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# TWISTED MAAß–KOECHER SERIES AND SPINOR ZETA FUNCTIONS

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**Abstract.** It is shown that a Siegel-Hecke eigenform of integral weight $k$ and genus 2 is uniquely determined by its Fourier coefficients indexed by $nT$ where $T$ runs over all half-integral positive definite primitive matrices of size 2 and $n$ over all squarefree positive integers. The proof uses analytic arguments involving Koecher-Maaß series and spinor zeta functions.

## §1. Introduction

A classical result of Hecke can be stated as follows: Let $f$ and $g$ be elliptic cuspidal Hecke eigenforms of the same weight with Fourier coefficients $a(n)$ resp. $b(n)$ ($n \in \mathbb{N}$). Assume that $a(1) = b(1)$ and that $a(p) = b(p)$ for every prime $p$. Then $f = g$.

Our aim in this paper is to give an analogous result in the case of modular forms of degree two. Let $S_k(\Gamma_2)$ be the complex vector space of Siegel cusp forms of weight $k \in \mathbb{Z}$ for the Siegel modular group $\Gamma_2 := \mathrm{Sp}(2;\mathbb{Z})$ of degree two. For $F \in S_k(\Gamma_2)$ we let $a(T)$ ($T$ half-integral, positive definite) be the Fourier coefficients of $F$. We will show

THEOREM. *Let $F, G \in S_k(\Gamma_2)$ be Hecke eigenforms with Fourier coefficients $a(T)$ resp. $b(T)$. If $a(nT) = b(nT)$ for all half-integral, positive definite, primitive matrices $T$ and all square-free integers $n \in \mathbb{N}$, then $F = G$.*

To prove the theorem we proceed as follows.

According to [An] there is an identity relating the Maaß–Koecher series of a Hecke eigenform $F$ with the spinor zeta function of $F$. Twisting this identity by Groessen characters and using the functional equations both for the twisted Maaß–Koecher series and for the spinor zeta functions, we obtain after some calculations that the twisted Maaß–Koecher series of $F$ and $G$ must coincide. (Note that a similar idea already has been used in [Ko] in the context of elliptic modular forms of half-integral weight.) By the converse theorem of Imai ([Im]) we therefore get $F = G$.

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## §2. Maaß–Koecher series

If $F \in S_k(\Gamma_2)$ ($k$ even) with Fourier coefficients $a(T)$ we denote by

$$D_F(s) := \sum_{T>0/\sim} \frac{a(T)}{\epsilon(T)(\det T)^s} \qquad (\mathrm{Re}\ s \gg 0)$$

the Maaß–Koecher series of $F$ (summation over a complete system of representatives of $\mathrm{GL}(2;\mathbb{Z})$-equivalence classes of half-integral, positive definite matrices $T$; $\epsilon(T) := \sharp\{U \in \mathrm{GL}(2;\mathbb{Z}) : U^tTU = T\}$). In the following we will use these series twisted by Groessen characters and for this we have to recall the Roelcke–Selberg spectral decomposition (cf. [Ku], [Im]).

Let $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}\ z > 0\}$ be the upper half plane and $\Delta$ the Laplace operator on $\mathbb{H}$. Denote by $< \cdot, \cdot >$ the Petersson scalar product on the space $\mathcal{L}^2(\Gamma_1 \setminus \mathbb{H})$ where $\Gamma_1$ is the elliptic modular group acting on $\mathbb{H}$ in the usual way.

Let $v_0 = \sqrt{\frac{3}{\pi}}$ and $\{v_j : j \in \mathbb{N}\}$ be an orthonormal basis consisting of cuspidal eigenfunctions for $\Delta$. Let $\Gamma_{1,\infty} := \{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \in \Gamma_1\}$ and let

$$E_u(z) := \frac{1}{2} \sum_{\gamma \in \Gamma_{1,\infty} \setminus \Gamma_1} (\mathrm{Im}\ \gamma z)^u \qquad (\mathrm{Re}\ u > 1, z \in \mathbb{H})$$

be the non-holomorphic Eisenstein series. We denote the analytic continuation of $E_u(z)$ to a meromorphic function on $\mathbb{C}$ with respect to the variable $u$ again by $E_u(z)$.

