GENERALISATION OF AN INTEGRAL DUE TO HARDY

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§ 1. *Introductory.* The integral

$$\int_0^\infty K_n(t)K_n\left(\frac{b}{t}\right)dt=\pi K_{2n}(2\surd b), \qquad \ldots\ldots\ldots(1)$$

where $b>0$, was given by Hardy (**1**). It was proved by applying Mellin's inversion formula. An alternative proof, based on the differential equation

$$x^2y''+xy'-(x^2+n^2)y=0 \qquad \ldots\ldots\ldots(2)$$

satisfied by $K_n(x)$, has been given by the author (**2**).

In § 2, a generalisation of this formula, namely

$$\prod_{s=1}^{p-1}\int_0^\infty K_n(t_s)\,t_s^{2s/p-1}\,dt_s\,K_n\left(\frac{b}{t_1t_2\ldots t_{p-1}}\right)=\pi^{p-1}K_{pn}(pb^{1/p}), \qquad \ldots\ldots\ldots(3)$$

where $b>0$, $p=2, 3, 4, \ldots$, will be established.

The following formulae will be required in the proof:

$$\int_0^\infty \lambda^{m-1}K_n(\lambda)\,d\lambda=2^{m-2}\Gamma\left(\frac{m+n}{2}\right)\Gamma\left(\frac{m-n}{2}\right), \qquad \ldots\ldots\ldots(4)$$

where $R(m\pm n)>0$ (**3**); and

$$\Gamma(z)\,\Gamma\left(z+\frac{1}{m}\right)\ldots\Gamma\left(z+\frac{m-1}{m}\right)=(2\pi)^{\frac{1}{2}m-\frac{1}{2}}\,m^{\frac{1}{2}-mz}\,\Gamma(mz), \qquad \ldots\ldots\ldots(5)$$

where m is a positive integer (**4**).

In § 3 a similar formula, involving Bessel Functions of the First Kind, will be obtained.

§ 2. *Proof of the Formula.* If the L.H.S. of (3) is denoted by $F(b)$, then

$$F'(b)=\prod_{s=1}^{p-1}\int_0^\infty K_n(t_s)\,t_s^{2s/p-1}\,dt_s\,K_n'\left(\frac{b}{t_1t_2\ldots t_{p-1}}\right)\frac{1}{t_1t_2\ldots t_{p-1}}$$

and

$$F''(b)=\prod_{s=1}^{p-1}\int_0^\infty K_n(t_s)\,t_s^{2s/p-1}\,dt_s\,K_n''\left(\frac{b}{t_1t_2\ldots t_{p-1}}\right)\frac{1}{(t_1t_2\ldots t_{p-1})^2}.$$

Then, from (2),

$$\begin{aligned}b^2F''(b)&=\prod_{s=1}^{p-1}\int_0^\infty K_n(t_s)\,t_s^{2s/p-1}\,dt_s\\&\quad\times\left[-\frac{b}{t_1t_2\ldots t_{p-1}}K_n'\left(\frac{b}{t_1t_2\ldots t_{p-1}}\right)+\left\{\left(\frac{b}{t_1t_2\ldots t_{p-1}}\right)^2+n^2\right\}K_n\left(\frac{b}{t_1t_2\ldots t_{p-1}}\right)\right]\\&=-bF'(b)+n^2F(b)+L,\end{aligned}$$

where

$$L\equiv b^2\prod_{s=1}^{p-1}\int_0^\infty K_n(t_s)\,t_s^{2s/p-3}\,dt_s\,K_n\left(\frac{b}{t_1t_2\ldots t_{p-1}}\right).$$

In this multiple integral change the order of integration so that the first integral becomes the last and replace t_1 by $b/(\lambda t_2 t_3 \dots t_{p-1})$, where λ is the new variable ; then

$$L = b^2 \prod_{s=2}^{p-1} \int_0^\infty K_n(t_s)\, t_s^{2s/p-3}\, dt_s$$
$$\times \int_0^\infty K_n(\lambda) \left(\frac{b}{\lambda t_2 \dots t_{p-1}}\right)^{2/p-3} K_n\left(\frac{b}{\lambda t_2 \dots t_{p-1}}\right) \frac{b\, d\lambda}{\lambda^2 t_2 \dots t_{p-1}}$$
$$= b^{2/p} \prod_{s=2}^{p-1} \int_0^\infty K_n(t_s)\, t_s^{2(s-1)/p-1}\, dt_s$$
$$\times \int_0^\infty K_n(\lambda)\, \lambda^{2(p-1)/p-1} K_n\left(\frac{b}{\lambda t_2 t_3 \dots t_{p-1}}\right) d\lambda.$$

Here write t_{s-1} for t_s and t_{p-1} for λ : then

$$L = b^{2/p} F(b) ;$$

so that

$$b^2 F''(b) + bF'(b) - (b^{2/p} + n^2) F(b) = 0.$$

Now, in (2) put $b = (x/p)^p$, and it becomes

$$b^2 \frac{d^2y}{db^2} + b\frac{dy}{db} - (b^{2/p} + n^2/p^2)y = 0 :$$

and therefore

$$F(b) = AK_{pn}(pb^{1/p}) + BI_{pn}(pb^{1/p}).$$

Here let $b \to \infty$ and it is seen that B must be zero. (For the purpose of the proof it may be assumed for the time being that $n \geqq 0$.)

