A bound for the fixed point index of area-preserving homeomorphisms of two-manifolds

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Abstract. The study of area preserving maps of manifolds has an extensive history in the theory of dynamical systems. One interest has been in the behaviour of such maps near an isolated fixed point. In 1974 Carl Simon proved the existence of an upper bound for the index of an isolated fixed point for C^k area preserving diffeomorphisms of a surface. We extend his result to homeomorphisms of an orientable two manifold. The proof utilizes the notion of free modification, developed by Morton Brown, and enlarges the scope of the problem to the consideration of 'nice' measures, i.e. uniformly equivalent to Lebesgue measure on compact sets. By suitably modifying the homeomorphism and the measure, we obtain the following theorem.

THEOREM. *Let* $h: M^2 \to M^2$ *be an orientation preserving homeomorphism of a smooth orientable two manifold which preserves area. If p is an isolated fixed point of h, then the index of p is* $\leq +1$.

0. Introduction

What restrictions does the hypothesis that a homeomorphism be area-preserving place on the dynamics of the map? In this paper we present a result about the possible dynamics of such a map in the neighbourhood of an isolated fixed point: the index of the point must be less than 2.

This result may be viewed as a generalization of Simon's theorem [6], which is concerned with C^k diffeomorphisms. Our method of proof is necessarily completely different from his, and appears to be applicable in a wide variety of situations. The method involves a procedure by which an area preserving homeomorphism is modified to produce a new homeomorphism which has the same fixed points, which preserves an equivalent measure, and which has a special canonical form. This procedure has its roots in the Brouwer Translation Arc Lemma, and employs the idea of a free modification of a homeomorphism developed by Brown [3], [4], and by Schmitt [5]. It should be noted that, although we are primarily concerned with homeomorphisms, the techniques we use work equally well in the setting of C^k diffeomorphisms.

Some of the applications of the results presented here derive from the fact that our theorem is a generalization of Simon's. In [6] Simon gives a number of applications of his theorem to mechanical (Hamiltonian) systems. In these applications a diffeomorphism of a surface is obtained from the flow in such a way that fixed points correspond to time periodic motions of the system. Simon then applies his result to conclude that the motions are nondegenerate, deducing, on topological grounds, the existence of other, independent, periodic motions.

As a result of theorem I below, his conclusions are true under the less restrictive assumption that the Hamiltonian governing the system is piecewise differentiable in such a way that a continuous, but not necessarily smooth flow is obtained. Piecewise smooth Hamiltonian systems, which are frequently easier to investigate numerically than their smooth analogues, have been receiving considerable attention in the physical literature.

In addition to their physical significance, area-preserving homeomorphisms have an intrinsic interest of their own. As an illustration of the usefulness of our theorem I in answering global versions of the question with which this section began, we mention a corollary to theorem I due to K. Boucher [1]:

COROLLARY. *Let* $h: S^1 \to S^1$ *be a homeomorphism which has at least n attracting (and therefore at least n repelling) fixed points. Then any extension of h to an orientation preserving, area-preserving homeomorphism of the disc D has at least Greatest integer* $(n/2+1)$ *fixed points in the interior of D.*

1. *Definitions and notation*

Let $\mathbb{R}^2$ denote the 2-plane with the standard norm $\| \quad \|$.

A *disc* $D \subset \mathbb{R}^2$ is a homeomorph of the unit ball $\{x \in \mathbb{R}^2 | \|x\| \leq 1\}$.

For $X \subset \mathbb{R}^2$ let int (X), bd (X), $\bar{X}$, X^c denote the *interior, boundary, closure* and *complement* of X respectively.

If h is a homeomorphism of $\mathbb{R}^2$, let Fix (h) denote the fixed point set of h.

By a *nice measure* μ on a 2-manifold we mean a measure which is absolutely continuous with respect to Lebesgue measure, zero on points and with Radon Nikodym derivative $d\mu/dx$ bounded above and below from 0 by constants on compact sets. For our purposes this last condition is equivalent to the existence of positive constants K_1, K_2 such that $K_1 m(A) \leq \mu(A) \leq K_2 m(A)$ where A is a Lebesgue measurable set contained in some compact set and m is Lebesgue measure.

A homeomorphism h of $\mathbb{R}^2$ is said to *preserve the measure* μ if $\mu(A) = \mu(h(A))$ for all measurable sets A.

Definition. Given a simple closed curve C in $\mathbb{R}^2$ and a homeomorphism $h: \mathbb{R}^2 \to \mathbb{R}^2$ with $C \cap$ Fix $(h) = \varnothing$, the *index of* C with respect to h, denoted ind (h, C), is the

degree (i.e. winding number) of the mapping $H : C \to S^1$ where

$$H(x) = \frac{h(x) - x}{\|h(x) - x\|}.$$

We shall need the following two basic properties of the index:

(i) Let h and g be two homeomorphisms such that Fix $(g) = $ Fix (h) and let C be a simple closed curve such that $C \cap$ Fix $(h) = \varnothing$. If there exists an isotopy Φ_t between h and g such that Fix $(h) = $ Fix (Φ_t) then ind $(h, C) = $ ind (g, C).

(ii) If C is invariant under h then ind $(h, C) = 1$.

Definition. If p is an isolated fixed point for h, then let the *index of p*, denoted ind (h, p), be ind (h, C) for a simple closed curve C which bounds a disc D containing p, where $D \cap$ Fix $(h) = p$. It follows easily that the index of a fixed point is well-defined.

By the Brouwer fixed point theorem one proves that if ind $(h, C) \neq 0$ for some simple closed curve C, then the disc D bounded by C contains a fixed point for h.