For any $f \in \mathcal{L}^2(\Gamma_1 \setminus \mathbb{H})$ the Roelcke–Selberg spectral decomposition is given by

$$f(z) = \sum_{j=0}^{\infty} < f, v_j > v_j(z) + \frac{1}{4\pi i} \int_{\mathrm{Re}\ \mathrm{u}=\frac{1}{2}} < f, E_u > E_u(z)\, du \qquad (z \in \mathbb{H}).$$

For $F \in S_k(\Gamma_2)$ let

$$F_t(W) := F(i\sqrt{t}W) \qquad (W > 0,\ \det W = 1;\ t \in \mathbb{R},\ t > 0).$$

For Re $s \gg 0$ we consider the Mellin transformation

$$\widetilde{F}_s(W) := \int_0^\infty F_t(W) t^{s-1}\, dt.$$

As in [Im], p. 917 we identify the set of all positive definite matrices $W$ of determinant 1 with $\mathbb{H}$ so that we easily see that $\widetilde{F}_s \in \mathcal{L}^2(\Gamma_1 \setminus \mathbb{H})$.

Define

$$\Lambda_k := \begin{cases} \{u \in \mathbb{C} : \mathrm{Re}\ u = \frac{1}{2}\} \cup \{j \in \mathbb{N}_0 : v_j \text{ even}\}, & \text{if } k \text{ is even,} \\ \{j \in \mathbb{N}_0 : v_j \text{ odd}\}, & \text{if } k \text{ is odd.} \end{cases}$$

For $\lambda \in \Lambda_k$ we let $\varphi_\lambda = \overline{\nu_j}$ or $\varphi_\lambda = E_{\overline{u}}$, respectively depending on whether $\lambda \in \mathbb{N}_0$ or not and define

$$D_{F,\lambda}(s) := \sum_{T>0/\sim} \frac{a(T)\varphi_\lambda\left(\frac{1}{\sqrt{\det T}}T\right)}{\epsilon(T)(\det T)^s} \qquad (\mathrm{Re}\ s \gg 0),$$

i.e. for $k$ even $D_{F,\lambda}(s)$ denotes the Maaß–Koecher series twisted by $\varphi_\lambda$. (Note that the constant implied in "$\gg$" can be chosen independently of $\lambda \in \Lambda_k$, cf. [Im], p. 932 ff.).

The following result is implicitly contained in [Im], however, for the reader's convenience we recall the proof.

LEMMA 2.1. *Let $F \in S_k(\Gamma_2)$. If $F \neq 0$, then there exists $\lambda \in \Lambda_k$ such that $D_{F,\lambda}(s)$ does not vanish identically.*

*Proof.* For Re $s \gg 0$ set $\Phi_{F,j}(s) :=< \widetilde{F}_s, v_j >$ $(j \in \mathbb{N}_0)$ resp. $\Phi_{F,u}(s) :=$ $< \widetilde{F}_s, E_u >$ $(u \in \mathbb{C}$, Re $u = \frac{1}{2})$. The Roelcke–Selberg spectral decomposition for $\widetilde{F}_s$ says

$$\widetilde{F}_s(z) = \sum_{j=0}^\infty \Phi_{F,j}(s) v_j(z) + \frac{1}{4\pi i} \int_{\mathrm{Re\ u}=\frac{1}{2}} \Phi_{F,u}(s) E_u(z)\, du.$$

Since $\widetilde{F}_s(z)$ is an even resp. odd function in $x = \mathrm{Re}\ z$ depending on whether $k$ is even or odd, in the sum only values $\lambda \in \Lambda_k$ can occur, and the integral is zero if $k$ is odd.

