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SPECIAL PRINCIPAL IDEAL RINGS AND ABSOLUTE SUBRETRACTS

ERIC JESPERS

ABSTRACT. A ring R is said to be an absolute subretract if for any ring S in the variety generated by R and for any ring monomorphism f from R into S, there exists a ring morphism g from S to R such that gf is the identity mapping. This concept, introduced by Gardner and Stewart, is a ring theoretic version of an injective notion in certain varieties investigated by Davey and Kovacs.

Also recall that a special principal ideal ring is a local principal ring with nonzero nilpotent maximal ideal. In this paper (finite) special principal ideal rings that are absolute subretracts are studied.

All rings in this paper are associative and commutative, but do not necessarily contain an identity. For a ring R we denote by $\mathrm{Var}(R)$ the variety generated by R (cf. [5]). Recall (cf. [4]) that a ring R with identity is called a *special principal ideal ring* if R is a local principal ideal ring with (nonzero) nilpotent maximal ideal M. Obviously $M = \beta(R)$, the prime radical of R.

In [3] several notions of injectiveness within a variety of rings are studied. Particular attention is given to *absolute subretracts.* A ring R is said to be an absolute subretract if for every ring S in $\mathrm{Var}(R)$ and for every ring monomorphism $f: R \longrightarrow S$ there exists a ring morphism $g: S \longrightarrow R$ such that gf is the identity mapping. Or equivalently, for every such morphism f there exists a two-sided ideal M of S such that $S = f(R) \oplus M$, a direct sum as $f(R)$-modules.

Gardner and Stewart in [3] characterize directly indecomposable absolute subretracts R with $R^2 = 0$. However very little is known for non-semiprime rings with $R^2 \neq 0$. Actually only one example of a ring R of this kind is included in [3], namely $R = \mathbf{Z}_2[X]/(X^2)$. Clearly R is a finite special principal ideal ring.

The aim of this paper is to give necessary and sufficient conditions for a finite special principal ideal ring R to be an absolute subretract. In general we obtain necessary conditions; but, if the characteristic of R, $\mathrm{char}(R)$, is not a power of 2, these conditions turn out to be sufficient too. This result gives us more examples of non-semiprime absolute subretracts with identity.

PROPOSITION 1. *Let R be a special principal ideal ring. If R is an absolute subretract, then $\beta(R)^3 = 0$. If, moreover, $\mathrm{char}\big(R/\beta(R)\big) \neq 2$, then $\beta(R)^2 = 0$.*

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PROOF. *Suppose $\beta(R) = Rx$, and let n be the index of nilpotency of x. That is $x^n = 0$ while $x^{n-1} \neq 0$. We consider two cases: (i) $n \geq 4$, and (ii) $n = 3$ and $\mathrm{char}(R/\beta(R)) \neq 2$. In each case we construct a ring T in $\mathrm{Var}(R)$ such that $T = f(R) \oplus f(R)\overline{Y}$, where $f: R \longrightarrow T$ is a ring monomorphism, $\overline{Y} \in T \setminus f(R)$, $f(x)\overline{Y} = 0$, and $\overline{Y}^2 = f(x)^{n-1}$.*

First assume $n \geq 4$. Let S be the following ring:

$$S = \{(a, a+j) \in R \oplus R \mid a \in R, j \in \beta(R)\},$$

and let $I = S(-x^{n-1}, -x^{n-1} + x^2)$, a principal two-sided ideal of S. Put $X = (x,x)$ and $Y = (-x^{n-2}, -x^{n-2} + x)$. Obviously $XY = (-x^{n-1}, -x^{n-1} + x^2) \in I$ and $Y^2 = (x^{2n-4}, x^{2n-4} - 2x^{n-1} + x^2)$. Since $2n - 4 \geq n$ and since $(x, x^{n-3}) \in S$ we obtain that $(x, x^{n-3})(-x^{n-1}, -x^{n-1} + x^2) = (0, -x^{2n-4} + x^{n-1}) = (0, x^{n-1}) \in I$. Consequently, $X^{n-1} - Y^2 = (x^{n-1}, x^{n-1}) - (0, -2x^{n-1} + x^2) = (x^{n-1}, x^{n-1} - x^2) + (0, 2x^{n-1}) \in I$. Hence X^{n-1} equals Y^2 modulo I. Furthermore, if $(a,a) \in I$, $a \in R$, then $(a,a) = (b, b+j)(-x^{n-1}, -x^{n-1} + x^2)$ for some $b \in R, j \in \beta(R)$. It follows that $a = b(-x^{n-1}) = (b+j)(-x^{n-1}+x^2)$. If $b \in \beta(R)$, then $a = 0$. In case $b \notin \beta(R)$ we obtain that $a \in \beta(R)^{n-1}$ and $a \in \beta(R)^2 \setminus \beta(R)^{n-1}$. This is impossible since $n \geq 4$. So we have shown that $\{(a,a) \mid a \in R\} \cap I = 0$. It also is clear that $Y \notin I + \{(a,a) \mid a \in R\}$.