The relation between index of a curve and index of a point is as follows:

LEMMA. *Let h be a homeomorphism of the plane with isolated fixed point set. If C is a simple closed curve with $C \cap$ Fix $(h) = \varnothing$, then $\sum$ ind $(h, p) = $ ind (h, C) where the sum is taken over those fixed points contained in the disc bounded by C.*

2. *The main theorem*

This section presents a statement of the main theorem, a reduction of this theorem, and introduces the notion of a free modification of a homeomorphism. Much of this section depends on the following form of the Brouwer lemma, which is due to Brown [2]:

BROUWER LEMMA. *Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be an orientation preserving homeomorphism of the plane, and let D be a disc with $h(D) \cap D = \varnothing$. If $h^n(D) \cap D \neq \varnothing$ for some $n \neq 0$, then there exists a simple closed curve C with $C \cap$ Fix $(h) \neq \varnothing$ such that the index of C is 1.*

THEOREM I. *Let $h : M^2 \to M^2$ be an orientation preserving homeomorphism of a smooth orientable 2 manifold M^2 which preserves area. If p is an isolated fixed point of h, then the index of h at p is less than or equal to 1.*

Proof. Assume that there exists an isolated fixed point p of h with ind $(h, p) = n > 1$. Let Fix $(h) = F \cup \{p\}$. Since Fix (h) is closed and p is isolated, F is a closed set. Consider the connected component X of $M^2 - F$ containing the point p. Since M^2 is orientable it follows that the universal cover $\tilde{X}$ of X is either $\mathbb{R}^2$ or S^2, and that h lifts to a homeomorphism $\tilde{h} : \tilde{X} \to \tilde{X}$ which fixes only the fiber $\pi^{-1}(p)$, where $\pi : \tilde{X} \to X$ is the covering projection. Since π is a local homeomorphism, $\pi^{-1}(p)$ consists of a countable collection of isolated points $p_1, p_2, \ldots$, and ind $(\tilde{h}, p_i) = n$ for all i such that p_i is fixed under $\tilde{h}$.

The case where the universal cover is S^2 will be handled separately. Since X is orientable $X \approx S^2$. Hence the only fixed point for h is p with index 2. However, by

using the Brouwer Translation Arc Lemma on $S^2 - p \approx \text{int}(D)$, there exists at least one other fixed point, a contradiction.

We will now define a measure μ on $\mathbb{R}^2$ which is invariant under $\tilde{h}$. If $A \subset \mathbb{R}^2$ is such that π is one to one on both A and $\tilde{h}(A)$, set $\mu(A) = \text{area}(\pi(A))$. Then μ is invariant because if A has the property that π is one to one on A and $\tilde{h}(A)$, then

$$\mu(\tilde{h}(A)) = \text{area}(\pi\tilde{h}(A)) = \text{area}(h\pi(A)) = \text{area}(\pi(A)) = \mu(A).$$

The fact that μ is nice follows from the fact that π can be chosen to be a smooth map of $\mathbb{R}^2$ to X. Thus without loss of generality assume that there exists an orientation preserving homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ which preserves a nice measure μ and such that:

    (i) $\text{Fix}(h) = \{p_1, p_2, \ldots\}$ is isolated; and,

    (ii) $\text{ind}(h, p_i) = n > 1$ for all i.

The proof of Theorem I then reduces to the following theorem.

THEOREM II. *Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be an orientation preserving homeomorphism of the plane which preserves a nice measure. If* $\text{Fix}(h) = \{p_1, p_2, \ldots\}$ *where each p_i is isolated and* $\text{ind}(h, p_i) = n$ *for all i, then $n \leq 1$.*

We prove this theorem by contradiction, assuming that $n > 1$. It will be necessary to consider free modifications of the given homeomorphism h. Each free modification hf of h has the same fixed point set as h and any simple closed curve C with $C \cap \text{Fix}(h) = \varnothing$ has $\text{ind}(h, C) = \text{ind}(hf, C)$.

We will show that under suitable conditions one can modify a homeomorphism to one which, though it does not preserve the original measure, nevertheless, preserves another nice measure. The following discussion concerning free modifications and reduction to canonical form is due to Brown [2], [3] and Schmitt [5].

Definition ([2]). Two homeomorphisms h and g of $\mathbb{R}^2$ are said to be *strongly equivalent* provided there exists a disc $D \subset \mathbb{R}^2$ such that $h(D) \cap D = \varnothing$ and $g^{-1}h$ is supported on D (i.e. $g^{-1}h(x) = x$ for all $x \in D$). We say that h is *freely equivalent* to g if there exists a finite sequence of homeomorphisms $h_0, \ldots, h_n$ such that $h = h_0$, $g = h_n$ and h_i is strongly equivalent to h_{i+1}. One observes that if h is freely equivalent to g, then $\text{Fix}(h) = \text{Fix}(g)$.

Given a homeomorphism h of $\mathbb{R}^2$ we will construct a freely equivalent homeomorphism by a finite sequence of free modifications.

Definition. A homeomorphism f of $\mathbb{R}^2$ is a *free modification* of h provided that there exists a disc D in $\mathbb{R}^2$ such that $h(D) \cap D = \varnothing$ and f is supported on $\text{int}(D)$.

By Alexander's Isotopy Theorem f is isotopic to the identity by an isotopy Φ_t such that $\Phi_t(x) = x$ for $x \notin \text{int}(D)$. We see that $\text{ind}(h, C) = \text{ind}(hf, C)$ for any simple closed curve C with $C \cap \text{Fix}(h) = \varnothing$.

When a homeomorphism is modified, however, one does not know a priori that there exists a nice measure which this new homeomorphism preserves. The construction of such a measure is possible under the hypotheses of theorem II.

PROPOSITION. *Let h be an orientation preserving homeomorphism of $\mathbb{R}^2$ preserving a nice measure μ and let* Fix (h) *be isolated with* ind $(h, p) = n > 1$ *for each $p \in$ Fix (h). Suppose that h is strongly equivalent to hf by a free modification f on a disc D, and that f is a C^2 diffeomorphism. Then there exists a nice measure ν preserved by hf.*

Proof. For a measurable set $A \subset \mathbb{R}^2$ define $A_j = h^{-j}(D) \cap A$ for $j = 0, 1, 2, \ldots$ and set $A_c = A - \bigcup_j A_j$. Then $A = A_c \cup \bigcup_j A_j$. We claim that we have expressed A as the disjoint union of subsets. If this were not the case, then by the Brouwer Lemma there would exist a simple closed curve C with index equal to 1. However, the index of each simple closed curve in $\mathbb{R}^2$ is either 0 or a positive integral multiple of n. Define the measure ν as follows:

$$\nu(A) = \mu(A_c) + \sum_{j=0}^{\infty} \mu((hf)^{j+1}(A_j)).$$

In order to show that ν is a nice measure, it suffices to obtain bounds on $\nu(A_j)$ in terms of the Lebesgue measure $m(A_j)$ which are uniform. Note that

$$\mu((hf)^{j+1}(A_j)) = \mu(hfh^j(A_j)) = \mu(fh^j(A_j)),$$

since h preserves μ.
Using the change of variables formula for integrals,

$$\mu(fh^j(A_j)) = \int \chi_{fh^j(A_j)}(x) \frac{d\mu}{dx}(x) \, dx = \int \chi_{h^j(A_j)}(f^{-1}x) \frac{d\mu}{dx}(x) \, dx$$

$$= \int \chi_{h^j(A_j)}(u) \frac{d\mu}{dx}(f(u)) |\det Df(u)| \, du.$$

Using the fact that $d\mu/dx$ and $|\det Df(u))|$ are bounded from zero and above by some constant on the compact set $h(D)$ shows that

$$K_1 \text{ Lebesgue } (A_j) \leq \nu(A_j) \leq K_2 \text{ Lebesgue } (A_j) \tag{$*$}$$

for constants K_1 and K_2 independent of j and A_j, and hence that the measure ν is also a nice measure.