In order to determine A the equation may be put in the form

$$\prod_{s=1}^{p-1} \int_0^\infty K_n(t_s)\, t_s^{2s/p-1}\, dt_s \frac{\pi}{2 \sin n\pi} \left\{ I_{-n}\left(\frac{b}{t_1 \dots t_{p-1}}\right) - I_n\left(\frac{b}{t_1 \dots t_{p-1}}\right)\right\}$$
$$= A \frac{\pi}{2 \sin (pn\pi)} \{I_{-pn}(pb^{1/p}) - I_{pn}(pb^{1/p})\}.$$

Now multiply by b^n and let $b \to 0$; then

$$\prod_{s=1}^{p-1} \int_0^\infty K_n(t_s)\, t_s^{2s/p-1}\, dt_s \frac{2^n \pi}{2 \sin n\pi} \frac{(t_1 t_2 \dots t_{p-1})^n}{\Gamma(1-n)} = A \frac{\pi (2/p)^{pn}}{2 \sin (pn\pi)} \frac{1}{\Gamma(1-pn)},$$

or, from (4),

$$\prod_{s=1}^{p-1} 2^{n+2s/p-2}\, \Gamma(n + s/p)\, \Gamma(s/p) 2^{n-1}\, \Gamma(n) = A\, 2^{pn-1} p^{-pn}\, \Gamma(pn).$$

Hence, from (5) and from (5) with $1/m$ in place of z,

$$2^{(p-1)n-(p-1)+n-1} (2\pi)^{\frac{1}{2}p-\frac{1}{2}} p^{\frac{1}{2}-pn}\, \Gamma(pn) (2\pi)^{\frac{1}{2}p-\frac{1}{2}} p^{-\frac{1}{2}} = A\, 2^{pn-1} p^{-pn}\, \Gamma(pn).$$

Therefore,

$$A = \pi^{p-1}.$$

Thus formula (3) has been established.

§ 3. *A Multiple Integral involving Bessel Functions of the First Kind.* The formula to be proved is

$$\prod_{s=1}^{p-1} \int_0^\infty J_n(t_s)\, t_s^{2s/p-1}\, dt_s\, J_n\left(\frac{b}{t_1 t_2 \dots t_{p-1}}\right) = J_{pn}(pb^{1/p}), \quad \dots\dots(6)$$

where $b > 0$, $R(n) > \frac{1}{2} - \frac{2}{p}$. For the particular case, when $p = 2$,

$$\int_0^\infty J_n(t)\, J_n\left(\frac{b}{t}\right) dt = J_{2n}(2\sqrt{b}), \quad \dots\dots(7)$$

where $b > 0$, $R(n) > -\frac{1}{2}$, see Watson's *Bessel Functions* (**5**).

The proof depends on the differential equation

$$x^2y'' + xy' + (x^2 - n^2)y = 0, \qquad \text{.......................................(8)}$$

satisfied by $J_n(x)$ and $J_{-n}(x)$.

The formula

$$\int_0^\infty J_n(\lambda)\lambda^{m-1}\,d\lambda = 2^{m-1}\,\Gamma\left(\frac{m+n}{2}\right)\Big/\Gamma\left(1+\frac{n-m}{2}\right), \qquad \text{.........................(9)}$$

where $R(n+m)>0$, $R(m)<\dfrac{3}{2}$, is required (**6**).

Denoting the L.H.S. of (6) by $\phi(b)$, we have, as in § 2, if $R(n)>\dfrac{3}{2}-\dfrac{2}{p}$,

$$b^2\phi''(b) = -b\phi'(b) + n^2\phi(b) - L,$$

where

$$L = b^2\prod_{s=1}^{p-1}\int_0^\infty J_n(t_s)\,t_s^{2s/p-3}\,dt_s\,J_n\left(\frac{b}{t_1t_2\ldots t_{p-1}}\right).$$

On proceeding as in § 2, it is found that

$$L = b^{2/p}\phi(b),$$

so that

$$b^2\phi''(b) + b\phi'(b) + (b^{2/p} - n^2)\phi(b) = 0.$$

Now, in (8), put $b = (x/p)^p$ and it becomes

$$b^2\frac{d^2y}{db^2} + b\frac{dy}{db} + \left(b^{2/p} - \frac{n^2}{p^2}\right)y = 0.$$

Therefore

$$\phi(b) = AJ_{pn}(pb^{1/p}) + BJ_{-pn}(pb^{1/p}).$$

Here multiply by b^n and let $b\to 0$; then clearly B must be zero.

Again, to determine A, multiply by b^{-n} and let $b\to 0$; then

$$\prod_{s=1}^{p-1}\int_0^\infty J_n(t_s)\,t_s^{2s/p-n-1}\,dt_s\,\frac{1}{2^n\Gamma(n+1)} = A\,\frac{p^{pn}}{2^{pn}\Gamma(pn+1)}.$$

But, from (9), the L.H.S. is equal to

$$\prod_{s=1}^{p-1}2^{2s/p-n-1}\,\Gamma\left(\frac{s}{p}\right)\Big/\Gamma\left(n+\frac{p-s}{p}\right)\times\frac{1}{2^n\Gamma(n+1)} = \frac{2^{-n(p-1)}(2\pi)^{\frac{1}{2}p-\frac{1}{2}}\,p^{-\frac{1}{2}}}{2^n\,n\,(2\pi)^{\frac{1}{2}p-\frac{1}{2}}\,p^{\frac{1}{2}-pn}\,\Gamma(pn)}.$$

Hence $A=1$, so that (6) has been proved. By applying analytical continuation the restriction $R(n)>\dfrac{3}{2}-\dfrac{2}{p}$ can be altered to $R(n)>\dfrac{1}{2}-\dfrac{2}{p}$.

REFERENCES

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(**3**) Gray, Mathews and MacRobert, *Bessel Functions*, p. 66.
(**4**) MacRobert, T. M., *Complex Variable*, p. 154.
(**5**) Watson, G. N., *Bessel Functions*, p. 437.
(**6**) MacRobert, T. M., *Complex Variable* (3rd ed.), p. 372.

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