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Now assume that $\Phi_{F,\lambda}(s) = 0$ for Re $s \gg 0$ and every $\lambda \in \Lambda_k$. Then $\widetilde{F}_s(z) = 0$ for $z \in \mathbb{H}$ and Re $s \gg 0$. Hence by Mellin inversion we find that $F_t(W) = 0$ for all $t$ and $W$ so that $F(iY) = 0$ for any $Y > 0$. This implies $F = 0$ in contrast to the assumption. Hence we can find $\lambda \in \Lambda_k$ such that $\Phi_{F,\lambda}(s)$ does not vanish identically.

If $\varphi_\lambda = \overline{v_j}$ ($j \in \mathbb{N}_0$) then by [Im], p. 927 we have for Re $s \gg 0$

$$\Phi_{F,j}(s) = 2(2\pi)^{-2s}\sqrt{\pi}\Gamma(s-a_j)\Gamma(s-b_j)D_{F,j}(s). \tag{1}$$

Here, $a_j$ and $b_j$ are determined by

$$a_j, b_j = \frac{1 \pm \sqrt{1+4\mu_j}}{4},$$

where $\Delta\overline{v_j} = \mu_j\overline{v_j}$. Since $\Phi_{F,j}(s)$ is an entire function of $s$ ([Im], p. 929) we conclude that $D_{F,j}(s)$ does not vanish identically.

Now let $\varphi_\lambda = E_{\overline{u}}$ with $u \in \mathbb{C}$, Re $u = \frac{1}{2}$. By the results of [Im] for Re $s \gg 0$ we have

$$\Phi_{F,u}(s) = 2(2\pi)^{-2s}\sqrt{\pi}\Gamma\left(s - \frac{\overline{u}}{2}\right)\Gamma\left(s + \frac{\overline{u}-1}{2}\right)D_{F,u}(s), \tag{2}$$

hence it follows that $D_{F,u}(s)$ does not vanish identically. $\square$

## §3. Andrianov identities

Let again $F \in S_k(\Gamma_2)$ with Fourier coefficients $a(T)$. We assume that $F$ is an eigenform of all Hecke operators $T(m)$ ($m \in \mathbb{N}$) with eigenvalues $\lambda_F(m)$.

According to Andrianov ([An], p. 84 f.), for any discriminant $D < 0$ we have the identity

$$L_D(s-k+2,\chi)\sum_{i=1}^{h(D)}\chi(N_i)\sum_{m=1}^{\infty}\frac{a(mN_i)}{m^s} = \psi_F^D(s,\chi)Z_F(s) \quad (\text{Re } s \gg 0). \tag{3}$$

Here $D = df^2$ with $d$ fundamental and $H(D) = H(\mathcal{O}_f)$ is the group of classes of similar modules for the coefficient ring $\mathcal{O}_f$ with discriminant $D$ in the quadratic field $\mathbb{Q}(\sqrt{d})$. Furthermore, $N_1, \ldots, N_{h(D)}$ is a set of representatives of such modules, $\chi$ is a character of $H(D)$ and $L_D(s,\chi)$ denotes the $L$-function of $\mathcal{O}_f$ twisted with the character $\chi$.

Note that we have a bijection $M \mapsto \widetilde{M}$ between $H(D)$ and the $\mathrm{SL}(2;\mathbb{Z})$-classes of half-integral, positive definite, *primitive* matrices with discriminant $D$. Putting $\widetilde{a}(m;M) := a(m\widetilde{M})$ for $m \in \mathbb{N}$, the function $\psi_F^D(s,\chi)$ is defined by

$$\psi_F^D(s,\chi) = \sum_{\delta_1|\delta|f} \frac{\mu(\delta)\mu(\delta_1)}{\delta^{s-k+2}\delta_1^{s-2k+3}} \sum_{i=1}^{h(D)} \chi(N_i)\widetilde{a}(\frac{\delta}{\delta_1};\mathcal{O}_{\frac{f}{\delta}}N_i), \tag{4}$$