The above condition that f be C^2 is not necessary. One could require only that f be L-bi-Lipschitzian, that is

$$(1/L)|x - y| \leq |f(x) - f(y)| \leq L|x - y|$$

for some constant $L \geq 1$. Then one can also obtain the inequality $(*)$.

Remark. To ensure that the measure ν is non-atomic it is essential that a bound be placed on the Jacobian as the following example shows:
Let $h(x, y) = (x/2, 2y)$ and D be the disc $\{(x, y) \,|\, |x| \leq \frac{1}{2}, \, |y - \frac{3}{2}| \leq \frac{1}{2}\}$. Let $f: D \to D$ be the free modification defined such that the vertical lines $x = \pm 2^{-n}$, $n = 1, 2, \ldots$ are mapped as shown in figure 1. Now extend f so that f restricted to the boundary of D is the identity.
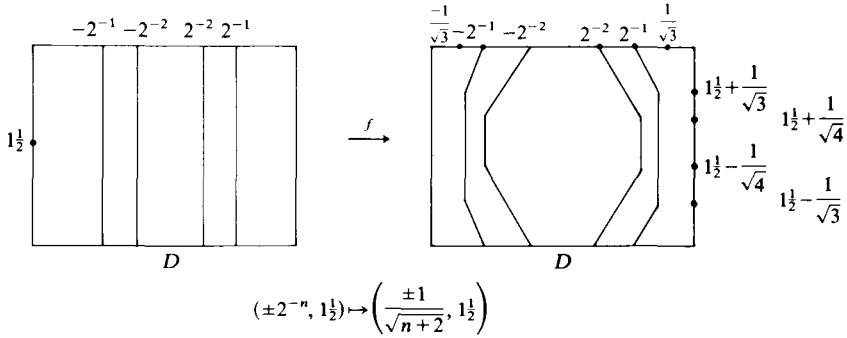
$$\left(\pm 2^{-n}, 1\tfrac{1}{2}\right) \mapsto \left(\frac{\pm 1}{\sqrt{n+2}}, 1\tfrac{1}{2}\right)$$

FIGURE 1

Let $E = \{(x, y) \,|\, |x| \leq \tfrac{1}{2}, \ |y| \leq \tfrac{1}{2}\}$. Then $F_n = h^n(h^{-n}(D) \cap E) = \{(x, y) \,|\, |x| \leq 2^{-n}, |y - \tfrac{3}{2}| \leq \tfrac{1}{2}\}$. See figure 2. After modifying h by f we have that

$$\nu(E) \geq \nu\left(\bigcup F_n\right) = \sum \nu(F_n) = \sum m(f F_n) > \sum 1/n = \infty.$$

Thus any neighbourhood of the origin will have infinite measure, and hence the measure ν will be atomic.
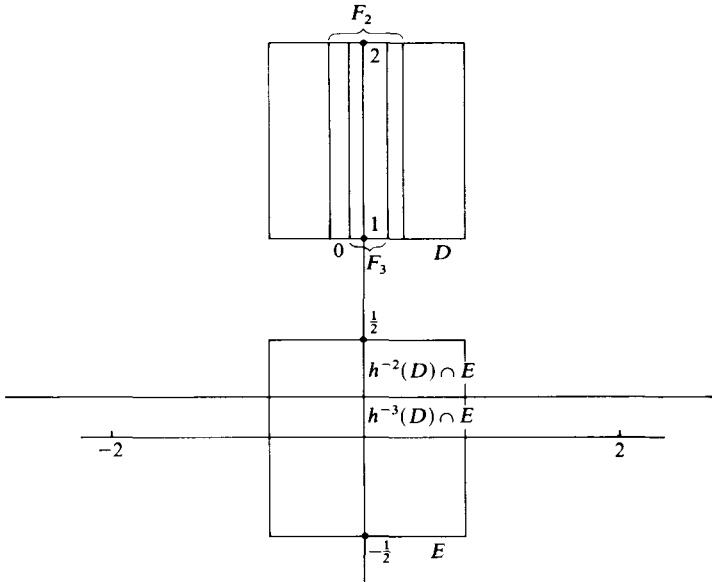
FIGURE 2

In reducing h to canonical form free modifications will be made on maps which preserve a nice measure. Each modification will be constructed using only maps which are C^2 diffeomorphisms with Jacobian bounded above and below from 0 (or L-bi-Lipschitzian maps). Thus, each of these modifications will result in a new homeomorphism which preserves a new (but nice) measure.

3. *Canonical form and the proof of Theorem* II

The index of a curve C with respect to an orientation preserving homeomorphism h can be calculated by considering the rotation of the vector $h(x) - x$ as x passes through two consecutive points on C whose images also lie on C. If the rotation is approximately $-\pi$ or π, one counts a contribution of $+1$ or -1 to the index. Summing over such pairs of points, and adding 1 for the total rotation of C yields the index of C. Some pairs of points which contribute 0 to the index are illustrated in figure 3. An arc on C with endpoints a and b having the property that $h(a)$ and $h(b)$ are
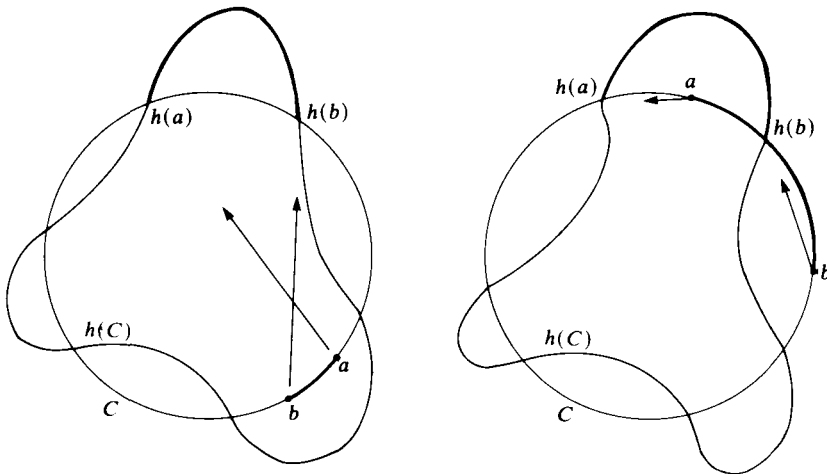
FIGURE 3

