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Periodic solutions for indefinite singular perturbations of the relativistic acceleration

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Using the Leray–Schauder degree, we study the existence of solutions for the following periodic differential equation with relativistic acceleration and singular nonlinearity:

$$\left(\frac{u'}{\sqrt{1-u'^2}}\right)' = \frac{h(t)}{u^\mu}, \qquad u(0)-u(T)=0=u'(0)-u'(T),$$

where $\mu > 1$ and the weight $h\colon [0,T] \to \mathbb{R}$ is a continuous sign-changing function. There are no *a priori* estimates on the set of positive solutions (a condition used in general to apply the Leray–Schauder degree), and we prove that no solution of the equation appears on the boundary of an unbounded open set during the deformation to an autonomous problem.

Keywords: singular differential equations; indefinite singularity; periodic solutions; degree theory; continuation theorem

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1. Introduction

We study the existence of T-periodic solutions for the following nonlinear differential equation with relativistic acceleration and singular nonlinearity:

$$\left(\frac{u'}{\sqrt{1-u'^2}}\right)' = \frac{h(t)}{u^\mu}, \qquad u(0)-u(T)=0=u'(0)-u'(T),$$

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where $\mu > 1$ and the weight $h\colon [0,T] \to \mathbb{R}$ is a continuous function. A *solution* of the above problem is a strictly positive function $u \in C^2([0,T])$ such that $\max_{[0,T]} |u'| < 1$ and u verifies the above periodic boundary-value problem.

The main result of this paper is the following theorem.

THEOREM 1.1. *Assume that there exists $a \in (0,T)$ such that $h(a) = 0$, h is strictly positive on $[0,a)$ and non-positive on $(a,T]$. If*

$$\int_0^T h(t)\,\mathrm{d}t < 0 \quad \textit{and} \quad \lim_{t\to a^-} \frac{h(t)}{(a-t)^{\mu-1}} = \infty,$$

the above problem has at least one solution.

First of all, we note that the weight function h must necessarily be sign indefinite, and $\int_0^T h(t)\,\mathrm{d}t < 0$.

The above periodic boundary-value problem, despite looking simple, is a difficult one. Because the nonlinearity has an indefinite weight and a singularity, there are no *a priori* estimates on the set of positive solutions, a condition used in general to apply one of the main tools of nonlinear functional analysis: the Leray–Schauder degree. We show only that no solution of the equation appears on the boundary of an unbounded open set during the deformation.

To overcome this problem, we introduce a new strategy, together with a homotopy with an autonomous equation. We prove a continuation theorem (theorem 2.3) for problems of the type

$$\left(\frac{u'}{\sqrt{1-u'^2}}\right)' = h(t)g(u), \qquad u(0) - u(T) = 0 = u'(0) - u'(T),$$

where the weight h is such that

$$\bar{h} = T^{-1}\int_0^T h(t)\,\mathrm{d}t \neq 0,$$

and $g\colon \mathbb{R} \to \mathbb{R}$ is continuous. The main idea is to consider for $\lambda \in [0,1]$ the homotopy

$$\left(\frac{u'}{\sqrt{1-u'^2}}\right)' = (\lambda h(t) - (1-\lambda)\bar{h})g(u) + (1-\lambda)\bar{h},$$
$$u(0) - u(T) = 0 = u'(0) - u'(T).$$

If there exists $0 < \varepsilon < R$ such that $g(\varepsilon) > 1 > g(R)$, and $J \subset [0,T]$ closed such that the above problem has no solution on ∂V for all $\lambda \in [0,1]$, where

$$V = \{u\colon [0,T] \to \mathbb{R}\colon u \text{ continuous with } u(0) = u(T),\ \varepsilon < u < R \text{ on } J\},$$

then the above problem for $\lambda = 1$ has at least one solution in V. The main tool in the proof is a continuation theorem from [14] (see also [2,11]) in which a homotopy is made to an arbitrary autonomous equation and one takes advantage of the S^1-invariance of the corresponding periodic problem to compute the associated Leray–Schauder degree.