where $\mathcal{O}_{\frac{f}{\delta}}$ is the order of discriminant $d\left(\frac{f}{\delta}\right)^2$ in $\mathbb{Q}(\sqrt{d})$ and $\mu$ denotes the Möbius function. Furthermore,

$$Z_F(s) = \zeta(2s-2k+4)\sum_{m=1}^{\infty}\frac{\lambda_F(m)}{m^s} \qquad (\mathrm{Re}\ s \gg 0)$$

is the spinor zeta function attached to the eigenform $F$.

Inverting the character sum on the left-hand side of (3) we find that for each $j = 1,\dots,h(D)$ we have

$$\sum_{m=1}^{\infty}\frac{a(mN_j)}{m^s} = \frac{1}{h(D)}\left(\sum_{\chi}\overline{\chi}(N_j)\frac{\psi_F^D(s,\chi)}{L_D(s-k+2,\chi)}\right)Z_F(s), \tag{5}$$

where the sum is taken over all characters $\chi$ of $H(D)$.

From (5) we can deduce for $\mathrm{Re}\ s \gg 0$ an identity between $D_{F,\lambda}(s)$ $(\lambda \in \Lambda_k)$ and the spinor zeta function, namely

$$\begin{aligned} D_{F,\lambda}(s) &= \sum_{T>0\ \text{primitive}/\sim}\ \sum_{m=1}^{\infty}\frac{a(mT)\varphi_\lambda\left(\frac{m}{\sqrt{\det mT}}T\right)}{\epsilon(mT)(\det mT)^s} \\ &= \sum_{T>0\ \text{primitive}/\sim}\frac{\varphi_\lambda\left(\frac{1}{\sqrt{\det T}}T\right)}{\epsilon(T)(\det T)^s}\sum_{m=1}^{\infty}\frac{a(mT)}{m^{2s}} \\ &= \frac{1}{2}\sum_{\substack{D\ \text{Diskr.}\\ D<0}}\frac{4^s}{|D|^s}\sum_{i=1}^{h(D)}\frac{\varphi_\lambda\left(\frac{1}{\sqrt{\det N_i}}N_i\right)}{\epsilon(N_i)}\sum_{m=1}^{\infty}\frac{a(mN_i)}{m^{2s}} \\ &= \frac{1}{2}K_{F,\lambda}(s)Z_F(2s) \end{aligned} \tag{6}$$

where

$$K_{F,\lambda}(s) = \sum_{\substack{D \text{ Diskr.} \\ D<0}} \frac{4^s}{h(D)|D|^s} \sum_{i=1}^{h(D)} \frac{\varphi_\lambda\left(\frac{1}{\sqrt{\det N_i}} N_i\right)}{\epsilon(N_i)} \sum_{\chi} \overline{\chi}(N_i) \frac{\psi_F^D(2s,\chi)}{L_D(2s-k+2,\chi)}.$$

## §4. Proof of Theorem

Let $F \in S_k(\Gamma_2)$ be an eigenform of all Hecke operators $T(m)$ $(m \in \mathbb{N})$. Given $\lambda \in \Lambda_k$, the corresponding function

$$s \mapsto \Phi_{F,\lambda}(s)$$

(notations as in §2) has a meromorphic continuation to the whole complex plane and satisfies the functional equation

$$\Phi_{F,\lambda}(k-s) = (-1)^k \Phi_{F,\lambda}(s) \tag{7}$$

([Im], p. 929).