also on C and that the arc on C from $h(a)$ to $h(b)$ contains the arc from a to b (see figure 4a) is called an elliptic arc. Elliptic arcs contribute $+1$ to the index of C. If the arc from $h(a)$ to $h(b)$ is contained in the arc from a to b, the arc is called hyperbolic (see figure 4b). Hyperbolic arcs contribute -1 to the index of C. We can
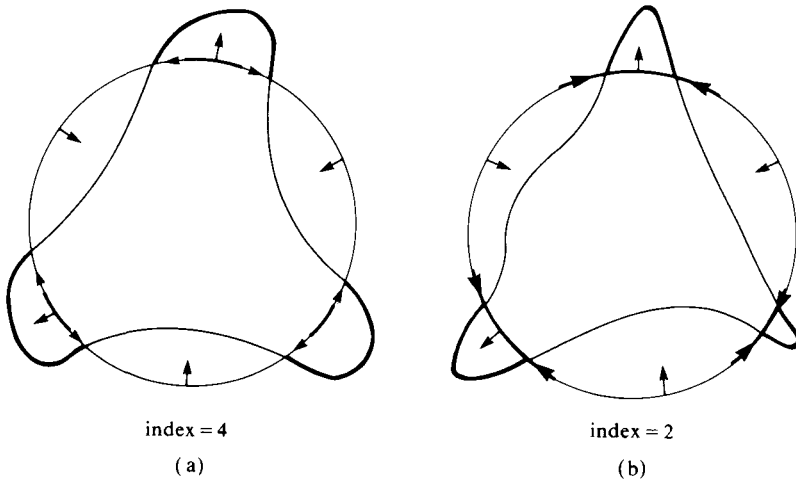

index = 4

(a)

index = 2

(b)

FIGURE 4

then compute that ind $(h, C) = 1 + e - h$, where e is the number of elliptic arcs on C, and h the number of hyperbolic arcs.

Definition. A homeomorphism f freely equivalent to h is a *canonical form of h* on the curve C if the number of intersections of C with $f(C)$ is exactly $|2(n-1)|$, where $n = \text{ind}(h, C)$.

Thus, if ind $(h, C) = 4$, a canonical form of h on C would have exactly 3 elliptic arcs.

In §4 it will be shown that, under the hypotheses of theorem II, every homeomorphism has a canonical form which preserves a nice measure. Using this fact. we first prove theorem II.

Proof of Theorem II. Assume that $n \geq 2$, and let C be a simple closed curve disjoint from Fix (h) and with the property that if D is the disc bounded by C, then $D \cap \text{Fix}(h) = \{p_1\}$. Without loss of generality, we may assume that h is in canonical form on C. Since ind $(h, C) = n > 1$, $h(C)$ consists of $n - 1$ elliptic arcs α_i with endpoints a_i, b_i such that the arc $a_i b_i \subset C \cap h(D)$. Let E_i be the disc bounded by $\alpha_i \cup a_i b_i$.

Let $x \in \text{int}(\alpha_1)$. We will construct an arc J lying in D such that J is a subset of an invariant line through x, and J intersects C at two points, $h^{-1}(x)$ and y. Then we will consider the simple closed curve $J \cup K$ where $K \subset C$ is an arc with endpoints $h^{-1}(x)$ and y. A simple computation will show that $1 \leq \text{ind}(h, J \cup K) \leq n - 1$, which will be a contradiction and complete the proof.

Let x be chosen as above, and F a disc in E_1 such that x, $h^{-1}(x) \in F$. Note that $h^{-1}(F) \subset D$. Since int $(F) \cap h^{-1}(\text{int}(F)) = \varnothing$, by the Brouwer Lemma int $(F) \cap h^n(\text{int}(F)) = \varnothing$ for all $n \neq 0$. However, $\mu(F) > 0$ and $\mu(D) < \infty$ which implies that there exists an $N > 0$ such that $h^{-N}(F) \cap C \neq \varnothing$. Choose the smallest such N with this property. Let l_1 be an arc from $h^{-N}(x)$ to some point $y \in C$ such that int $(l_1) \subset \text{int}(h^{-N}(F) \cap D)$. Let l_2 be an arc from y to $h^{-(N+1)}(x)$ such that int $(l_2) \subset \text{int}(h^{-N}(F))$ and $l_1 \cup l_2 = y$. (See figure 5.) The arc $J' = h(l_1) \cup \bigcup_{i=2}^{N} h^i(l_1 \cup l_2)$ lies
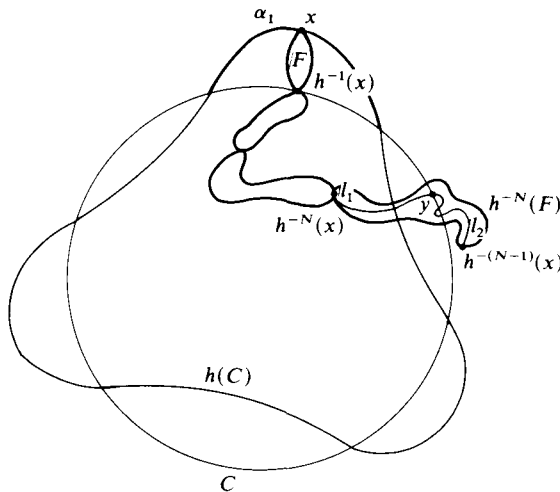

FIGURE 5

in $h(D)$ and $J' \cap h(C) = \{h(y), x\}$. Let $J = h^{-1}(J')$ and let K_1, $K_2 \subset C$ be the arcs with endpoints $\{y, h^{-1}(x)\}$. Then $J \cup K_i$ is a simple closed curve for $i = 1, 2$. (See figure 6.)
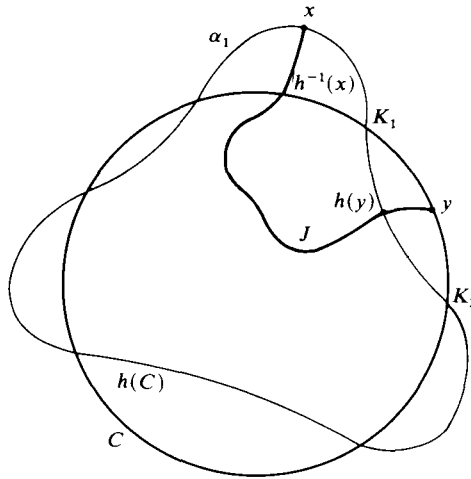

FIGURE 6

LEMMA. $1 \leq \mathrm{ind}\,(h, J \cup K_i) \leq n - 1$ *for* $i = 1, 2$.