There is a large literature concerning nonlinear super- and sublinear problems with a weight function having an indefinite sign (see, for example, [1,6,7,12]).

However, fewer results concerning problems with singularity and indefinite weight seem to be available. Our paper is motivated by the papers [8, 15]. In both these papers, pure ordinary differential equation strategies are used in the proofs. In the first one the weight satisfies strong symmetry conditions, and in the second the weight has only non-degenerate zeros or is piecewise constant. The existence of periodic solutions for singular nonlinearities with indefinite weight and Newtonian acceleration is considered for the first time in [9].

Our paper is structured as follows. In § 2 we prove our continuation theorem. In § 3 we apply the continuation theorem to a modified problem, and in § 4 we prove that the solution of a particular modified problem is also solution of the main periodic problem. For results concerning periodic solutions of nonlinear perturbation of the relativistic acceleration, see, for example, [4, 5, 10].

2. Homotopy and degree

To construct the fixed-point operator we need some notation. Let C denote the Banach space of continuous functions on $[0, T]$ endowed with the uniform norm $\|\cdot\|_\infty$. We consider the closed subspace

$$C_T = \{u \in C \colon u(0) = u(T)\}.$$

The open ball of centre 0 and radius r is denoted by B_r. We denote by $P, Q \colon C \to C$ the continuous projectors

$$Pu(t) = u(0), \qquad Qu(t) = T^{-1}\int_0^T u(s)\,\mathrm{d}s = \bar{u} \quad (t \in [0, T]).$$

On the other hand, let $H \colon C \to C$ be the continuous linear operator given by

$$Hu(t) = \int_0^t u(s)\,\mathrm{d}s \quad (t \in [0, T]).$$

Throughout the paper we use the following notation:

$$\phi(s) = \frac{s}{\sqrt{1 - s^2}} \quad (s \in (-1, 1)).$$

Let $Q_\phi \colon C \to \mathbb{R}$ be the continuous function determined by the relation

$$Q \circ \phi^{-1} \circ (I - Q_\phi) \circ u = 0 \quad \text{for all } u \in C.$$

We need the following fixed-point lemma (see [3, proposition 2]).

Lemma 2.1. *Assume that $F \colon C \to C$ is continuous and takes bounded sets into bounded sets. Then, u is a solution of the abstract periodic problem*

$$(\phi(u'))' = F(u), \qquad u(0) - u(T) = 0 = u'(0) - u'(T),$$

if and only if $u \in C_T$ is a fixed point of the completely continuous operator $M \colon C_T \to C_T$ given by

$$M = P + QF + H \circ \phi^{-1} \circ (I - Q_\phi) \circ [H(I - Q)F].$$

Using exactly the same proof as in [14, theorem 4.1], one has the following continuation result.

LEMMA 2.2. *Consider the periodic problems*

$$(\phi(u'))' = f(t, u, \lambda), \qquad u(0) - u(T) = 0 = u'(0) - u'(T),$$

where $f\colon [0,T] \times \mathbb{R} \times [0,1]$ *is continuous and*

$$f(t, u, 0) = f_0(u)$$

is independent of t. *Let* Ω *be an open bounded subset in* C_T *such that for each* $\lambda \in [0,1]$ *the above problem has no solution on* $\partial\Omega$. *Then,*

$$d_{\mathrm{LS}}[I - M_1, \Omega, 0] = -d_{\mathrm{B}}[f_0, \Omega \cap \mathbb{R}, 0],$$

where M_1 *is the fixed-point operator of the above problem for* $\lambda = 1$.

