. Soc. **128**(2000), 2517–2528.

[Dav] H. Davenport, *Multiplicative Number Theory.* Second edition, Springer-Verlag, 1980.

[GS] A. Granville and H. M. Stark, *ABC implies no "Siegel zeros" for L-functions of characters with negative discriminant.* Invent. Math. **139**(2000), 509–523.

[HH] K. Horie and M. Horie, CM*-fields and exponents of their ideal class groups.* Acta Arith. **55**(1990), 157–170.

[Hof] J. Hoffstein, *Some analytic bounds for zeta functions and class numbers.* Invent. Math. **55**(1979), 37–47.

[Hor] K. Horie, *On a ratio between relative class numbers.* Math. Z. **211**(1992), 505–521.

[Lan] S. Lang, *Algebraic Number Theory.* Graduate Texts in Math. **110**, second edition, Springer-Verlag, 1994.

[Lef] Y. Lefeuvre, *Corps diédraux à multiplication complexe principaux.* Ann. Inst. Fourier **50**(2000), 67–103.

[Lem] F. Lemmermeyer, *Ideal class groups of cyclotomic number fields, I.* Acta Arith. **72**(1995), 347–359.

[LL] Y. Lefeuvre and S. Louboutin, *The class number one problem for the dihedral* CM*-fields.* In: Proceedings of the ICM 1998 Satellite Conference, Algebraic Number Theory and Diophantine Analysis (Gras, August 31-September 4, 1998).

[LLO] F. Lemmermeyer, S. Louboutin and R. Okazaki, *The class number one problem for some non-abelian normal* CM*-fields of degree* 24. J. Théor. Nombres Bordeaux **11**(1999), 387–406.

[LO1] S. Louboutin and R. Okazaki, *Determination of all non-normal quartic* CM*-fields and of all non-abelian normal octic* CM*-fields with class number one.* Acta Arith. **67**(1994), 47–62.

[LO2] ———, *The class number one problem for some non-abelian normal* CM*-fields of* 2*-power degrees.* Proc. London Math. Soc. (3) **76**(1998), 523–548.

[LO3] ———, *Determination of all quaternion* CM*-fields with ideal class groups of exponent* 2. Osaka J. Math. **36**(1999), 229–257.

[LOO] S. Louboutin, R. Okazaki and M. Olivier, *The class number one problem for some non-abelian normal* CM*-fields.* Trans. Amer. Math. Soc. **349**(1997), 3657–3678.

[Lou1] S. Louboutin, *Majorations explicites de* $|L(1,\chi)|$. C. R. Acad. Sci. Paris **316**(1993), 11–14.

[Lou2] ———, *Lower bounds for relative class numbers of* CM*-fields.* Proc. Amer. Math. Soc. **120**(1994), 425–434.

[Lou3] ———, *Determination of all nonquadratic imaginary cyclic number fields of* 2*-power degrees with ideal class groups of exponents* ≤ 2. Math. Comp. **64**(1995), 323–340.

[Lou4] ———, *Majorations explicites de* $|L(1,\chi)|$ *(suite).* C. R. Acad. Sci. Paris **323**(1996), 443–446.

[Lou5] ———, *Determination of all quaternion octic* CM*-fields with class number* 2. J. London Math. Soc. **54**(1996), 227–238.

[Lou6] ———, *The class number one problem for the non-abelian normal* CM*-fields of degre* 16. Acta Arith. **82**(1997), 173–196.

[Lou7] ———, CM*-fields with cyclic ideal class groups of* 2*-power orders.* J. Number Theory **67**(1997), 1–10.

[Lou8] ———, *Majorations explicites du résidu au point* 1 *des fonctions zêta des corps de nombres.* J. Math. Soc. Japan **50**(1998), 57–69.

[Lou9] ———, *Upper bounds on* $|L(1,\chi)|$ *and applications.* Canad. J. Math. **50**(1999), 794–815.

[Lou10] ———, *The class number one problem for the dihedral and dicyclic* CM*-fields.* Colloq. Math. **80**(1999), 259–265.

[Lou11] ———, *Explicit bounds for residues of Dedekind zeta functions, values of L-functions at* $s = 1$ *and relative class numbers.* J. Number Theory **85**(2000), 263–282.

[Lou12] ———, *Majorations explicites de* $|L(1,\chi)|$ *(troisième partie).* C. R. Acad. Sci. Paris **332**(2001), 95–98.

[Lou13] ———, *Computation of* $L(0,\chi)$ *and of relative class numbers of* CM*-fields.* Nagoya Math. J. **161**(2001), 171–191.

[LP] S. Louboutin and Y.-H. Park, *Class number problems for dicyclic* CM*-fields.* Publ. Math. Debrecen **57**(2000), 283–295.

[LPL] S. Louboutin, Y.-H. Park and Y. Lefeuvre, *Construction of the real dihedral number fields of degree* $2p$*. Applications.* Acta Arith. **89**(1999), 201–215.

[LPP] H. W. Lenstra, J. Pila and C. Pomerance, *A hyperelliptic smoothness test, II.* To appear.

[LYK] S. Louboutin, Y.-S. Yang and S.-H. Kwon, *The non-normal quartic* CM*-fields and the dihedral octic* CM*-fields with ideal class groups of exponent* ≤ 2. Preprint, Univ. Caen, November 1999.

[Mel] H. Mellin, *Abriss einer einheitlichen der Gamma und hypergeometrischen Funktionen.* Math. Ann. **68**(1910), 305–337.

[MM] M. R. Murty and V. K. Murty. *Non-vanishing of L-functions and applications.* Progr. Math. **157**, Birkhäuser Verlag, 1997.

[Mur] V. K. Murty, *Stark zeros in certain towers of fields.* Math. Research Letters **6**(1999), 511–519.

[Odl] A. Odlyzko, *Some analytic estimates of class numbers and discriminants.* Invent. Math. **29**(1975), 275–286.

[Oka] R. Okazaki, *Inclusion of* CM*-fields and divisibility of relative class numbers.* Acta Arith. **92**(2000), 319–338.

[Rad] H. Rademacher, *Topics in Analytic Number Theory.* Springer-Verlag, 1973.

[Ram] O. Ramaré, *Approximate formulae for* $L(1,\chi)$. Acta Arith., to appear.

[Ser] J. P. Serre, *Minorations de discriminants.* In: Collected Papers, Vol. III (1972–1984), Springer-Verlag.

[Sta] H. M. Stark, *Some effective cases of the Brauer-Siegel Theorem.* Invent. Math. **23**(1974), 135–152.

[Was] L. C. Washington, *Introduction to Cyclotomic Fields.* Graduate Texts in Math. **83**, second edition, Springer-Verlag, 1997.

[Yam] K. Yamamura, *The determination of the imaginary abelian number fields with class number one.* Math. Comp. **62**(1994), 899–921.

[YK] H.-S. Yang and S.-H. Kwon, *The non-normal quartic* CM*-fields and the octic dihedral* CM*-fields with relative class number two.* J. Number Theory **79**(1999), 175–193.

Institut de Mathématiques de Luminy
UPR 906
163, avenue de Luminy
Case 907
13288 Marseille Cedex 9
France
email: loubouti@iml.univ-mrs.fr