Proof. Either by direct computation, noting that the index is determined by the number of elliptic arcs which intersect $h(K_i)$, or by using free modifications to eliminate all but one intersection of $h(J)$ with J.

Remark. The above method works as well with n hyperbolic arcs. The index of h on C is then $1 - n$. The last lemma then would state that the index of $J \cup K_i$ is between 0 and $-n$. The arc J must then pass through C between α_1 and either α_2 or α_n. We then have that the simple closed curve $J \cup K_i$ has index 0 which presents no contradiction. This is precisely the case for the smooth hyperbolic maps of the plane. As an example consider the smooth map $(x, y) \rightarrow (x/2, 2y)$.

4. *Reduction to canonical form*

The following procedure will remove pairs of intersection points which contribute 0 to the total index. These types are indicated in figure 3.

We first assume that $h(C)$ intersects C transversely and only finitely often, and that the disc bounded by C intersects its image in only one component. Let D be the disc bounded by C, and consider the arcs $\alpha_1, \ldots, \alpha_m$ with $\bigcup \alpha_i = \overline{h(C) \cap D^c}$. Let α_i have endpoints a_i, b_i and $a_i b_i$ be the arc on C such that $a_i b_i \subset h(D) \cap C$. (See figure 7.)
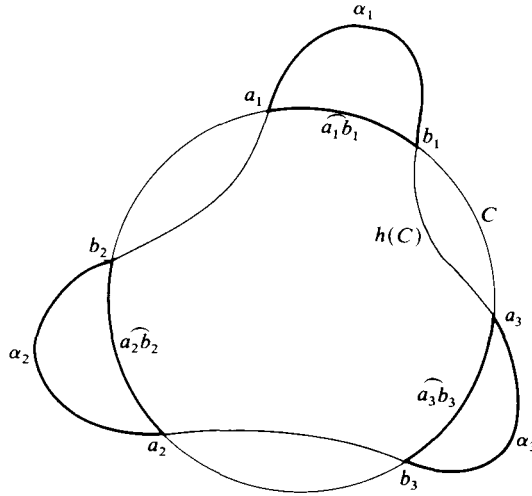
FIGURE 7

Case I. Assume that $h^{-1}(\alpha_i) \cap a_i b_i = \varnothing$. Since $a_i b_i - \{a_i, b_i\}$ lies entirely within int $(h(D))$ we have that $h^{-1}(a_i b_i - \{a_i, b_i\})$ lies entirely within int (D). Thus we can find a disc $D(\alpha_i)$ containing $\alpha_i \cup a_i b_i$ such that $h^{-1}(D(\alpha_i)) \cap D(\alpha_i) = \varnothing$. We then employ a free modification f on $h^{-1}(D(\alpha_i))$ such that $hf(h^{-1}(\alpha_i)) \subset$ int (D). (See figure 8.) With this modification $hf(C)$ has two fewer points of intersection with C.
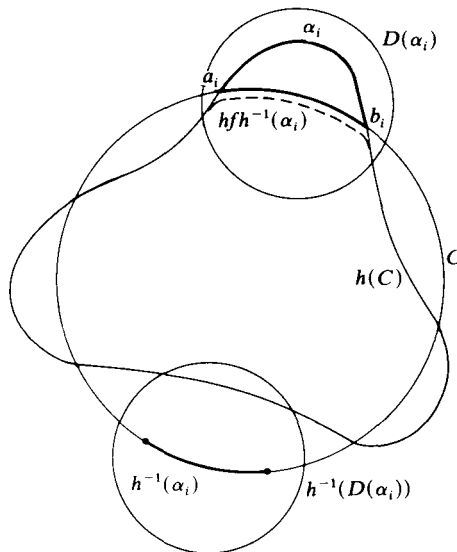
FIGURE 8

Case II. Assume that $h^{-1}(\alpha_i) \cap a_i b_i \neq \varnothing$ and precisely one of a_i or b_i lie in $h^{-1}(\alpha_i)$. Assume b_i lies in $h^{-1}(\alpha_i)$. Since h is orientation preserving $h^{-1}(a_i) \in a_i b_i$ (figure 9a). Let $d_i \in \alpha_i$ be a point near b_i and $\tilde{\alpha}_i \subset \alpha_i$ the arc from a_i to d_i. Pick a disc $D(\tilde{\alpha}_i)$
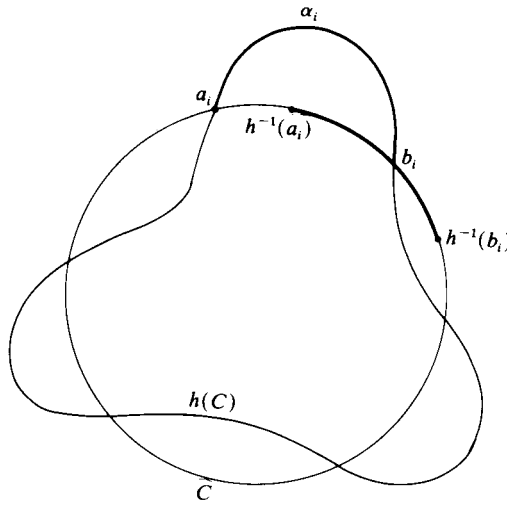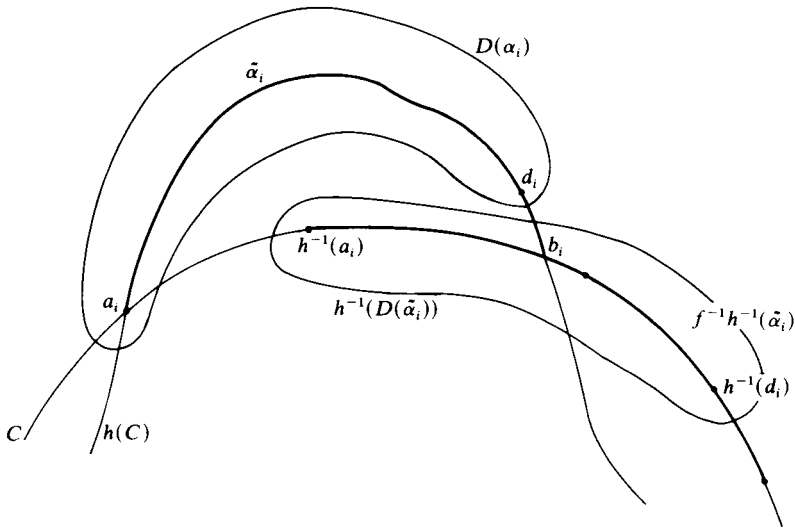
FIGURE 9(a).