Next, for each $\lambda \in [0,1]$, we consider the periodic problem

$$\left.\begin{aligned} (\phi(u'))' = (\lambda h(t) - (1-\lambda)\bar{h})g(u) + (1-\lambda)\bar{h}, \\ u(0) - u(T) = 0 = u'(0) - u'(T), \end{aligned}\right\} \tag{2.1}$$

where $h \in C$ and $g\colon \mathbb{R} \to \mathbb{R}$ is continuous. The main result of this section is the following theorem.

THEOREM 2.3. *Assume that* $\bar{h} \neq 0$ *and there exists* $0 < \varepsilon < R$ *such that* $g(\varepsilon) > 1 > g(R)$. *Let* $J \subset [0,T]$ *be closed and*

$$V = \{u \in C_T \colon \varepsilon < u < R \text{ on } J\}.$$

If, for each $\lambda \in [0,1]$, *(2.1) has no solution on* ∂V, *then, for* $\lambda = 1$, *(2.1) has at least one solution in* V.

Proof. Let $u \in \bar{V}$ be a solution of (2.1). Then, by the fundamental theorem of calculus, we have

$$|u(t)| \leqslant R + \int_0^T |u'(s)|\,\mathrm{d}s < R + T$$

for each $t \in [0,T]$. Taking the open bounded set $\Omega = V \cap B_{R+T}$, it follows that (2.1) has no solution on $\partial\Omega$, Hence, the homotopy invariance of Leray–Schauder degree, together with the above lemma, implies that

$$d_{\mathrm{LS}}[I - M_1, \Omega, 0] = d_{\mathrm{LS}}[I - M_0, \Omega, 0] = -d_{\mathrm{B}}[\bar{h}(1-g), (\varepsilon, R), 0],$$

where M_λ is the fixed-point operator corresponding to (2.1). It follows that $d_{\mathrm{LS}}[I - M_1, \Omega, 0] \neq 0$, and the existence property of the Leray–Schauder degree concludes the proof. □

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3. A modified problem

In this section we assume that

$$\bar{h} < 0.$$

For each $0 < \delta < 1$ we define a truncation function $g_\delta \colon \mathbb{R} \to \mathbb{R}$ such that g_δ is smooth on $\mathbb{R}$ and is non-increasing and bounded by $1 + \delta^{-\mu}$, and $g_\delta(u) = u^{-\mu}$ for any $u \geqslant \delta$. Consider the following family of periodic problems:

$$\left.\begin{aligned} (\phi(u'))' = (\lambda h(t) - (1-\lambda)\bar{h}) g_\delta(u) + (1-\lambda)\bar{h}, \\ u(0) - u(T) = 0 = u'(0) - u'(T). \end{aligned}\right\} \tag{3.1}$$

Lemma 3.1. *There exists $R > 1 + T$ such that, for any solution u of (3.1), one has*

$$\max_{[0,T]} u < R.$$

Proof. Let

$$H_+ = \int_0^a h(t)\,\mathrm{d}t, \qquad H_- = -\int_a^T h(t)\,\mathrm{d}t.$$

We take $R > 1 + T$ sufficiently large such that

$$H_+ - \left(\frac{x - T}{x}\right)^\mu H_- + T\bar{h}(1 - (x - T)^\mu) > 0, \qquad H_+ - \left(\frac{x - T}{x}\right)^\mu H_- < 0,$$

for all $x \geqslant R$. Assume that u is a solution of (3.1) such that $M_u = \max_{[0,T]} u \geqslant R$. By integrating both sides of (3.1) over $[0, T]$ and taking into account that $\max_{[0,T]} u - \min_{[0,T]} u < T$, we obtain that

$$0 \leqslant \lambda H_+ - \lambda\left(\frac{M_u - T}{M_u}\right)^\mu H_- - (1-\lambda)T\bar{h} + (1-\lambda)\bar{h}T(M_u - T)^\mu.$$

