THE MIKI-GESSEL BERNOULLI NUMBER IDENTITY

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Abstract. A generalization due to Gessel [3] of Miki’s identity between Bernoulli numbers is shown to be a direct consequence of a functional equation for the generating function.

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1. Introduction. Recall that the Bernoulli numbers $B_n$ are defined by the formal power series:

$$T(X) = \frac{X}{e^X - 1} = \sum_{n \geq 0} \frac{B_n}{n!} X^n \in \mathbb{Q}[[X]].$$

More generally, the Bernoulli polynomials $B_n(\lambda) \in \mathbb{Q}[\lambda]$ are given by:

$$T_\lambda(X) = \frac{X e^{\lambda X}}{e^X - 1} = \sum_{n \geq 0} \frac{B_n(\lambda)}{n!} X^n \in (\mathbb{Q}[\lambda])[X].$$

In [3] Gessel established the following generalization of an identity due to Miki [4].

PROPOSITION 1.1. (Miki, Gessel et al.). For $k \geq 1$,

$$\sum_{i+j=k; i,j \geq 1} \frac{B_i(\lambda)}{i} \cdot \frac{B_j(\lambda)}{j} = \frac{2}{k} \left( B_k(\lambda) \cdot H_{k-1} + \sum_{r+2s=k; s \geq 1} \binom{k}{r} B_r(\lambda) \cdot \frac{B_{2s}}{2s} \right),$$

where $H_{k-1}$ is the harmonic number $\sum_{l=1}^{k-1} 1/l$.

The original result of Miki is obtained by setting $\lambda = 0$; the case $\lambda = 1/2$ reduces to an identity of Faber and Pandharipande which appeared in [2] (with a proof supplied by D. Zagier). See [3] for the details.

We shall deduce Proposition 1.1 from a functional equation for the generating function $T_\lambda(X)$. The proof is essentially a distillation of an argument given by Dunne and Schubert in the recent paper [1].

2. Proof. Let us write

$$\mathcal{L}(X) = \frac{X}{\tanh(X/2)} = \frac{X}{2} \left( \frac{e^X + 1}{e^X - 1} \right) = \sum_{m \geq 0} \frac{B_{2m}}{(2m)!} X^{2m} = T(X) + \frac{X}{2} \in \mathbb{Q}[[X]].$$
(The notation ‘\(T_\alpha\)’ and ‘\(\mathcal{L}\)’ is taken from Topology: for the Todd and Hirzebruch-L genera.)

It is then an elementary exercise to establish the functional equation:

\[
T_\alpha(tX) \cdot T_\alpha((1-t)X) = T_\alpha(X)((1-t)\mathcal{L}(tX) + t\mathcal{L}((1-t)X)) \quad (t \in \mathbb{R}),
\]

which may be read as an identity in the ring \((A[\lambda]][X]\), where \(A\) is the ring of polynomial functions \(\mathbb{R} \rightarrow \mathbb{R}\).

Now recall that

\[
\int_0^1 t^i (1-t)^j \frac{dt}{i!} = \frac{(1-i)! (j-1)!}{(i+j)!} \quad \text{for positive integers } i, j \geq 1,
\]
as follows, for example, from the identity \((X - Y) \int_0^1 e^{tX} e^{(1-t)Y} \ dt = e^X - e^Y\) in \(\mathbb{Q}[[X, Y]]\). It is also easy to verify that, for any integer \(k \geq 1\),

\[
\int_0^1 (-t^k + 1 - (1-t)^k) \frac{dt}{i!} = \int_0^1 \frac{(t-t^k) + (1-t-(1-t)^k)}{t(1-t)} \ dt = 2 \sum_{1 \leq l < k} \frac{1}{l} = 2H_{k-1}.
\]

By rearranging the functional equation (2.1) as

\[
(T_\alpha((tX) - 1) \cdot (T_\alpha((1-t)X) - 1) = (1 - T_\alpha(tX) + T_\alpha(X)) - T_\alpha((1-t)X))
\]

we obtain from the coefficient of \(X^k\), for \(k \geq 1\), the identity:

\[
\sum_{i+j=k; \ i, j \geq 1} B_i(\lambda) \cdot B_j(\lambda) \frac{t^i (1-t)^j}{i! j!} = (-t^k + 1 - (1-t)^k) \frac{B_k(\lambda)}{k!}
\]

\[
+ \sum_{r+s=k; \ r \geq 1} \frac{B_r(\lambda)}{r!} \cdot ((1-t)^s + t(1-t)^{2s}) \frac{B_{2s}}{(2s)!}
\]

in \(A[\lambda]\). Multiplying this identity by \(dt/(t(1-t))\) and integrating between 0 and 1 we deduce that

\[
\sum_{i+j=k; \ i, j \geq 1} \frac{(i-1)! (j-1)!}{(k-1)! i! j!} B_i(\lambda) \cdot B_j(\lambda) = 2H_{k-1} \frac{B_k(\lambda)}{k!} + 2 \sum_{r+s=k; \ s \geq 1} \frac{(2s-1)!}{(2s)! (2s)!} \frac{B_r(\lambda)}{r!} \cdot B_{2s}.
\]

This equality is easily recast in the form (1.1). \(\square\)

**Remark 2.2.** Further identities, of the type considered in [1] involving the gamma function, may be obtained by multiplying (2.1) by \(t^r (1-t)^s \ dt/(t(1-t))\) for any choice of real numbers \(x, y \geq 0\).

**References**