FIGURE 9(b).

containing $\tilde{\alpha}_i$ such that $D(\tilde{\alpha}_i) \cap h^{-1}(D(\tilde{\alpha}_i)) = \varnothing$. Modify h on $h^{-1}(D(\tilde{\alpha}_i))$ by f such that $f(h^{-1}(\tilde{\alpha}_i)) \cap \alpha_i = \varnothing$. (Figure 9b.) We now have that $(fh)^{-1}(\alpha_i) \cap a_i b_i = \varnothing$ and use Case I to eliminate two intersection points. Similarly if $h^{-1}(\alpha_i) \cap a_i b_i \neq \varnothing$ and both a_i and b_i lie in $h^{-1}(\alpha_i)$, but $h^{-1}(\alpha_i) \neq a_i b_i$, use the above method to eliminate two intersection points. (See figure 10.)
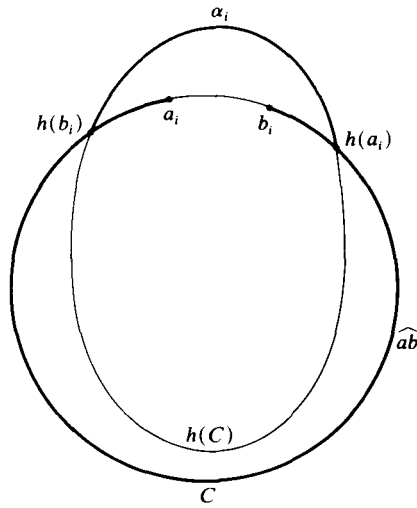
FIGURE 10

Case III. We are now left with arcs α_i such that either $h^{-1}(\alpha_i) \subset a_i b_i$, $a_i b_i \subset h^{-1}(\alpha_i)$ or $h(C) \cap C = \varnothing$. Since h is orientation preserving and fixed point free on C, these containments are proper. Call those α_i such that $h^{-1}(\alpha_i) \subset a_i b_i$ *elliptic* arcs and call those α_i such that $a_i b_i \subset h^{-1}(\alpha_i)$ *hyperbolic* arcs.

Assume that an elliptic arc is adjacent to a hyperbolic arc along C. Let a_1, b_1, a_2, b_2 be the four successive intersections of $h(C)$ with C, where a_1, b_1 are the endpoints of the elliptic arc and a_2, b_2 are the endpoints of the hyperbolic arc. Consider the arc β along $h(C)$ from b_1 to a_2. From the point of view of h^{-1}, this arc is in Case II, wherein we 'pull' β outside of D. We now have an arc with endpoints a_1, b_2 which can be removed as in Case II. This procedure removes four intersection points. (See figure 11.)
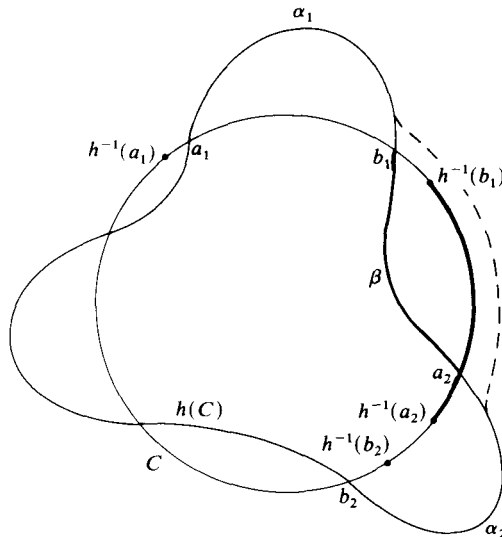

FIGURE 11

We are left with a curve C, after a finite number of free modifications, which is either disjoint from its image, is composed of a finite number of elliptic arcs, or is composed of a finite number of hyperbolic arcs. One computes the index as 1, $1+e$ or $1-h$, where e is the number of elliptic arcs and h is the number of hyperbolic arcs.

In the more general setting where $h(C) \cap C$ consists of nontransverse intersections we consider a Lebesgue number δ for h restricted to C such that if I is any connected arc on C with diameter less than δ, then $h(I) \cap I = \varnothing$. Pick a finite collection of such arcs which cover C. For each such I we will modify h on a disc $D(I)$ containing I in its interior which will replace the nontransverse intersection by a transverse one on I. (See figure 12.) The dotted line represents the image of an open arc containing I in $D(I)$ under the free modification f. Since C is covered by finitely many such intervals, we need perform this operation only finitely many times.
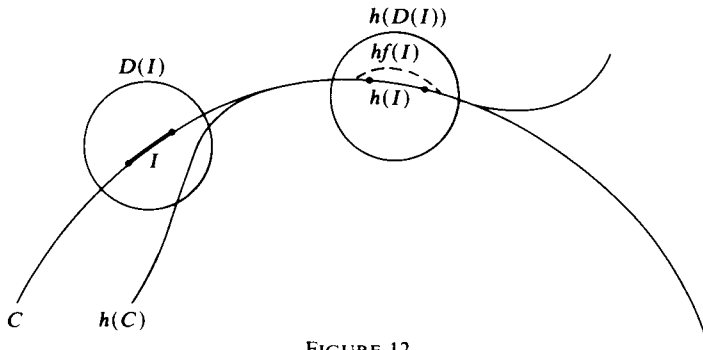