It follows that

$$0 \leqslant H_+ - \left(\frac{M_u - T}{M_u}\right)^\mu H_-,$$

which is a contradiction of the choice of R. □

Lemma 3.2. *Assume that there exists $R > 0$ such that $u \in B_R$ for each solution u of (3.1). There exists $K > 0$ such that, for any solution u of (3.1), one has*

$$\left\| \frac{\phi(u')}{g_\delta(u)} \right\|_\infty \leqslant K.$$

Proof. Let $u \in B_R$ be a solution of (3.1). Let $t_m \in [0, T]$ be the point where u attains the maximum value on $[0, T]$. From the periodic boundary conditions one has that $u'(t_m) = 0$. Multiplying both sides of (3.1) by $g_\delta(u)^{-1}$ and integrating by parts, we deduce that

$$\frac{\phi(u')}{g_\delta(u)} = \int_{t_m}^t \phi(u') \left[\frac{1}{g_\delta(u(s))}\right]' \mathrm{d}s + \int_{t_m}^t (\lambda h(s) - (1-\lambda)\bar{h})\,\mathrm{d}s + (1-\lambda)\bar{h} \int_{t_m}^t \frac{\mathrm{d}s}{g_\delta(u)}$$

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for all $t \in [0, T]$. Taking into account that g_δ is a non-increasing function, one can easily to check that

$$\phi(u'(s)) \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{g_\delta(u(t))}\right]_{t=s} \geqslant 0$$

for all $s \in [0, T]$. It follows that

$$\frac{\phi(u')}{g_\delta(u)} \leqslant K \quad \text{on } [0, t_m], \qquad \frac{\phi(u')}{g_\delta(u)} \geqslant -K \quad \text{on } [t_m, T],$$

where $K := \|h\|_1 + T|\bar{h}|(1 + R^\mu)$. The conclusion now follows from the periodic boundary conditions. $\square$

Lemma 3.3. *There exists* $0 < \varepsilon < 1$ *such that if* $0 < \delta \leqslant \varepsilon$ *and* u *is a solution of (3.1) then* $\min_{[0,a/2]} u \neq \varepsilon$.

Proof. Let $M_0 > 0$ be such that

$$\int_{a/2}^{3a/4} \phi^{-1}\left(\frac{(h_0 + \bar{h}M^\mu)(s - \frac{1}{2}a)}{M^\mu}\right) \mathrm{d}s > M,$$
$$\int_{0}^{a/2} \phi^{-1}\left(\frac{(h_0 + \bar{h}M^\mu)(\frac{1}{2}a - s)}{M^\mu}\right) \mathrm{d}s > M,$$

for all $M < M_0$, where $h_0 := \min\{\min_{[0,3a/4]} h, |\bar{h}|\} > 0$. Moreover, we can assume that

$$\frac{h_0}{M_0^\mu} + \bar{h} > 0.$$

We take $R > T+1$ as in lemma 3.1 and $K > 0$ verifying lemma 3.2, and we consider $0 < \varepsilon < \min\{M_0, 1\}$ such that

$$K < h_0 M_0^\mu \int_\varepsilon^{M_0} \frac{\mathrm{d}s}{s^\mu} - |\bar{h}|(T+1)^{\mu+1}.$$

Let $\delta \in (0, \varepsilon]$ be fixed. Assume that there exists a solution u of (3.1) such that $\min_{[0,a/2]} u = \varepsilon$. Let $t_* \in [0, \frac{1}{2}a]$ be such that $u(t_*) = \varepsilon$. Using lemma 3.1, one has that $\max_{[0,T]} u < R$, and then

$$\min_{[0,T]} u > \max_{[0,T]} u - T > \varepsilon - T > -R.$$

Hence,

$$\|u\|_\infty < R.$$

We distinguish two cases.