Secondly, assume $n = 3$ and $\mathrm{char}(R/\beta(R)) \neq 2$. In this case, let $S = \{(a, a+j) \mid a \in R, j \in \beta(R)\}$ and $I = S(x^2, -x^2)$. Put $X = (x,x)$ and $Y = (x + x^2, -x + x^2)$. It follows that $XY = (x^2, -x^2) \in I$ and $Y^2 = (x^2, x^2) = X^2$. Furthermore, if $(a,a) \in I$, $a \in R$, then $(a,a) = (b, b+j)(x^2, -x^2)$ for some $b \in R, j \in \beta(R)$. Hence $a = bx^2 = (b+j)(-x^2)$. Hence $2bx^2 = 0$. Since 2 is a unit in R, we obtain that $bx^2 = 0$, and therefore $a = 0$. So again we have $I \cap \{(a,a) \mid a \in R\} = 0$, and clearly $Y \notin I + \{(a,a) \mid a \in R\}$.

In both cases let $T = S/I$. Clearly T belongs to $\mathrm{Var}(R)$. Further, let $f: R \longrightarrow T: r \longmapsto \bar{r} = (r,r) + I$ be the natural homomorphism, and put $\overline{X} = f(x)$ and $\overline{Y} = Y + I$. Then f is a monomorphism, $f(R) \subset T = f(R) + f(R)\overline{Y}$, $\overline{X}\,\overline{Y} = 0$ and $\overline{Y}^2 = \overline{X}^{n-1}$. We claim that $Tt \cap f(R) \neq \{0\}$, for every $t \in T \setminus f(R)$. Indeed, since $t \notin f(R)$, we have that $t = \bar{a} + \bar{b}\,\overline{Y}$, with $\bar{b}$ a unit in $f(R)$ and $\bar{a} \in f(R)$. If $\bar{a} \notin \beta\big(f(R)\big)$, then t is a unit since it is a sum of a unit and a nilpotent. This yields $f(R) \subset Tt$. However, if $\bar{a} \in \beta(R)$, then $\bar{a} = \overline{a'}\,\overline{X}$ for some $\overline{a'} \in f(R)$. Hence, $t\overline{Y} = \bar{a}\,\overline{Y} + \bar{b}\,\overline{Y}^2 = \overline{a'}(\overline{X}\,\overline{Y}) + \bar{b}\,\overline{Y}^2 = \bar{b}\,\overline{X}^{n-1} \neq 0$. Thus $Tt \cap f(R) \neq 0$. This proves the claim. Consequently $T \neq f(R) \oplus M$ for every ideal M of T; and thus R is not an absolute subretract. The result follows. ∎

PROPOSITION 2. *Let R be a finite special principal ideal ring. If $\beta(R)^2 = 0$, then R is an absolute subretract.*

PROOF. Let S be in $\mathrm{Var}(R)$ and $f: R \longrightarrow S$ a ring monomorphism. We have to prove that $S = f(R) \oplus M$ for some ideal M of S. For this we may identify R with $f(R)$. Furthermore, since $1 \in R$ it follows that

$$R \subseteq S1 \oplus \{s - s1 \mid s \in S\} = S,$$

a direct sum of ideals of S. Hence we may assume $S1 = S$, that is 1 is also the identity of S. Let M be an ideal of S maximal with respect to $M \cap R = \{0\}$. It is sufficient to prove that $S = R + M$.

Let $T = S/M$ and identify R with its natural image in T. Because of this identification $1 \in T \cap R$. We have to show that $T = R$. Clearly every nonzero ideal of T intersects R non-trivially. Hence, since R has a minimal nonzero ideal, we obtain that T has a minimal nonzero ideal, say $H(T)$.

We now show that T, and thus also $T/\beta(T)$, has only trivial idempotents. Indeed, suppose e is a non-trivial idempotent of T. Then there exists $t \in T$ with $te - t \neq 0$. Hence $H(T) \subseteq Te$ and $H(T) \subseteq T(te - t)$. However this implies that $H(T) = 0$, a contradiction.