FIGURE 12

In case $h(C) \cap C$ consists of an infinite number of transverse intersections, cover $h(C) \cap C$ with a finite number of discs $D_1, D_2, \ldots, D_k$ having the properties that (1) $h^{-1}(D_i) \cap D_i = \varnothing$ for each i, and (2) $h(C) \cap D_i$ is connected for each i. Connect the endpoints of $C \cap h^{-1}(D_i)$ by a smooth arc α which lies in $h^{-1}(D_i)$ and which intersects $h^{-1}(C \cap D_i)$ a finite number of times. (This can be done by an arc having at most one point of intersection.) Modify h on $h^{-1}(D_i)$ by a map which takes $C \cap h^{-1}(D_i)$ into the arc α. After a finite number of such modifications we obtain a map freely equivalent to h under which C intersects its image a finite number of times. (Figure 13.)
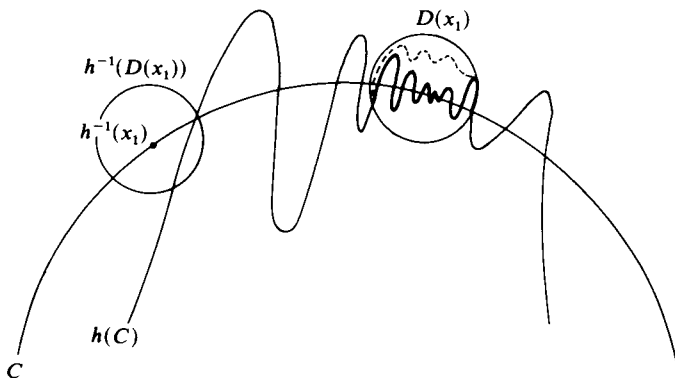

FIGURE 13

Finally, consider the case where $D \cap h(D)$ is not connected. By the above, we can assume that $D \cap h(D)$ is the union of a finite number of components K_i, $i = 1, 2, \ldots, n$, with p_1 in K_1. We will construct a simple closed curve C_1 bounding a disc $D_1 \subset D$ with p_1 in D_1 and a modified homeomorphism h' so that $h'(D_1) \cap D_1$ has fewer components. (See figure 14.)
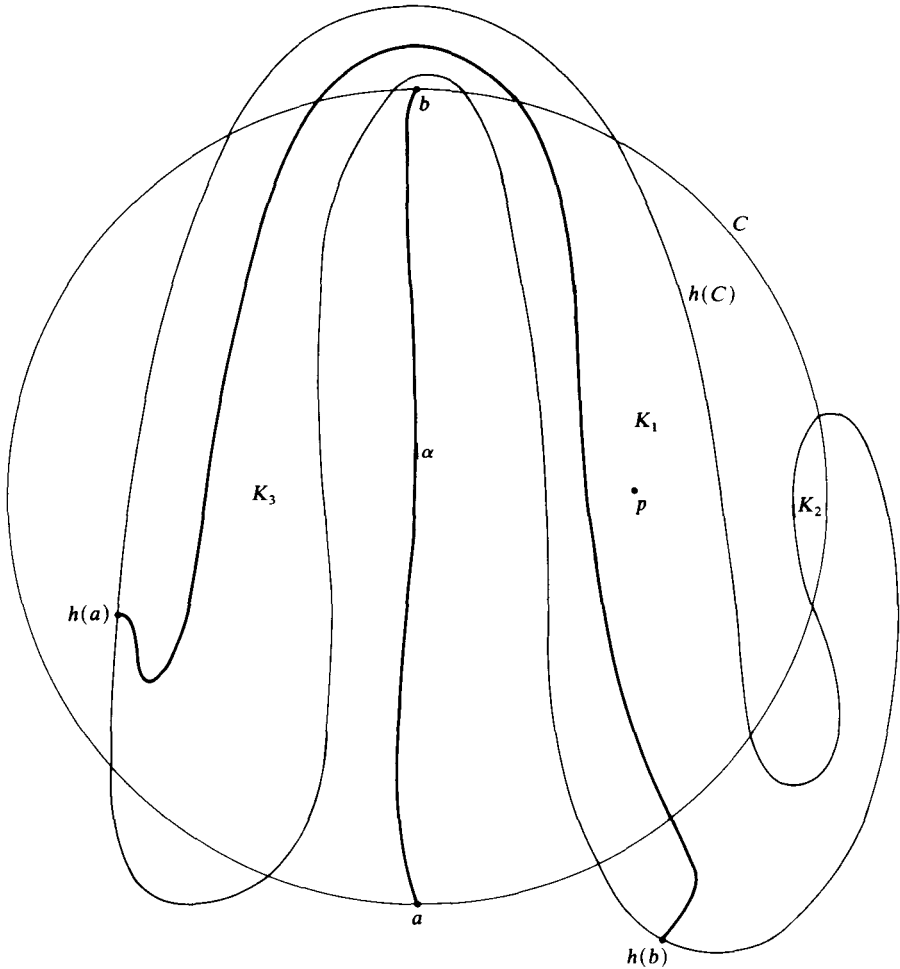

FIGURE 14

Pick $a, b \in D \cap (\bigcup K_i)^c$ such that a, b are endpoints of an arc α in $D \cap (\bigcup K_i)^c$ and so that $\alpha \cap \mathrm{bd}\, (D \cap h(D)^c) = \{a, b\}$ and such that α separates K_1 from at least one K_i in D. Since $h(\alpha) \cap \alpha = \varnothing$, we may assume that $h(\alpha)$ intersects C a finite number of times. (Otherwise, a free modification of h on a disc containing α will produce this condition.)

Let β_1 and $\beta_2 \subset C$ be the two arcs with endpoints a and b. If $h(\alpha) \cap K_i$ is connected for each i, let D_1 be the disc bounded by α and either β_1 or β_2 – whichever is such that $K_1 \subset D_1$. Then $D_1 \cap h(D_1)$ has fewer than n components.