Case 1 ($t_* \in [0, \frac{1}{2}a)$). Observe that $u'(t_*) \geqslant 0$. We claim that

$$\max_{[t_*, 3a/4]} u \geqslant M_0.$$

Indeed, if we assume that $u < M_0$ on $[t_*, \frac{3}{4}a]$, then one has that $(\phi(u'))' > 0$ on $[t_*, \frac{3}{4}a]$. Since $u'(t_*) \geqslant 0$, we have $u'(t) \geqslant 0$ and $\varepsilon \leqslant u(t)$ for $t \in [t_*, \frac{3}{4}a]$ and

$u(\frac{3}{4}a) = \max_{[t_*,3a/4]} u$. Multiplying both sides of (3.1) by u^μ and integrating from t_* to $t \in [t_*, \frac{3}{4}a]$, we deduce that

$$\phi(u'(t)) \geqslant \frac{(h_0 + \bar{h}u(\frac{3}{4}a)^\mu)(t - t_*)}{u^\mu(t)} \quad \text{for all } t \in [t_*, \tfrac{3}{4}a].$$

Since ϕ is an increasing homeomorphism, it follows that

$$u'(t) \geqslant \phi^{-1}\biggl(\frac{(h_0 + \bar{h}u(\frac{3}{4}a)^\mu)(t - t_*)}{u(\frac{3}{4}a)^\mu}\biggr) \quad \text{for all } t \in [t_*, \tfrac{3}{4}a].$$

Integrating the latter inequality over $[t_*, \frac{3}{4}a]$, we obtain that

$$\begin{aligned} u\biggl(\frac{3a}{4}\biggr) &\geqslant \int_{t_*}^{3a/4} \phi^{-1}\biggl(\frac{(h_0 + \bar{h}u(\frac{3}{4}a)^\mu)(t - t_*)}{u(\frac{3}{4}a)^\mu}\biggr)\,\mathrm{d}t \\ &\geqslant \int_{a/2}^{3a/4} \phi^{-1}\biggl(\frac{(h_0 + \bar{h}u(\frac{3}{4}a)^\mu)(t - \frac{1}{2}a)}{u(\frac{3}{4}a)^\mu}\biggr)\,\mathrm{d}t, \end{aligned}$$

a contradiction of the choice of M_0.

Since $\max_{[t_*,3a/4]} u \geqslant M_0$, we can define $b_* \in (t_*, \frac{3}{4}a]$ such that $u(b_*) = M_0$ and $\varepsilon \leqslant u \leqslant M_0$ on $[t_*, b_*]$. Then $(\phi(u'))' > 0$ and $u' \geqslant 0$ (since $u'(t_*) \geqslant 0$) on $[t_*, b_*]$. Moreover,

$$(\phi(u'))'u' \geqslant \frac{h_0 u'}{u^\mu} + (1 - \lambda)\bar{h}u' \quad \text{on } [t_*, b_*].$$

Integrating from t_* to $t \in [t_*, b_*]$, we obtain

$$\phi(u'(t)) \geqslant h_0 \int_\varepsilon^{u(t)} \frac{\mathrm{d}s}{s^\mu} + \bar{h}(T + 1) \quad \text{for all } t \in [t_*, b_*].$$

Since $\|u\|_\infty < R$ and $u^\mu = g_\delta(u)^{-1}$ on $[t_*, b_*]$, multiplying both sides of the above inequality by u^μ and applying lemma 3.2 yields

$$K \geqslant h_0 u^\mu(t) \int_\varepsilon^{u(t)} \frac{\mathrm{d}s}{s^\mu} - |\bar{h}|(T + 1)^{\mu+1} \quad \text{for all } t \in [t_*, b_*],$$

which contradicts the choice of K taking $t = b_*$.