Clearly $R/\beta(R)$ is a finite field, say of order p^k, p a prime number and $k \geq 1$. Since, moreover, $\beta(R)^2 = 0$ and $T \in \mathrm{Var}(R)$, both rings R and T satisfy the identity $(X^{p^k} - X)^2 = 0$. Consequently $T/\beta(T)$ satisfies the identity $X^{p^k} - X = 0$. In particular, for every $0 \neq x \in T/\beta(T)$ the set $\{x^i \mid i \geq 1\}$ is finite. So Theorem 1.9 in [1] implies that x^n is an idempotent for some $n \geq 1$. But since $T/\beta(T)$ has only trivial idempotents, we obtain that $x^n = 1$. Consequently, $T/\beta(T)$ is a field satisfying the equation $X^{p^k} - X = 0$ and containing the finite field $R + \beta(T)/\beta(T)$ (a copy of the field $R/\beta(R)$) satisfying the same equation. Therefore $R + \beta(T)/\beta(T) = T/\beta(T)$, yielding $T = \beta(T) + R$.

We claim that $\beta(T)^2 = 0$. Indeed, since R, and therefore also T, satisfies the equations

$$(X^{p^{mk}} - X)(Y^{p^{mk}} - Y) = 0,$$

for all $m \geq 1$ and k as above, it follows immediately that $xy = 0$ for all $x, y \in \beta(T)$.

Assume now the result is false, that is R is strictly contained in $T = \beta(T) + R$. Then there exists $t \in \beta(T) \setminus R$. Because $\beta(T)^2 = 0$ it follows that

$$Tt = \beta(T)t + Rt = Rt = U(R)t \cup \{0\},$$

where $U(R)$ is the set of all invertible elements of R. However since $1 \in T \cap R$, we obtain that $U(R)t \cap R = \{0\}$ and thus $Tt \cap R = \{0\}$, in contradiction with the construction of T. This finishes the proof. ∎

An immediate consequence of Propositions 1 and 2 is the following.

COROLLARY 3. *Let R be a finite special principal ideal ring with* $\mathrm{char}\big(R/\beta(R)\big) \neq 2$. *Then R is an absolute subretract if and only if $\beta(R)^2 = 0$.* ∎

The example of Gardner and Stewart, namely $R = \mathbf{Z}_2[X]/(X^2)$, is a contracted monoid algebra (cf. [4]). That is $R = k_0[S] = k[S]/k\theta$, where S is a monoid with identity e and zero element $\theta \neq e$, k is a field and $k[S]$ is the monoid algebra over S. In the example $S = \{e, X, \theta\}$ with $X^2 = \theta$. Note also that any group algebra $k[G]$ of a group G is a contracted monoid algebra $k_0[T]$ where T is the monoid obtained from G by adjoining a zero element.

Using a result in [2] we are able to characterize the contracted monoid algebras over a finite field, with characteristic different from 2, that are both special principal ideal rings and absolute subretracts.

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COROLLARY 4. *Let k be a finite field with* $\mathrm{char}(k) = p$ *and S a commutative monoid with identity e and zero element $\theta \neq e$. Then, the contracted monoid algebra $k_0[S]$ is both a special principal ideal ring with $\beta(k_0[S])^2 = 0$ and an absolute subretract if and only if one of the following conditions is satisfied*

1. *$S = \{e, s, \theta\}$, $s \neq \theta$ and $s^2 = \theta$, in particular $k_0[S] \cong k[X]/(X^2)$.*
2. *$p = 2$, $S = \{e, s, \theta\}$, $s \neq e$ and $s^2 = e$, in particular $k_0[S] \cong k[\mathbf{Z}_2]$ a group ring of the cyclic group of order* 2.

PROOF. It is shown in [2] that $k_0[S]$ is a special principal ideal ring if and only if one of the following conditions is satisfied: (i) S is a cyclic monoid of finite order m generated by one element, say s, with $s \neq \theta$ and $s^n = \theta$, or (ii) $S = G \cup \{\theta\}$, where G is a cyclic group of order $p = \mathrm{char}(k)$. Now one easily verifies that then $\beta(k_0[S])$ is of nilpotency index 2 if and only if $n = 2$ in case (i), and $p = 2$ in case (ii). The result now follows from Proposition 2. ∎

COROLLARY 5. *Let k and S be as in Corollary 4. Assume, moreover that $p \neq 2$. Then, $k_0[S]$ is both a special principal ideal ring and an absolute subretract if and only if $k_0[S] \cong k[X]/(X^2)$.*

PROOF. This follows at once from Proposition 1 and Corollary 4. ∎

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Memorial University of Newfoundland
St. John's, Newfoundland A1C 5S7

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