Thus we assume that $h(\alpha) \cap K_i$ is not connected for at least some i. (See figure 15.) Let $x_1, x_2, \ldots, x_m$ be the intersection points of $h(\alpha)$ with C. Assume that the
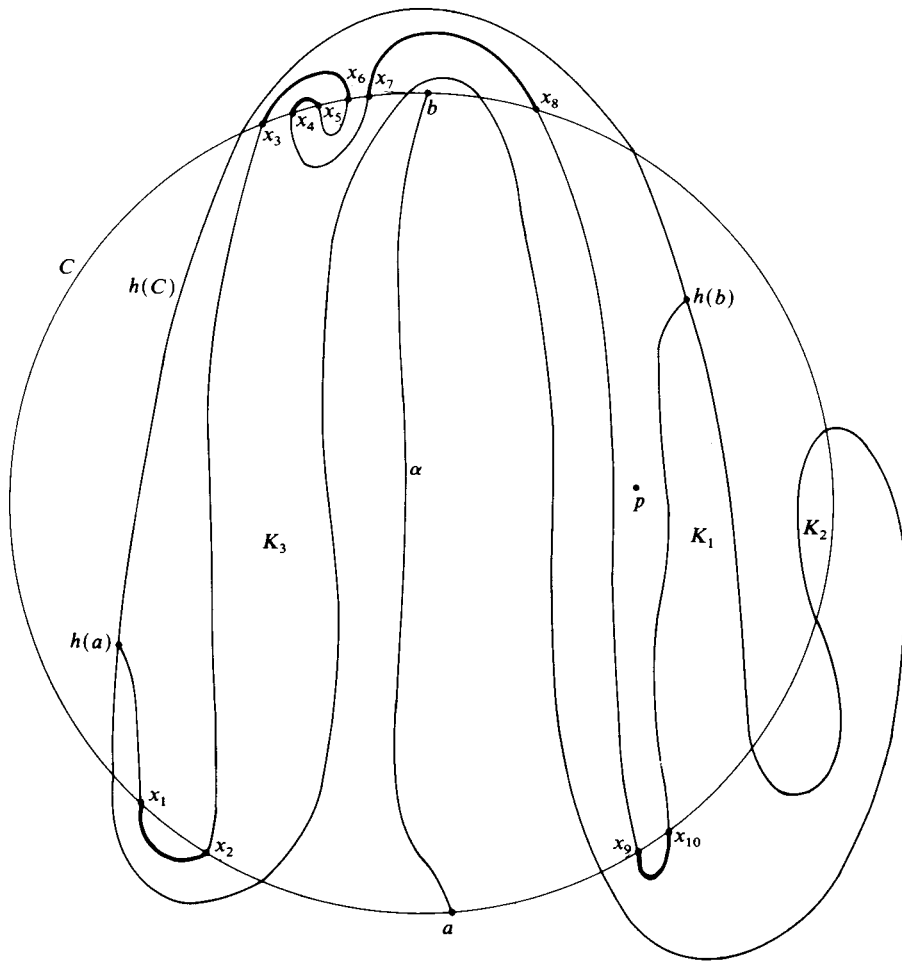

FIGURE 15

subscripts give an order to these points which is inherited from α. Let $x_i x_{i+1} \subset h(\alpha)$ be the arc with endpoints x_i and x_{i+1}.

Consider only those arcs which lie in the complement of the interior of D. Since $h(D)$ is contractible there exists at least one such arc $x_j x_{j+1}$ with endpoints lying in bd (K_i) for some i.

Pick x_j so that no other x_k lies between x_j and x_{j+1} on $C \cap \mathrm{bd}\,(K_i)$, where $x_k x_{k+1}$ is another such arc. We will modify the homeomorphism h in such a way as to move the arc $x_j x_{j+1}$ into the interior of D. Since we will focus upon this particular arc, we will rename the arc xy, and the component K. (See figure 16.)
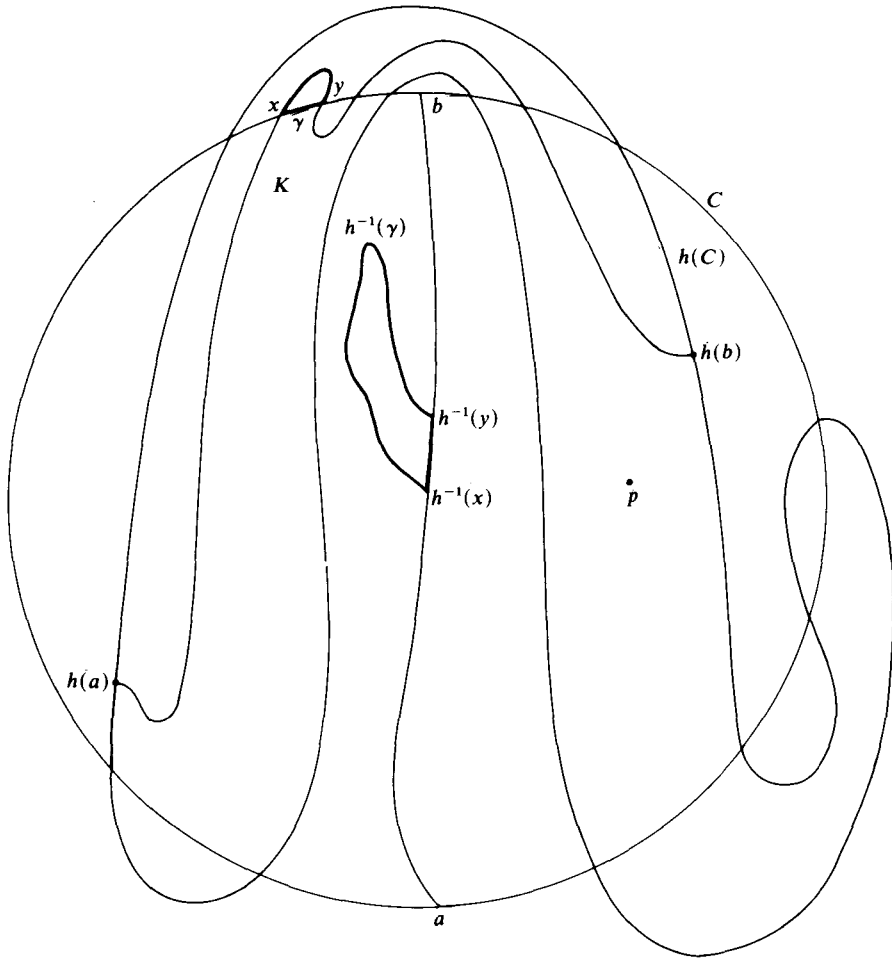

FIGURE 16

Let γ be the arc on $C \cap \mathrm{bd}\,(K)$ with endpoints x and y. We observe that $(\gamma \cup xy) \cap h^{-1}(\gamma \cup xy) \neq \emptyset$. Hence, by using a disc E containing $h^{-1}(\gamma \cup xy)$ in its interior, we can modify h to h' where $h'(h^{-1}xy)) \subset \mathrm{int}\,(D)$. We have pushed the arc xy into the interior of D. Thus we can modify the map h to obtain a map h' so that $h'(\alpha) \cap K_i$ is connected for all i.

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