Case 2 ($t_* = \frac{1}{2}a$). Observe that $u'(t_*) \leqslant 0$. We claim that

$$\max_{[0,t_*]} u \geqslant M_0.$$

Assume that $u < M_0$ on $[0, t_*]$. This implies that $(\phi(u'))' > 0$ on $[0, t_*]$. Since $u'(t_*) \leqslant 0$, we have $u'(t) \leqslant 0$ and $\varepsilon \leqslant u(t)$ for all $t \in [0, t_*]$. Arguing as in case 1, multiplying both sides of (3.1) by u^μ, integrating from $t \in [0, t_*]$ to t_* and using that $u(0) = \max_{[0,t_*]} u$, we obtain

$$-u'(t) \geqslant \phi^{-1}\biggl(\frac{(h_0 + u(0)^\mu \bar{h})(t_* - t)}{u(0)^\mu}\biggr) \quad \text{for all } t \in [0, t_*].$$

Integrating the latter inequality over $[0, t_*]$, we obtain a contradiction of the choice of M_0.

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Finally, since $\max_{[0,t_*]} u \geqslant M_0$, we can define $a_* \in [0, t_*)$ such that $u(a_*) = M_0$ and $\varepsilon \leqslant u(t) \leqslant M_0$ for all $t \in [a_*, t_*]$. Then $(\phi(u'))' > 0$ and $u'(t) \leqslant 0$ (since $u'(t_*) \leqslant 0$) on $[a_*, t_*]$. Moreover,

$$(\phi(u'))'u' \leqslant \frac{h_0 u'}{u^\mu} + (1-\lambda)\bar{h}u' \quad \text{on } [a_*, t_*].$$

By integrating from $t \in [a_*, t_*]$ to t_* we have

$$\phi(u'(t)) \leqslant -h_0 \int_\varepsilon^{u(t)} \frac{\mathrm{d}s}{s^\mu} + |\bar{h}|(T+1) \quad \text{for all } t \in [a_*, t_*].$$

Since $\|u\|_\infty < R$ and $u^\mu = g_\delta(u)^{-1}$ on $[a_*, t_*]$, multiplying both sides of the above inequality by u^μ yields

$$-K \leqslant -h_0 u^\mu(t) \int_\varepsilon^{u(t)} \frac{\mathrm{d}s}{s^\mu} + |\bar{h}|(T+1)^{1+\mu} \quad \text{for all } t \in [a_*, t_*],$$

which contradicts the choice of K taking $t = a_*$. □

The main result of this section is the following existence result concerning the modified problem.

Proposition 3.4. *There exists* $0 < \varepsilon < 1$ *such that if* $0 < \delta \leqslant \varepsilon$, *then the problem*

$$(\phi(u'))' = h(t) g_\delta(u), \qquad u(0) - u(T) = 0 = u'(0) - u'(T),$$

has at least one solution u *with* $\min_{[0,a/2]} u > \varepsilon$.

Proof. This follows immediately from lemmas 3.1 and 3.3 and theorem 2.3. □

4. Proof of the main result

We take $R > T+1$ as in lemma 3.1, $K > 0$ verifying lemma 3.2 and $\varepsilon > 0$ given by proposition 3.4. Using the assumptions upon h, there exists $a_* \in (\frac{1}{2}a, a)$ such that

$$h_0 \varepsilon^\mu \int_{2\delta}^{\varepsilon} \frac{\mathrm{d}s}{s^\mu} > K,$$

where $h_0 = \min_{[0,a_*]} h$, and $\delta = a - a_*$ is such that $2\delta < \varepsilon$. Consider u given by proposition 3.4 with δ defined above. We will show that $u \geqslant \delta$ on $[0, T]$. This means that u is a solution to our main periodic problem.

Assume by contradiction that $\min_{[0,T]} u < \delta$. Using that $\min_{[0,a/2]} u > \varepsilon$ and the periodic boundary conditions, it follows that there exists $t_* \in (\frac{1}{2}a, T)$ such that $\min_{[0,T]} u = u(t_*)$.

We have three cases.