On the other hand, by [An], p. 88,

$$Z_F^*(s) := (2\pi)^{-2s}\Gamma(s)\Gamma(s-k+2)Z_F(s)$$

has a meromorphic continuation to the whole complex plane and satisfies the functional equation

$$Z_F^*(2k-2-s) = (-1)^k Z_F^*(s). \tag{8}$$

Proposition 4.1. *Let $F, G \in S_k(\Gamma_2)$ be eigenforms of all Hecke operators $T(m)$ $(m \in \mathbb{N})$ with Fourier coefficients $a(T)$ resp. $b(T)$.*

*If $a(nT) = b(nT)$ for every half-integral, positive definite, primitive matrix $T$ and every square-free integer $n \in \mathbb{N}$, then*

$$K_{F,\lambda}(s) = K_{G,\lambda}(s) \qquad (\mathrm{Re}\ s \gg 0).$$

*for any $\lambda \in \Lambda_k$.*

*Proof.* We only have to check that $\psi_F^D(s,\chi) = \psi_G^D(s,\chi)$ for any discriminant $D < 0$ and any character $\chi$ of $H(D)$. In the definition (4) we may consider $\mathcal{O}_{\frac{f}{\delta}} N_i$ as an element of $H(d\left(\frac{f}{\delta}\right)^2)$ ([An], p. 72). Substituting $m := \frac{\delta}{\delta_1}$ we must sum over all $m|\delta|f$, and due to the factor $\mu(\delta)$ only square-free values of $m$ give a non-zero contribution. By assumption, for such values $m$ we have

$$\widetilde{a}(m; \mathcal{O}_{\frac{f}{\delta}} M_i) = a(m\widetilde{\mathcal{O}_{\frac{f}{\delta}} M_i}) = b(m\widetilde{\mathcal{O}_{\frac{f}{\delta}} M_i}) = \widetilde{b}(m; \mathcal{O}_{\frac{f}{\delta}} M_i),$$

which proves what we wanted. □

Now let $F, G$ be as in Proposition 4.1 and $\lambda \in \Lambda_k$.

First suppose that $D_{F,\lambda}(s)$ is not identically zero. By (6) and Proposition 4.1 the same holds for $D_{G,\lambda}(s)$ and we have

$$C(s) := \frac{D_{F,\lambda}(s)}{D_{G,\lambda}(s)} = \frac{Z_F(2s)}{Z_G(2s)} \qquad (s \in \mathbb{C}),$$

hence also

$$C(s) = \frac{\Phi_{F,\lambda}(s)}{\Phi_{G,\lambda}(s)} = \frac{Z_F^*(2s)}{Z_G^*(2s)} \qquad (s \in \mathbb{C}).$$

By (7)

$$C(k-s) = C(s)$$

and by (8)

$$C(s) = C(k-1-s).$$

Together we obtain

$$C(s) = C(k-s) = C(k-1-(s-1)) = C(s-1),$$

i.e. $C$ is periodic modulo 1. Since $Z_G(2s)$ has no zeroes in some half plane $\mathrm{Re}\ s \gg 0$ due to the convergent Euler product we conclude that $C(s)$ is holomorphic in this region, hence due to the periodicity, $C(s)$ is an entire function.

Clearly there exist $c > 0$ and $K_1, K_2 > 0$ such that

$$K_1 < |Z_F(2s)|, |Z_G(2s)| < K_2$$

for any $s \in \mathbb{C}$ with $c \leq \mathrm{Re\ s} \leq c+1$. Therefore

$$|C(s)| = \frac{|Z_F(2s)|}{|Z_G(2s)|} \leq \frac{K_2}{K_1}$$

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for such $s$ and hence for all $s \in \mathbb{C}$. Thus $C(s)$ is constant, hence $C(s) = 1$. Therefore, $D_{F,\lambda}(s) = D_{G,\lambda}(s)$.

On the other hand, if $D_{F,\lambda}(s)$ is identically zero, then by (6) and Proposition 4.1 also $D_{G,\lambda}(s)$ is zero.

Hence, $D_{F,\lambda}(s) = D_{G,\lambda}(s)$ for every $\lambda \in \Lambda_k$ which by Lemma 2.1 implies $F = G$ (alternatively, for $k$ even one also could use [Ki]). $\square$

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