Case 1 ($t_* \in (\frac{1}{2}a, a_*]$). Since $u(\frac{1}{2}a) > \varepsilon$, there exists $\tilde{t} \in (\frac{1}{2}a, t_*)$ such that $u(\tilde{t}) = \varepsilon$. Moreover, using the modified problem, one has that $(\phi(u'))' \geqslant 0$ on $[\frac{1}{2}a, t_*]$, which implies that u' is non-increasing on $[\frac{1}{2}a, t_*]$. Then, since $u'(t_*) = 0$, it follows that $u' \leqslant 0$ on $[\frac{1}{2}a, t_*]$ and

$$(\phi(u'))'u' \leqslant h_0 g_\delta(u) u' \quad \text{on } [\tfrac{1}{2}a, t_*].$$

Integrating, we deduce that

$$\phi(u'(t)) \leqslant -h_0 \int_{u(t_*)}^{u(t)} g_\delta(s)\,\mathrm{d}s \quad \text{for all } t \in [\tfrac{1}{2}a, t_*].$$

Multiplying by $g_\delta(u)^{-1}$ in the latter inequality and applying lemma 3.2 (note that $\|u\|_\infty < R$) we deduce that

$$-K \leqslant \frac{-h_0}{g_\delta(u(t))} \int_{\delta}^{u(t)} g_\delta(s)\,\mathrm{d}s \quad \text{for all } t \in [\tfrac{1}{2}a, t_*],$$

which contradicts the choice of δ taking $t = \tilde{t}$.

Case 2 ($t_* \in (a_*, a]$). In this case we use a strategy inspired by [13]. Our first task will consist in observing that

$$u(a_*) \geqslant 2\delta.$$

We use a contradiction argument and assume that $u(a_*) < 2\delta$. Since $(\phi(u'))' \geqslant 0$ on $[0, a]$ and $u'(t_*) = 0$, we have $u'(t) \leqslant 0$ for all $t \in [0, t_*]$. Using the modified problem, multiplying by u' and integrating from $t \in [0, a_*]$ to a_*, we obtain

$$\phi(u'(t)) \leqslant -h_0 \int_{u(a_*)}^{u(t)} g_\delta(s)\,\mathrm{d}s \quad \text{for all } t \in [0, a_*].$$

On the other hand, since $u(\frac{1}{2}a) > \varepsilon$, there exists $\tilde{t} \in (\frac{1}{2}a, a_*)$ such that $u(\tilde{t}) = \varepsilon$. Multiplying by $g_\delta(u)^{-1}$ in both sides of the inequality above and applying lemma 3.2, we deduce that

$$-K \leqslant \frac{-h_0}{g_\delta(u(t))} \int_{2\delta}^{u(t)} g_\delta(s)\,\mathrm{d}s \quad \text{for all } t \in [0, a_*].$$

This is a contradiction of the choice of a_* taking $t = \tilde{t}$.

Next, let $t_0 \in [a_*, t_*)$ be such that $u(t_0) = 2\delta$. A direct integration in the modified problem shows that

$$-\phi(u'(t)) = \int_t^{t_*} h(s) g_\delta(u)\,\mathrm{d}s \leqslant (1 + \delta^{-\mu}) \int_t^{t_*} h(s)\,\mathrm{d}s$$

for all $t \in [0, t_*]$. Thus,

$$-u'(t) \leqslant \phi^{-1}\left((1 + \delta^{-\mu}) \int_{t_*}^{t} h(s)\,\mathrm{d}s \right) \quad \text{for all } t \in [t_0, t_*].$$

We integrate over $[t_0, t_*]$ obtaining that

$$2\delta - u(t_*) < t_* - t_0 \leqslant a - a_* = \delta.$$

However, this contradicts that $u(t_*) < \delta$.

Case 3 ($t_* \in (a, T]$). From the modified problem it follows that $(\phi(u'))' \leqslant 0$ on $[a, T]$, which implies that u is constant on $[a, T]$. Hence,

$$\delta > u(t_*) = u(T) = u(0) > \varepsilon,$$

which is a contradiction. The proof is complete.

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